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On the Minimum Time Problem in Linear Discrete Systems with the Discrete Set of Admissible Controls

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In this paper the minimum time problem for discrete linear systems with a discrete set of admissible controls is formulated and discussed. Neccessary conditions are stated in the form of a basic theorem. Using this theorem, a simple algorithm is developed for approximate solution of such problems.

1. INTRODUCTION

This paper can be divided into 3 main sections (§2, 3, 4). In §2 our basic problem is formulated. In the case of discrete set of admissible controls it is not possible to formulate the minimum time problem in the usual way [2]. Two possibilities of problem formulation are suggested with respect to two main methods for treatment of this problem. In §3 necessary conditions for the solution of the basic problem are stated in the form of a fundamental theorem. This theorem makes it possible to construct an algorithm which finds "desirable points" satisfying the necessary conditions in the finite number of iterations. This is done in §4. The suggested method is demonstrated on a simple example.

2. PROBLEM FORMULATION AND PRELIMINARIES

The following notation is employed: E^n is the Euclidean n-space, $\|\cdot\|$ is the Euclidean norm, $\langle\cdot,\cdot\rangle$ is the scalar product, x^j is the j-th coordinate of a vector x, A^T is the transpose of a matrix A, E is the unit matrix.

Given a linear dynamical system described by the difference equation

$$(1)\qquad x_{i+1} = Ax_i + Bu_i\,, \quad x_i \in E^n\,, \quad u_i \in E^m\,, \quad i = 0, 1, \ldots,$$

where x_i is the state of the system and u_i is the control applied to the system at time i, A resp. B is a $[n \times n]$ resp. $[n \times m]$ constant matrix. Let

$$(2) \qquad x_0 = \hat{x}_0$$

be the given initial state of the system and

$$(3) \qquad U = \left\{ u \,\middle|\, u^j = \pm \frac{l}{k_0}, \quad l = 0, 1, \ldots, k_0, \quad j = 1, 2, \ldots, m \right\},$$

where $k_0 \geqq 1$ is a given integer, be the set of admissible controls.

Basic problem (BP). Given the system (1), its initial state (2) and a certain "quasi-terminal" state $\hat{x}_T$. Let $O(\hat{x}_T)$ be any compact set with the point $\hat{x}_T$ in its interior. Find the sequence of admissible controls $\{\hat{u}_0, \hat{u}_1, \ldots, \hat{u}_{k_T-1}\}$ for which

$$(4) \qquad A^{k_T}\hat{x}_0 + \sum_{i=0}^{k_T-1} A^{k_T-i-1}B\hat{u}_i \in O(\hat{x}_T),$$

where $k_T \geqq 1$ is a finite integer and for all k, $1 \leqq k < k_T$

$$A^k\hat{x}_0 + \sum_{i=0}^{k-1} A^{k-i-1}Bu_i \notin O(\hat{x}_T)$$

for any admissible control sequence $\{u_0, u_1, \ldots, u_{k-1}\}$.

The set $O(\hat{x}_T)$ can be defined in various ways with respect to the physical interpretation of this mathematical model. Here we present two most important types of sets $O(\hat{x}_T)$. It is the ball

$$(5) \qquad B(\hat{x}_T, \delta) = \{x \mid \|x - \hat{x}_T\| \leqq \delta\}$$

and the polyhedron

$$(6) \qquad C(\hat{x}_T, \delta) = \{x \mid \sum_{j=1}^{n} |x^j - \hat{x}_T^j| \leqq \delta\}.$$

If $O(\hat{x}_T)$ is given by (6), it is possible to recast BP into the form of the sequence of mixed integer linear programs of the vast dimensionality (See [2]). If $O(\hat{x}_T)$ is given by (5), it is possible to transcript BP into the form of the sequence of pure integer quadratic programs, where the dimensionality is less than in the last case, but still very high.

Let us assume for simplicity, that $\hat{x}_T = 0$ and $O(\hat{x}_T)$ is given by (5). Then we can replace BP by the following sequence of problems. Find

$$\min \langle x_k, x_k \rangle \tag{7}$$

subject to

$$x_k = A^k \hat{x}_0 + \sum_{i=0}^{k-1} A^{k-i-1} B u_i ,$$

$$u_i \in U$$

for $k = 0, 1, \ldots$.

Clearly, $\langle x_k, x_k \rangle \leqq \delta$ for $k = k_T$ and $\langle x_k, x_k \rangle > \delta$ for all k, $1 \leqq k < k_T$. Let some k be fixed. Then a necessary condition for the sequence $\{\hat{u}_0, \hat{u}_1, \ldots, \hat{u}_{k-1}\}$ of admissible controls to be the solution of the problem 7 is stated in the following theorem.

3. BASIC THEOREM

Let $\{\hat{u}_0, \hat{u}_1, \ldots, \hat{u}_{k-1}\}$ be the solution of the problem 7 for some fixed integer k and $\hat{x}_k$ be the corresponding terminal state of the system (1). Then

$$\langle \hat{p}_{i+1}, B\,\Delta u_i \rangle + \langle B\,\Delta u_i, \hat{\Psi}_{i+1} B\,\Delta u_i \rangle \leqq 0 \tag{8}$$

for all Δu_i for which $\hat{u}_i + \Delta u_i \in U$, $i = 0, 1, \ldots, k - 1$. The vector $\hat{p}_i$ is the solution of the adjoint equation

$$\hat{p}_i = A^T \hat{p}_{i+1} , \quad i = 0, 1, \ldots, k - 1 , \tag{9}$$

with the terminal condition

$$\hat{p}_k = -2\hat{x}_k \tag{10}$$

and the matrix $\hat{\Psi}_i[n \times n]$ is the solution of the matrix adjoint equation

$$\hat{\Psi}_i = A^T \hat{\Psi}_{i+1} A , \quad i = 0, 1, \ldots, k - 1 , \tag{11}$$

with the terminal condition

$$\hat{\Psi}_k = -E . \tag{12}$$

Proof. Suppose that (8) is false. Then there must exist a vector Δu_i, $\hat{u}_i + \Delta u_i \in U$, such that

$$\langle \hat{p}_{i+1}, B\,\Delta u_i \rangle + \langle B\,\Delta u_i, \hat{\Psi}_{i+1} B\,\Delta u_i \rangle > 0 .$$

The Δx_k corresponding to Δu_i is

$$\Delta x_k = A^{k-i-1} B \, \Delta u_i \, .$$

For the corresponding variation of the cost functional we can write

$$\Delta \langle x_k, x_k \rangle = 2 \langle \hat{x}_k, A^{k-i-1} B \, \Delta u_i \rangle + \langle A^{k-i-1} B \, \Delta u_i, A^{k-i-1} B \, \Delta u_i \rangle =$$
$$= -[\langle \hat{p}_k, A^{k-i-1} B \, \Delta u_i \rangle + \langle A^{k-i-1} B \, \Delta u_i, \hat{\Psi}_k A^{k-i-1} B \, \Delta u_i \rangle] =$$
$$= -[\langle \hat{p}_{i+1}, B \, \Delta u_i \rangle + \langle B \, \Delta u_i, \hat{\Psi}_{i+1} B \, \Delta u_i \rangle] < 0 \, .$$

It is evident, that if we apply the control sequence $\{\hat{u}_0, \hat{u}_1, \ldots, \hat{u}_i + \Delta u_i, \ldots, \hat{u}_{k-1}\}$, then

$$\langle \hat{x}_k + \Delta x_k, \hat{x}_k + \Delta x_k \rangle < \langle \hat{x}_k, \hat{x}_k \rangle$$

for the corresponding state $\hat{x}_k + \Delta x_k$ of the system (1). However, this is a contradiction with the optimality of $\{\hat{u}_0, \hat{u}_1, \ldots, \hat{u}_{k-1}\}$, q. e. d.

The state $\hat{u}_k$ corresponding to the sequence $\{\hat{u}_0, \hat{u}_1, \ldots, \hat{u}_{k-1}\}$ is compared with states x_k, which we get by admissible variation of $\{\hat{u}_0, \hat{u}_1, \ldots, \hat{u}_{k-1}\}$ only at one stage i. Let us vary the sequence $\{\hat{u}_0, \hat{u}_1, \ldots, \hat{u}_{k-1}\}$ in all stages $i = 0, 1, \ldots, k - 1$ by admissible variations Δu_i. Then

$$\Delta \langle x_k, x_k \rangle = - \sum_{i=0}^{k-1} [\langle \hat{p}_{i+1}, B \, \Delta u_i \rangle + \langle B \, \Delta u_i, \hat{\Psi}_{i+1} B \, \Delta u_i \rangle + \tag{13}$$
$$+ 2 \langle A \, \Delta x_i, \hat{\Psi}_{i+1} B \, \Delta u_i \rangle] \, ,$$

where $\hat{p}_i$ resp. $\hat{\Psi}_i$ are given by (9) resp. (11). This variation of the cost functional cannot be decomposed into the convenient form (8). If it were possible, we could get the sufficient condition for the solution of the problem (7) with fixed k in the very constructive form. However, we shall use the basic theorem as it is and hope, that this necessary conditon will be strong enough to obtain a good approximate solution.

4. NUMERICAL METHOD FOR SOLVING THE PROBLEM (7) WITH FIXED k

Algorithm A

1. Select some sequence of admissible controls $\{\bar{u}_0, \bar{u}_1, \ldots, \bar{u}_{k-1}\}$.
2. Set $i = 0$.

3. Compute the corresponding trajectory $\{\hat{x}_0, \bar{x}_1, \bar{x}_2, \ldots, \bar{x}_k\}$ resp. multipliers $\bar{p}_1, \bar{p}_2, \ldots, \bar{p}_k$ resp. matrices $\bar{\Psi}_1, \bar{\Psi}_2, \ldots, \bar{\Psi}_k$ according to (1), (2) resp. (9), (10) resp. (11), (12).
4. Solve the following problem of discrete quadratic programming

$$\max\left(\langle \bar{p}_{i+1}, B\,\Delta u_i\rangle + \langle B\,\Delta u_i, \bar{\Psi}_{i+1} B\,\Delta u_i\rangle\right)$$

subject to

$$\bar{u}_i + \Delta u_i \in U$$

and store the solution $\Delta\bar{u}_i$.

5. Set $\bar{u}_i = \bar{u}_i + \Delta\bar{u}_i$.
6. If $i < k - 1$, set $i = i + 1$ and go to step 3, else go to step 7.
7. If $\Delta\bar{u}_i = 0$ for $i = 0, 1, \ldots, k - 1$, go to step 8, else go to step 2.
8. Set $\hat{u}_i = \bar{u}_i$ for $i = 0, 1, \ldots, k - 1$, $\hat{x}_i = \bar{x}_i$ for $i = 1, 2, \ldots, k$ and stop.

Theorem. Algorithm A constructs the finite sequence of sequences $\{\bar{u}_0, \bar{u}_1, \ldots, \bar{u}_{k+1}\}$, where the last one satisfies the necessary condition stated in the basic theorem.

Proof. It is evident that the computation terminates if and only if the sequences $\{\bar{u}_0, \bar{u}_1, \ldots, \bar{u}_{k-1}\}$ and $\{\hat{x}_0, \bar{x}_1, \bar{x}_2, \ldots, \bar{x}_k\}$ satisfy the basic theorem. The algorithm cannot cycle, because only such suitable variations Δu_i are performed for which the value of the cost function $\langle x_k, x_k\rangle$ decreases. As the number of feasible controls is finite, the algorithm will terminate in a finite number of iterations, q.e.d.

Remark. One iteration of the algorithm A terminates by the jump from the step 7 to the step 2.

The algorithm A has two disadvantages. First, we do not known any suitable estimate for the distance between $\hat{x}_k$ and the exact solution x_E of problem (7). However, we can profit from the following fact. Let

$$U_1 = \{u \mid -1 \leqq u^j \leqq 1\,,\quad j = 1, 2, \ldots, m\} \tag{14}$$

and x_* be the solution of the problem (7) with fixed k and U replaces by U_1. The point x_* can be easily obtained for instance by the contact function method [3; 4]. Now let $R_k(\hat{x}_0)$ be the reachable set of the system (1) at the k-th stage in case of the set of admissible controls U_1. Now, clearly, if $0 \notin R_k(\hat{x}_0)$, then

$$\|x_*\| \leqq \|x_E\| \leqq \|\hat{x}_k\| \tag{15}$$

and if $0 \in R_k(\hat{x}_0)$, then

$$0 \leqq \|x_E\| \leqq \|\hat{x}_k\|\,. \tag{16}$$

Thus, we can estimate the quality of the approximate solution $\hat{x}_k$ in comparison with the exact solution x_* of the problem (7) with U replaced by U_I. The second disadvantage of the algorithms A is a rather complicated implementation of the step 4. We must solve the pure integer quadratic program of low dimensionality k-times at each iteration. With respect to this low dimensionality, the bound and branch method can be applied in connection with some effective algorithms for quadratic programming [1]. For alternative approach see [6].

Now we shall demonstrate the suggested method on a simple example. Let

$$(17)\qquad x_{i+1} = \begin{bmatrix} 0{\cdot}9 & 0{\cdot}1 \\ 0{\cdot}3 & 0{\cdot}6 \end{bmatrix} x_i + \begin{bmatrix} 3 & 3 \\ 3 & 2 \end{bmatrix} u_i\,, \quad \hat{x}_0 = \begin{bmatrix} 10 \\ 10 \end{bmatrix},$$

$$u_j^i \in \{-1, -0{\cdot}8, -0{\cdot}6, -0{\cdot}4, -0{\cdot}2, 0, 0{\cdot}2, 0{\cdot}4, 0{\cdot}6, 0{\cdot}8, 1\}\,, \quad j = 1, 2\,.$$

We have to solve BP, where $O(\hat{x}^T) = \{x \in E^2 \mid \| \times \| \leqq 0{\cdot}3\}$.

For $k = 1$ and $\bar{u}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ the following problem is obtained from the basic theorem

$$\min\,(0{\cdot}72I^2 + 0{\cdot}52J^2 + 1{\cdot}12IJ + 22{\cdot}8I + 19{\cdot}2J)$$

subject to $-5 \leqq I, J \leqq 5, I, J$ integers.

This problem is solved by $I = J = -5$. Therefore

$$\hat{u}_0 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \quad \hat{x}_1 = \begin{bmatrix} 4 \\ 4 \end{bmatrix} = x_E = x_* \notin O(\hat{x}_T)\,.$$

For $k = 2$ and $\bar{u}_0 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$, $\bar{u}_1 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$ and $i = 0$, we have to find

$$\min\,(0{\cdot}65I^2 + 0{\cdot}51J^2 + 1{\cdot}15IJ - 3{\cdot}91I - 3{\cdot}5J)$$

subject to $0 \leqq I, J \leqq 10, I, J$ integers.

This problem is solved by $I = 1$, $J = 2$. Therefore corresponding

$$\bar{u}_0 = \begin{bmatrix} -0{\cdot}8 \\ -0{\cdot}6 \end{bmatrix}, \quad \bar{u}_1 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \quad \bar{x}_2 = \begin{bmatrix} -0{\cdot}24 \\ -0{\cdot}02 \end{bmatrix}.$$

After recomputing the equation (9) for the new terminal condition, we get for $i = 1$

$$\min\,(0{\cdot}72I^2 + 0{\cdot}52J^2 + 1{\cdot}2IJ - 0{\cdot}31I - 0{\cdot}3J)$$

subject to $0 \leqq I, J \leqq 10, I, J$ integers.

This problem is solved by $I = 0$, $J = 0$.

In the second iteration we get for $i = 0$

$$\min\,(0{\cdot}65I^2 + 0{\cdot}51J^2 + 1{\cdot}15IJ - 0{\cdot}31I - 0{\cdot}29J)$$

subject to $-1 \leqq I \leqq 9,\ -2 \leqq J \leqq 8,\ I,\ J$ integers.

This problem is solved by $I = 0$, $J = 0$ and we are done. Clearly, $\hat{u}_0 = \bar{u}_0$, $\hat{u}_1 = \bar{u}_1$, $\hat{x}_2 = \bar{x}_2 \in O(\hat{x}_T)$, $0 \leqq \|x_E\| \leqq 0{\cdot}24$.

On this simple example we can easily realise how much work we spare by using the basic theorem. If we wanted to solve this example for $k = 2$ by complete testing of all admissible sequences $\{u_0, \ldots, u_{k-1}\}$, we should have to compute 14 641 (11^4) norms of obtained states x_2 of the system (17). By application of the basic theorem we reduced this vast amount to 363. The additional work (solving equations (9), (11)) is quite negligible.

5. CONCLUSION

The BP can be easily recast into the form of discrete programming problem, as it was mentioned above. From this point of view the basic theorem is based on same ideas as all kinds of discrete maximum principles. It makes use of the special structure of the constrained set in the discrete programming, given by the system description (1), to decompose the original high dimensional problem of discrete programming into the series of low dimensional ones. "Finite" variations must be used as well as in the quasimaximum principle [5]. On the other hand, the basic theorem is the second order condition, because the first order condition would be quite useless.

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