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INFORMATION CHANNELS COMPOSED OF MEMORYLESS COMPONENTS

KAREL WINKELBAUER

Explicit bounds for the maximum length of n -dimensional codes at any admitted level of the probability of error are derived, valid for all n , in case that the channels considered are composed of a finite number of memoryless components. The special case studied by the author in [2], is discussed in this more general frame.

BASIC NOTATIONS

Given a finite non-empty set M , the symbol W_M means the class of all shift-invariant probability measures in the space M^I , where I denotes the set of all integers. A measure $m \in W_M$ (satisfying the relation $m \circ T_M^{-1} = m$) is defined on the σ -algebra F_M of Borel sets in M^I which may be generated by the class of "elementary" cylinders (a base of the topology) of the form $T_M^i[z]$, $i \in I$,

$$[z] = \bigcap_{0 \leq i < n} \{ \zeta \in M^I : \zeta_i = z_i \} \quad \text{for } z = (z_i, 0 \leq i < n) \in M^n;$$

here T_M is the shift (defined by $(T_M \zeta)_i = \zeta_{i+1}$). Define

$$\tilde{W}_M = \{ m \in W_M : m \text{ is ergodic w.r. to } T_M \}.$$

All the information channels considered here are supposed to have finite alphabets A, B (card $A \geq 2$, card $B \geq 2$), A the output alphabet, B the input alphabet. Denoting by xy the element in $(A \times B)^n$ for $x \in A^n, y \in B^n$ given by $(xy)_i = (x_i, y_i)$, $0 \leq i < n$, associate with any measure ω in $W_{A \times B}$ the information rate (convention: $\log = \log_2$)

$$J(\omega) = \lim_n (1/n) \sum_{x \in A^n, y \in B^n} \omega[xy] \log \frac{\omega[xy]}{\omega^A[x] \omega^B[y]},$$

where $\omega^A \in W_A, \omega^B \in W_B$ are the marginal measures determined by the conditions

$$\omega^A[x] = \sum_y \omega[xy], \quad \omega^B[y] = \sum_x \omega[xy].$$

A point $\zeta \in Z = (A \times B)^I$ is called regular if there is a (uniquely determined) measure $m_\zeta \in \tilde{W}_{A \times B}$ such that

$$m_\zeta[z] = \lim_n (1/n) \sum_{i=0}^{n-1} \chi_{Iz}(T_{A \times B}^i \zeta), \quad z \in \bigcup_n (A \times B)^n;$$

χ_E designates the characteristic function of $E \subset Z$. The set of all regular points in Z will be denoted by R . Since $\omega(R) = 1$, we may define, for $\omega \in W_{A \times B}$,

$$q(\Theta, \omega) = \min \{t \geq 0: \omega\{z \in R: J(m_\zeta) \leq t\} \geq \Theta\}, \quad 0 < \Theta \leq 1;$$

$$\bar{q}(\Theta, \omega) = \max \{t \geq 0: \omega\{z \in R: J(m_\zeta) \geq t\} \geq 1 - \Theta\}, \quad 0 \leq \Theta < 1.$$

The latter quantities are the lower and the upper Θ -quantiles of the random variable $(J(m_\zeta), \zeta \in R)$ w.r. to ω .

In the entire paper a channel (a discrete information channel, stationary and of zero past history; cf. [1]) is defined as a family $v = (v_\eta, \eta \in B^I)$ of probability measures v_η on F_A satisfying the relations

$$v_{\eta'}(T_A^i[x]) = v_\eta[x] \quad \text{for } \eta' \in T_B^i[y], \quad \eta \in [y], \quad x \in A^n, \quad y \in B^n, \quad n = 1, 2, \dots$$

Since $v_\eta[x]$ is constant for $\eta \in [y]$, define for $E \subset A^n$

$$v[E | y] = \sum_{x \in E} v[x | y], \quad v[x | y] = v_\eta[x], \quad \eta \in [y].$$

If $\mu \in W_B$ then $v\mu$ will denote the measure in $W_{A \times B}$ satisfying

$$v\mu[x y] = v[x | y] \mu[y], \quad x \in A^n, \quad y \in B^n, \quad n = 1, 2, \dots$$

The quantile function q_v of a channel v (cf. [5]) is defined by

$$q_v(\Theta) = \sup \{q(\Theta, v\mu): \mu \in \tilde{W}_B\}, \quad 0 < \Theta \leq 1.$$

As an auxiliary function we define

$$\bar{q}_v(\Theta) = \sup \{\bar{q}(\Theta, v\mu): \mu \in \tilde{W}_B\}, \quad 0 \leq \Theta < 1.$$

The quantile functions q_v and \bar{q}_v of a memoryless channel v are constant, both identically equal to the transmission-rate capacity of the channel (cf. [1]); recall that v is memoryless iff

$$v[x | y] = \prod_{0 \leq i < n} v[x_i | y_i].$$

If $0 < \varepsilon < 1$, $Y \subset B^n$ then a family $Q = (Q(y), y \in Y)$ of mutually disjoint sets $Q(y) \subset A^n$ is, by definition, an n -dimensional ε -code for a channel v of length $l_Q = \text{card } Y$ iff $v[Q(y) | y] > 1 - \varepsilon$ for all $y \in Y$. The maximum length of n -dimensional ε -codes will be denoted by $S_n(\varepsilon, v)$; in symbols:

$$S_n(\varepsilon, v) = \max \{l_Q: Q \text{ is an } n\text{-dimensional } \varepsilon\text{-code for } v\}.$$

The set of all families $p = (p(b), b \in B)$ of non-negative real numbers which add to one, will be denoted by P (the set of probability vectors on alphabet B). Let μ^p

be the measure in W_B satisfying the condition

$$\mu^n[y] = \prod_{0 \leq i < n} p(y_i), \quad y \in B^n (p \in P).$$

If $N(b | y) = \{i: y_i = b (0 \leq i < n)\}$, $s_p(b) = (p(b)(1 - p(b))^{1/2})$,

$$d = \max(\text{card } A, \text{card } B),$$

define

$$F_n(p) = \bigcap_{y \in B^n} \{y \in B^n: |N(b | y) - n p(b)| \leq 2s_p(b)(nd)^{1/2}\}, \quad p \in P.$$

An n -dimensional ε -code $Q = (Q(y), y \in Y)$ for a channel v is said to be a (p, ε) -code for p in P iff $Y \subset F_n(p)$. Define

$$S_n^*(\varepsilon, v, \mu^p) = \max \{l_Q: Q \text{ is an } n\text{-dimensional } (p, \varepsilon)\text{-code for } v\}.$$

The behaviour of the latter auxiliary quantity will be studied by means of the n -dimensional information density

$$I_n(xy; v\mu^p) = (1/n) \log (v[x | y]/(v\mu^p)^A [x]).$$

COMPOSED CHANNEL

We shall make the following assumptions: k is a natural number, $(v^\alpha, \alpha \in \mathcal{K})$ for $\mathcal{K} = \{1, \dots, k\}$ is a family of memoryless channels, $\xi = (\xi_1, \dots, \xi_k)$ is a probability vector satisfying the condition

$$\xi_0 = \min \{\xi_\alpha: \alpha \in \mathcal{K}\} > 0,$$

and v is the composed channel defined by

$$v_\eta(E) = \sum_{\alpha=1}^k \xi_\alpha v_\eta^\alpha(E), \quad \eta \in B^l, \quad E \in F_A;$$

the latter relation will be written as

$$v = \sum_{\alpha \in \mathcal{K}} \xi_\alpha v^\alpha.$$

Together with the given channel v we shall consider its "subchannel" $v^{\mathcal{A}}$ for $\mathcal{A} \subset \mathcal{K}$ (\mathcal{A} non-empty) defined by

$$v^{\mathcal{A}} = \sum_{\alpha \in \mathcal{A}} (\xi_\alpha / \xi(\mathcal{A})) v^\alpha, \quad \xi(\mathcal{A}) = \sum_{\alpha \in \mathcal{A}} \xi_\alpha,$$

particularly in case that \mathcal{A} belongs to the class

$$\mathbf{A}(\Theta) = \{\mathcal{A} : \mathcal{A} \subset \mathcal{K}, \quad \xi(\mathcal{A}) \geq \Theta\}, \quad 0 < \Theta \leq 1.$$

Let $\mathcal{R}_\alpha(p) = \sum \{p(b) v^\alpha[a | b] I_1(ab; v^\alpha \mu^p): a \in A, b \in B\}$,

$$r_p(\Theta, \xi) = \min_{\mathcal{A} \in \mathbf{A}(\Theta)} \max_{\alpha \in \mathcal{A}} \mathcal{R}_\alpha(p), \quad 0 < \Theta \leq 1,$$

$$r'_p(\Theta, \xi) = \max_{\mathcal{A} \in \mathbf{A}(1-\Theta)} \min_{\alpha \in \mathcal{A}} \mathcal{R}_\alpha(p), \quad 0 \leq \Theta < 1, \quad p \in P.$$

Then the quantile function of the composed channel is expressed by

$$q_v(\theta) = \max_{p \in P} r_p(\theta, \xi) = \lim_{\lambda \downarrow 0} \bar{q}_v(\theta - \lambda), \quad 0 < \theta \leq 1$$

(cf. Theorem 4 in [4] and Lemma 3 in [2]), analogously

$$\bar{q}_v(\theta) = \max_{p \in P} r'_p(\theta, \xi) = \lim_{\lambda \downarrow 0} q_v(\theta + \lambda), \quad 0 \leq \theta < 1,$$

and both q_v, \bar{q}_v are monotonically increasing, having the same set \mathcal{D}_v of discontinuity points in the open interval $(0, 1)$ satisfying the relations

$$\mathcal{D}_v = \{\theta: 0 < \theta < 1, \quad q_v(\theta) < \bar{q}_v(\theta)\}, \quad \mathcal{D}_v \subset \{\xi(\mathcal{A}): \mathcal{A} \subset \mathcal{X}\}.$$

In every open subinterval (θ_1, θ_2) not containing any discontinuity point from \mathcal{D}_v , q_v and \bar{q}_v are constant and equal to each other (cf. Theorem 4 in [3]).

Hence it follows that the set

$$P(\theta) = \{p \in P: r'_p(\theta, \xi) = \bar{q}_v(\theta)\} \quad \text{for } 0 \leq \theta < 1$$

is non-empty. Let

$$\begin{aligned} w_0 &= \min \{v^2[a \mid b]: a \in \mathcal{X}, \quad a \in A, \quad b \in B, \quad v^2[a \mid b] > 0\}, \\ p_0 &= \min \{p(b): b \in B, \quad p(b) > 0\}, \quad p \in P, \\ \delta_\varepsilon &= \frac{1}{4} \min \{|\xi(\mathcal{A}) - \varepsilon|: \mathcal{A} \subset \mathcal{X}, \quad \xi(\mathcal{A}) \neq \varepsilon\}, \\ K_\varepsilon &= (2d)^3 \sqrt{(d)(w_0 \delta_\varepsilon)^{-1/2}}, \quad p_\varepsilon = \sup \{p_0: p \in P(\varepsilon)\}, \quad 0 < \varepsilon < 1. \end{aligned}$$

The composed channel v is, by definition (cf. [2]), non-singular if $v^2[a \mid b] > 0$ for all $a \in \mathcal{X}$, $a \in A$, $b \in B$.

Theorem 1. If ε is a continuity point of the quantile function q_v (i.e., $0 < \varepsilon < 1$, $\varepsilon \notin \mathcal{D}_v$), then the maximum length of n -dimensional ε -codes for the channel v satisfies the inequalities

$$\begin{aligned} \log S_n(\varepsilon, v) &< n q_v(\varepsilon) + \log(2\delta_\varepsilon \xi_0)^{-1} + K_\varepsilon \sqrt{n}, \\ \log S_n(\varepsilon, v) &> n q_v(\varepsilon) - \log(2\delta_\varepsilon \xi_0)^{-1} - K_\varepsilon (p_\varepsilon)^{-1} \sqrt{n}, \end{aligned}$$

for $n = 1, 2, \dots$; if v is non-singular, then

$$|\log S_n(\varepsilon, v) - n q_v(\varepsilon)| < \log(2\delta_\varepsilon \xi_0)^{-1} + K_\varepsilon \sqrt{n}.$$

Remark. The non-singular case was treated in [2], but the method used there for finding the bounds must be modified for the case considered here, as will be seen below. On the other hand, Theorem 4 in [3] guarantees only the existence of a constant c_ε such that $|\log S_n(\varepsilon, v) - n q_v(\varepsilon)| < c_\varepsilon \sqrt{n}$, but yields no direct method for its computing.

LEMMATA

To show the validity of Theorem 1, we shall first prove two lemmas under the assumption that we are given a real number $\varepsilon' (0 < \varepsilon' < 1)$, a probability vector p in P , and a non-empty set $\mathcal{A} \subset \mathcal{X}$. Let $\omega^\alpha = (v^\alpha \mu^p)^\alpha$ for $\alpha \in \mathcal{A}$, $L(t) = t \log t^{-1}$ ($0 < t \leq 1$), $L(0) = 0$,

$$\begin{aligned} L_0 &= \max \{L(t): 0 \leq t \leq 1\} = e^{-1} \log e, \\ K' &= K'(\varepsilon') = d(\varepsilon')^{-(1/2)}, \\ \bar{K} &= \bar{K}(\varepsilon') = L_0 d^2(5\sqrt{d} + 2K' + 4K'(1 + 2\sqrt{d})^{1/2}), \\ K_0 &= K_0(\varepsilon') = K' L_0 d(d+1)(w_0)^{-(1/2)}, \\ K'_0 &= K'_0(\varepsilon') = (K' + 2\sqrt{d}) L_0(d^2 + 1)(w_0)^{-(1/2)}, \\ \lambda_n &= \lambda_n(\varepsilon') = (1/n)(\log(\xi_0)^{-1} + (\bar{K} + K_0)\sqrt{n}), \\ \lambda'_n &= \lambda'_n(\varepsilon', p_0) = (1/n)(\log(\xi_0)^{-1} + (\bar{K} + K'_0)(p_0)^{-1}\sqrt{n}), \\ \lambda''_n &= \lambda''_n(\varepsilon') = (1/n)(\log(\xi_0)^{-1} + (\bar{K} + K'_0)\sqrt{n}). \end{aligned}$$

If $N(a, b \mid x, y) = \{i: x_i = a, y_i = b(0 \leq i < n)\}$, $s_\alpha(a \mid b) = (v^\alpha[a \mid b] (1 - v^\alpha[a \mid b]))^{1/2}$, define for $\alpha \in \mathcal{A}$, $y \in B^n$,

$$\begin{aligned} \Gamma_n^\alpha(y; a, b) &= \{x \in A^n: |N(a, b \mid x, y) - N(b \mid y) v^\alpha[a \mid b]| \leq K' s_\alpha(a \mid b) \sqrt{N(b \mid y)}\}, \\ \Gamma_n^\alpha(y) &= \bigcap_{a \in \mathcal{A}, b \in B} \Gamma_n^\alpha(y; a, b), \quad \Gamma_n^{\omega^\alpha}(y) = \bigcup_{\alpha \in \mathcal{A}} \Gamma_n^\alpha(y). \end{aligned}$$

Lemma 1. If $x \in \Gamma_n^{\omega^\alpha}(y)$ for $y \in F_n(p)$ then

$$\min_{\alpha \in \mathcal{A}} \mathcal{R}_\alpha(p) - \lambda'_n < I_n(x; y; v^{\omega^\alpha} \mu^p) < \max_{\alpha \in \mathcal{A}} \mathcal{R}_\alpha(p) + \lambda_n.$$

Moreover, if v is non-singular, $I_n(x; y; v^{\omega^\alpha} \mu^p) + \lambda''_n > \min_{\alpha \in \mathcal{A}} \mathcal{R}_\alpha(p)$.

Proof. I. Given $\alpha \in \mathcal{A}$, suppose that $x \in \Gamma_n^\alpha(y)$; then $v^2[x \mid y] \geq w_0^n$, $\omega^\alpha[x] \geq (p_0 w_0)^n$ because of $\mu^n[y] \geq (p_0)^n$, and if $v^2[a \mid b] > 0$ and $\omega^\alpha[a] > 0$, respectively, then

$$\begin{aligned} &|N(a, b \mid x, y) - n p(b) v^\alpha[a \mid b]| < \\ &< \sqrt{n} (\sqrt{d} v^\alpha[a \mid b] + K' (v^\alpha[a \mid b])^{1/2}), \\ &|N(a \mid x) - n \omega_\alpha[x]| < \\ &< \sqrt{n} d(2\sqrt{d} (\omega^\alpha[a])^{1/2} + \bar{K}' (\omega^\alpha[a])^{1/4}) \end{aligned}$$

where $N(a \mid x) = \sum_{b \in B} N(a, b \mid x, y)$; $\bar{K}' = K'(1 + 2\sqrt{d})^{1/2}$; cf. [6], Chapter 2, and notice that $s_p(b) \leq \frac{1}{2}$. Similarly as in [6], loc. cit., we have

$$\begin{aligned} \log \frac{v^\alpha[x \mid y]}{\omega^\alpha[x]} &= \sum_{a,b} N(a, b \mid x, y) \log v^\alpha[a \mid b] - \sum_a N(a \mid x) \log \omega^\alpha[a] \leq \\ &\leq n \mathcal{R}_\alpha(p) \pm L_0 d^2(\sqrt{d} + 2K' + 4\sqrt{d} + 4\bar{K}') \sqrt{n}. \end{aligned}$$

Thus it is proved under the above assumptions that

$$(1) \quad |I_n(xy; v^\alpha \mu^p) - \mathcal{R}_s(p)| < \bar{K} \cdot n^{-(1/2)}.$$

II. Given α, β in \mathcal{A} , suppose that $x \in \Gamma_n^\alpha(y)$, and that $v^\beta[x | y] > 0$; then $v^\beta[a | b] = 0$ implies that $N(a, b | x, y) = 0$. Let

$$A_b = \{a \in A: v^\alpha[a | b] > 0, v^\beta[a | b] > 0\},$$

$$V_b = \sum_{a \in A_b} N(a, b | x, y) \log \frac{v^\beta[a | b]}{v^\alpha[a | b]} \quad \text{for } b \in B_y,$$

where $B_y = \{b \in B: N(b | y) > 0\}$. Since (cf. [2], (2.14))

$$\sum_{a \in A_b} v^\alpha[a | b] \log \frac{v^\beta[a | b]}{v^\alpha[a | b]} \leq L(v^\alpha[A_b | b]),$$

$$|N(a, b | x, y) - N(b | y) v^\alpha[a | b]| < \frac{1}{2} K' \sqrt{[N(b | y)]} \quad \text{for } b \in B_y,$$

we obtain the inequality

$$V_b < N(b | y) L(v^\alpha[A_b | b]) + \frac{1}{2} K' \log(w_0)^{-1} \cdot d \sqrt{[N(b | y)]}, \quad b \in B_y.$$

Case 1: If, for all a in A , $v^\beta[a | b] = 0$ implies that $v^\alpha[a | b] = 0$, then $v^\alpha[A_b | b] = 1$, and

$$V_b < \sqrt{[N(b | y)]} K' d \log(w_0)^{-(1/2)} \quad \text{for } b \in B_y.$$

Case 2: Let $v^\beta[a_b | b] = 0$ and $v^\alpha[a_b | b] > 0$, $a_b \in A$, $b \in B_y$; then

$$N(a_b, b | x, y) = 0 > N(b | y) v^\alpha[a_b | b] - K' (N(b | y) v^\alpha[a | b])^{1/2}$$

so that $\sqrt{[N(b | y)]} < K' (w_0)^{-(1/2)}$. Thus in both cases

$$V_b < \sqrt{[N(b | y)]} (K' (w_0)^{-(1/2)} L_0 + K' d \log(w_0)^{-(1/2)}) \quad \text{for } b \in B_y.$$

From here it follows that

$$(2) \quad \sum_{b \in B_y} V_b = \log \frac{v^\beta[x | y]}{v^\alpha[x | y]} < \sqrt{(n)} K_0.$$

III. Given α, β in \mathcal{A} , suppose that $x \in \Gamma_n^\alpha(y)$, and that $\omega^\beta[x] > 0$; then $\omega^\beta[a] = 0$ implies that $N(a | x) = 0$. Let

$$A_0 = \{a \in A: \omega^\alpha[a] > 0, \omega^\beta[a] > 0\},$$

$$V_0 = \log \frac{\omega^\beta[x]}{\omega^\alpha[x]} = \sum_{a \in A_0} N(a | x) \log \frac{\omega^\beta[a]}{\omega^\alpha[a]}.$$

Since $|N(a | x) - n \omega^\alpha[a]| < \frac{1}{2}(K' + 2\sqrt{(d)}) d \sqrt{(n)}$, we find that

$$V_0 < n L(\omega^\alpha[A_0]) + \frac{1}{2}(K' + 2\sqrt{(d)}) d^2 \log(p_0 w_0)^{-1},$$

because under the assumptions made, if $\omega^\alpha[a] > 0$ and $\omega^\beta[a] > 0$, respectively, then $\omega^\alpha[a] \geq p_0 w_0$, $\omega^\beta[a] \geq p_0 w_0$.

Case 1: If, for all a in A , $\omega^\beta[a] = 0$ implies that $\omega^\alpha[a] = 0$, then $\omega^\alpha[A_0] = 1$, and $V_0 < \sqrt{(n)} \cdot \frac{1}{2}(K' + 2\sqrt{(d)}) d^2 \log(p_0 w_0)^{-1}$.

Case 2: Let $\omega^\beta[a_0] = 0$ and $\omega^\alpha[a_0] > 0$, $a_0 \in A$; then $N(a_0 | x) = 0$. Choose b_0 in B such that

$$p(b_0) v^\alpha[a_0 | b_0] > 0;$$

then

$$N(a_0, b_0 | x, y) = 0 > n p(b_0) v^\alpha[a_0 | b_0] - K''(n p(b_0) v^\alpha[a | b])^{1/2} - K'(v^\alpha[a | b] N(b_0 | y))^{1/2}$$

where $K'' = 2\sqrt{(d)}$. Thus $\sqrt{(n)} < (K' + K'')(p_0)^{-1} (w_0)^{-(1/2)}$ so that

$$V_0 < \sqrt{(n)} \frac{K' + 2\sqrt{(d)}}{p_0 \sqrt{(w_0)}} (L_0 + d^2 L(p_0 \sqrt{(w_0)})).$$

In both cases we obtain that

$$(3) \quad \log \frac{\omega^\beta[x]}{\omega^\alpha[x]} < \sqrt{(n)} K'_0 \frac{1}{p_0}.$$

If v is non-singular, then $\omega^\alpha[a] \geq w_0$ for $\omega^\alpha[a] > 0$, and $\omega^\beta[a] \geq w_0$ for $\omega^\beta[a] > 0$, which yields (cf. Case 1) the inequalities

$$(4) \quad \log \frac{\omega^\beta[x]}{\omega^\alpha[x]} < \sqrt{(n)} (K' + 2\sqrt{(d)}) d^2 \log(w_0)^{-(1/2)} < \sqrt{(n)} K'_0.$$

IV. Let $\mathcal{A}^* = \{\alpha \in \mathcal{A}: x \in \Gamma_n^\alpha(y)\}$; by assumption, \mathcal{A}^* is non-empty. Write

$$I_n = I_n(x, y; v^\alpha \mu^\beta), \quad I_n^* = I_n(x, y; v^{\alpha^*} \mu^\beta),$$

$$\mathcal{B} = \{\beta \in \mathcal{A}: \beta \notin \mathcal{A}^*, v^\beta[x | y] > 0\}, \quad \mathcal{B}' = \{\beta \in \mathcal{A}: \beta \notin \mathcal{A}^*, \omega^\beta[x] > 0\},$$

$$r = \max_{\alpha \in \mathcal{A}} \mathcal{R}_\alpha(p), \quad r' = \min_{\alpha \in \mathcal{A}} \mathcal{R}_\alpha(p),$$

$$\varrho = (1/n)(r + \bar{K} \sqrt{n}), \quad \varrho' = (1/n)(r' - \bar{K} \sqrt{n}),$$

$$V = \sum_{\beta \in \mathcal{B}} \xi_\beta \left(\sum_{\alpha \in \mathcal{A}^*} \xi_\alpha \frac{v^\alpha[x | y]}{v^\beta[x | y]} \right)^{-1},$$

$$V' = \sum_{\beta \in \mathcal{B}'} \xi_\beta \left(\sum_{\alpha \in \mathcal{A}^*} \xi_\alpha \frac{\omega^\alpha[x]}{\omega^\beta[x]} \right)^{-1}.$$

If $\alpha \in \mathcal{A}^*$ then, according to (1),

$$2^{ne'} \omega^\alpha[x] < v^\alpha[x | y] < \omega^\alpha[x] 2^{ne}$$

so that $\varrho' < I_n^* < \varrho$. Applying the inequality (2), we obtain

$$\exp_2(n I_n) = \exp_2(n I_n^*) \cdot (1 + V)(1 + V')^{-1} < 2^{ne}(1 + (1 - \xi_0)(\xi_0)^{-1} \exp(K_0 \sqrt{n}))$$

so that $I_n < r + \lambda_n$. Analogously, making use of (3), we get $I_n > r' - \lambda'_n$. If v is non-singular, then $I_n > r' - \lambda'_n$ by (4). \square

Lemma 2. If $0 < \varepsilon < 1 - \varepsilon'$ then

$$\log S_n^*(\varepsilon, v^{\delta} \mu^p) < n(\max_{\alpha \in \mathcal{A}} \mathcal{R}_\alpha(p) + \lambda_n(\varepsilon')) + \log(1 - \varepsilon - \varepsilon')^{-1};$$

if $\varepsilon' < \varepsilon < 1$ then

$$\log S_n^*(\varepsilon, v^{\delta} \mu^p) > n(\min_{\alpha \in \mathcal{A}} \mathcal{R}_\alpha(p) - \lambda_n(\varepsilon', p_0)) - \log\left(\frac{4}{3}(\varepsilon - \varepsilon')^{-1}\right).$$

If v is non-singular, then it holds for $\varepsilon' < \varepsilon < 1$ that

$$n(\min_{\alpha \in \mathcal{A}} \mathcal{R}_\alpha(p) - \lambda_n(\varepsilon')) < \log S_n^*(\varepsilon, v^{\delta} \mu^p) + \log\left(\frac{4}{3}(\varepsilon - \varepsilon')^{-1}\right).$$

The proof is the same as that of Lemma 2 in [2], performed by making use of the preceding Lemma 1.

Theorem 2. Let $K_1(\delta) = \bar{K}(\delta) + K_0(\delta) + d \log e$, $K_2(\delta) = \bar{K}(\delta) + K_0(\delta)$, $0 < \delta < \frac{1}{2}$. Then the maximum length of n -dimensional ε -codes ($0 < \varepsilon < 1$) for the composed channel v satisfies the inequalities

$$\log S_n(\varepsilon, v) < n q_v(\varepsilon + 4\delta) + \log(2\delta\xi_0)^{-1} + \sqrt{(n)K_1(\delta)}$$

for $4\delta \leq 1 - \varepsilon$, and

$$\log S_n(\varepsilon, v) > n \bar{q}_v(\varepsilon - 4\delta) - \log(2\delta\xi_0)^{-1} - \sqrt{(n)(p_0)^{-1}K_2(\delta)}$$

for $\varepsilon \geq 4\delta$ and any $p \in P(\varepsilon - 4\delta)$. If v is non-singular, then

$$\log S_n(\varepsilon, v) > n \bar{q}_v(\varepsilon - 4\delta) - \log(2\delta\xi_0)^{-1} - \sqrt{(n)K_2(\delta)}, \quad \varepsilon \geq 4\delta.$$

Proof. Repeating the proof of Theorem 1 in [2] with the aid of the preceding lemma, we obtain that

$$\log S_n(\varepsilon, v) < n q_v(\Theta) + n \lambda_n(\varepsilon') + \log(1 - \varepsilon \cdot \Theta^{-1} - \varepsilon')^{-1} + d \log(n + 1)$$

for $\varepsilon < \Theta \leq 1$, $\varepsilon' > 0$, $\varepsilon' + \varepsilon \cdot \Theta^{-1} < 1$. Since $1 - \varepsilon \cdot \Theta^{-1} - \varepsilon' \geq \frac{1}{2}(\Theta - \varepsilon)$ for $\varepsilon' \leq \frac{1}{2}(\Theta - \varepsilon)$, $\log(n + 1) < \sqrt{(n) \log e}$, the first inequality follows from the preceding one by setting $\delta = \varepsilon' = \frac{1}{4}(\Theta - \varepsilon)$. Taking p in $P(\Theta')$ and using the second inequality of Lemma 2 above in the proof of Theorem 2 in [2], we derive the inequality

$$\log S_n(\varepsilon, v) > n \bar{q}_v(\Theta') - n \lambda'(\varepsilon', p_0) - \log\left(\frac{4}{3}(\varepsilon - \varepsilon')^{-1}\right)$$

for $0 \leq \Theta' < \varepsilon$, $0 < \varepsilon' < \varepsilon - \Theta'$. From here the second inequality given in the theorem follows for $\delta = \varepsilon' = \frac{1}{4}(\varepsilon - \Theta')$. The third inequality is verified analogously. \square

Theorem 1 is a corollary to Theorem 2 for $\delta = \delta_\varepsilon$ because $\bar{q}_v(\varepsilon - 4\delta_\varepsilon) = q_v(\varepsilon) = q_v(\varepsilon + 4\delta_\varepsilon)$ for $\varepsilon \notin \mathcal{D}_v$, $K_1(\delta_\varepsilon) < K_\varepsilon$, $K_2(\delta_\varepsilon) < K_\varepsilon$. In verifying the latter inequalities it is necessary to use the following relations: $\delta < \frac{1}{4}$, $d \geq 2$, $L_0 < (\frac{3}{4})^2$.

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