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**BANG-BANG CONTROL OF A SECOND-ORDER  
NON-LINEAR STABLE PLANT  
WITH SECOND-ORDER NONLINEARITY**

I. VAKILZADEH, A. A. KESHAVARZ

In this paper the design of a controller for a relay-controlled second-order non-linear stable plant with second-order nonlinearity is considered. The task of the controller is the simultaneous reduction of plant's output and output derivative to zero with the input to the feedback system being at  $z=0$ . It will be shown that for all initial values of output and output derivative it would be possible to bring them to zero, simultaneously and in the shortest possible time with at most one switching reversal of the relay. Through simple transformation, it would be shown that the equation of switching curve can be made independent of constant gain of the plant as well as the coefficient of its non-linear term.

**1. INTRODUCTION**

Second-order relay-controlled plants possessing two poles at the origin, two distinct negative real poles, two pure imaginary poles and two complex poles with negative real parts have been thoroughly studied in [1]. In [2], a second-order relay-controlled plant possessing one pole at the origin and one negative real pole have been studied. Reference [3] considers a relay-controlled plant possessing one negative real pole of order two. References [4] and [5] consider unstable relay-controlled plants, the former having a pole at the origin and a pole on the positive real axis while the latter having one positive and one negative real pole. In Chapter 7 of reference [1], time-optimal control of a class of second-order non-linear systems is investigated and it is shown that for non-linear systems, like linear ones, bang-bang control is indeed time optimal. A number of examples are considered, but no analytical treatment for obtaining the equation of switching curve is presented. Reference [6] considers analytical treatment of bang-bang control of a second-order non-linear plant with second-order and third-order nonlinearity, for both stable and unstable plants – see references [7], [8] and [9]. The results obtained in this paper and in reference [8] are very similar to those obtained for linear stable plants considered in references

[1], [2] and [3]. Similarly, results obtained in references [7] and [9] are analogous to those obtained in references [4] and [5] – i.e. provided the initial values of output and output derivative fall in a ‘controllable region’, then it is possible to reduce both of them to zero, simultaneously and in the shortest possible time with at most ‘one’ switching reversal of the relay.

## 2. ANALYSIS

Consider a plant whose input-output is governed by the second-order non-linear differential equation

$$(1) \quad \ddot{e} + a\dot{e}|e| = Ku(t) \quad a > 0, \quad |u(t)| \leq 1.$$

$a > 0$  means the second-order non-linear plant is a stable one. On the other hand,  $a < 0$  means the plant is an unstable one which has been fully treated in [7]. The plant represents a moving mass with frictional force which is proportional to the square of velocity. Our objective is the design of a controller for a feedback control system such that in the absence of any input to the system, i.e.  $r(t) = 0$ , will bring any output  $c(t)$  and output derivative  $\dot{c}(t)$  to the origin  $(0, 0)$  simultaneously and in the shortest possible time. As mentioned before, in reference [1] it is shown that bang-bang control is indeed the condition for time-optimality requirement, i.e.  $u(t) = \pm 1$ . Therefore, with  $r(t) = 0$ , the system’s error differential equation is given by

$$(2) \quad \ddot{e} + a\dot{e}|e| = -Ku \quad u = \pm 1.$$

The reduction of error and error derivative to zero will mean the reduction of output  $c(t)$  and output derivative  $\dot{c}(t)$  to zero.

Now

$$(3) \quad \ddot{e} = \frac{d\dot{e}}{dt} = \dot{e} \frac{d\dot{e}}{de}.$$

From Eqs. (2) and (3), we have

$$(4) \quad \dot{e} \frac{d\dot{e}}{de} + a\dot{e}|e| = -Ku.$$

For the solution of the non-linear differential equation (4), two cases for  $\dot{e}$  are considered:

(i)  $\dot{e} > 0$ .

In this case, Eq. (4) can be written as

$$(5) \quad \dot{e} \frac{d\dot{e}}{de} + a\dot{e}^2 = -Ku$$

or

$$(6) \quad \frac{de}{d\dot{e}} = \frac{-\dot{e}}{ae^2 + Ku}.$$

Multiplying both sides of Eq. (6) by (2a), we have

$$(7) \quad 2a \, de = \frac{-2a\dot{e}}{ae^2 + Ku} \, d\dot{e}.$$

Let

$$(8) \quad y_1 = 2ae$$

$$(9) \quad y_2 = \sqrt{\frac{a}{K}} \, \dot{e}.$$

Therefore, from Eqs. (7), (8) and (9), we have

$$(10) \quad dy_1 = \frac{-2y_2}{y_2^2 + u} \, dy_2.$$

Integrating both sides of Eq. (10), we get

$$(11) \quad y_1 - \xi_1 = -\ln \frac{y_2^2 + u}{\xi_2^2 + u}$$

where  $\xi_1 = y_1(0)$  and  $\xi_2 = y_2(0)$ . From Eq. (11), we have

$$(12) \quad y_2 = + [(\xi_2^2 + u) \exp(-y_1 + \xi_1) - u]^{1/2}, \quad u = \pm 1$$

(ii)  $\dot{e} < 0$ .

In this case, Eq. (4) can be written as

$$(13) \quad \dot{e} \frac{d\dot{e}}{de} - ae^2 = -Ku$$

or, as in Eqs. (6) to (9)

$$(14) \quad dy_1 = \frac{2y_2}{y_2^2 - u} \, dy_2.$$

Integrating both sides of Eq. (14), we get

$$(15) \quad y_1 - \xi_1 = \ln \frac{y_2^2 - u}{\xi_2^2 - u}$$

where, as before,  $\xi_1 = y_1(0)$  and  $\xi_2 = y_2(0)$ . From Eq. (15), we have

$$(16) \quad y_2 = - [(\xi_2^2 - u) \exp(y_1 - \xi_1) + u]^{1/2}, \quad u = \pm 1.$$

### 3. CONSTRUCTION OF SWITCHING CURVE

Figure 1 shows the phase-plane trajectories. The solid curves are for control  $u = +1$  and the dashed ones for control  $u = -1$ , the arrows show the direction of increasing time. These trajectories originate from the initial state  $(\xi_1, \xi_2)$  and terminate to the final state  $(-\infty, -1)$  for control  $u = +1$ , and  $(+\infty, +1)$  for control  $u = -1$ .

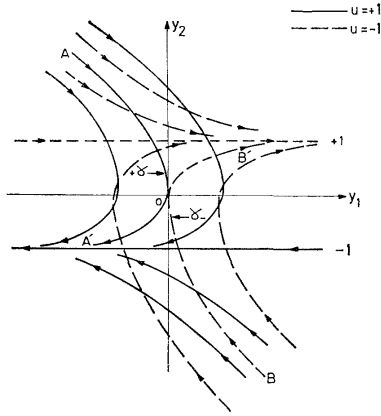


Fig. 1. The  $u = +1$  and  $u = -1$  forced trajectories in the  $y_1 - y_2$  plane.

$u = -1$ . The two trajectories which pass through the origin  $(0, 0)$  are marked  $AOA'$ , or  $\gamma_+$ , which is for control  $u = +1$  and  $BOB'$ , or  $\gamma_-$ , which is for control  $u = -1$ .

From Eqs. (12) and (16), we have

$$(17) \quad \text{Equation of AO: } y_2 = + [\exp(-y_1) - 1]^{1/2}$$

$$(18) \quad \text{Equation of OA': } y_2 = - [-\exp(y_1) + 1]^{1/2}$$

$$(19) \quad \text{Equation of BO: } y_2 = - [\exp(y_1) - 1]^{1/2}$$

$$(20) \quad \text{Equation of OB': } y_2 = + [-\exp(-y_1) + 1]^{1/2}$$

The switching curve is the union of segments AO and BO. From Eqs. (17) and (19), the equation of switching curve is given by

$$(21) \quad y_2 = - \frac{y_1}{|y_1|} [\exp(|y_1|) - 1]^{1/2}.$$

As seen from Fig. 1, for all states initially above the switching curve we must first apply control  $u = +1$  and when the state reaches the switching curve then the control  $u = -1$ . Conversely, for all states initially below the switching curve we must first apply control  $u = -1$  and when the state reaches the switching curve then the control  $u = +1$ . Therefore, all the initial states  $(\xi_1, \xi_2)$  can be brought to the origin  $(0, 0)$  simultaneously and in the shortest possible time by the application of one of the four possible control sequences:

$$[+1], [-1], [+1, -1], [-1, +1].$$

Figure 2 shows the relay-controlled plant with the designed controller.

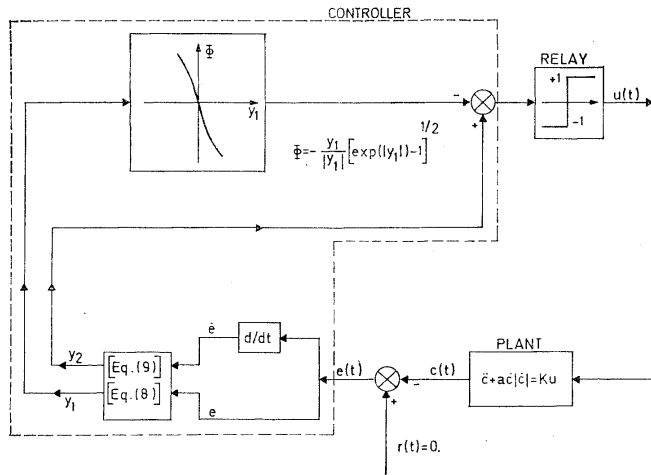


Fig. 2. Block diagram of bang-bang control of the non-linear plant with designed controller.

#### 4. CONCLUSION

In this paper, bang-bang control of a second-order non-linear stable plant with second-order nonlinearity was studied. Through simple transformation, an equation for the switching curve was obtained which is independent of both the d.c. gain ( $K$ ) and the coefficient of second-order nonlinearity ( $a$ ) of the plant. Hence, if  $r(t) = 0$ , then the designed controller will bring any initial values of output  $c(t)$  and output derivative  $\dot{c}(t)$  to zero, simultaneously and in the shortest possible time with maximum 'one' switching reversal of the relay.

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