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A UNIFIED APPROACH FOR DESIGNING ROBUST LINEAR FEEDBACK CONTROLLERS

JAN LUNZE

A general approach for designing robust multivariable controllers is presented, which can be applied in the time and frequency domains for both continuous and discrete-time systems. The model uncertainties are dealt with as unknown-but-bounded uncertainties by means of multi-input multi-output comparison systems. On this basis, important generalizations of the description of incompletely known original systems, and a general methodology for designing robust controllers are presented. These results show important methodical similarities of a variety of recently published design procedures, which proved to be specializations of this new approach, and exhibit the way towards an improvement of the robustness analysis of feedback controllers.

1. INTRODUCTION

In the theory of multivariable feedback control, numerous synthesis and design principles have been elaborated. The Internal Model Principle yields a controller structure that makes command tracking and disturbances rejection possible [8], [15]. Design principles such as pole assignment, linear optimal control or the Direct and Inverse Nyquist Array Methods are elaborated to select controller parameters so as to ensure the stability and a well suited dynamical input-output (I/O) behaviour of the closed-loop system [30].

These methods are based on the assumption that the process to be regulated is completely known and can be described by a nearly precise linear model. This assumption is crucial because it enables the design engineer to check the stability and performance of the closed-loop system with confidence by algebraical tests or by simulation.

However, in many practical applications this assumption is not satisfied. Either the plant is not completely known, it must be dealt with on the basis of a simplified model, or cannot at all be described by any linear equations. In any case, the model has severe uncertainties, and the plant must be referred to as 'incompletely known system'.

The design of robust controllers is one way to overcome the difficulties that arise from the differences between the model and the real plant. A linear controller is determined so as to satisfy the given requirements on the closed-loop system although the model has severe uncertainties. This is to be ensured without adaptation, i.e. by appropriately selecting the parameters of a linear time invariant control law.

The main theoretical problems of robust control are to find out suitable models for the incompletely known plant, to evaluate the I/O-behaviour of the closed-loop system, and to develop methods for designing linear controllers in the presence of severe model uncertainties.

In the recent few years, different ways of solution to these problems have been published. Many of them start from a description of the I/O-error of the model by norm or sector bounds. Accordingly, the plant is thought of as a parallel connection of the approximate model and some error model, which describes the output error $f = y - y^{\wedge}$ by inequalities (Fig. 1(a)). In these models, the model errors are dealt with as unknown-but-bounded uncertainties. In the following, we will use such a kind of description of the model error, although alternative approaches are known, where the plant is described by a number of parameter sets of the model [1].

To characterize the robustness of the controller, two different methods have been used. The authors of [33], [34], [36] generalized the notion of stability margin as known from single-input single-output systems to multivariable control and investigated this property in connection with LQ-regulators. On the other hand, expressions for the I/O-behaviour of the original closed-loop system in terms of the closed-loop approximate model and the given open-loop model uncertainty bound are given in [7], [9], [10], [17], [19], [27]. This provides a basis to check the behaviour of the closed-loop system according to the given design specifications and to assess the possible effects of the plant uncertainties.

A comparison of all these different approaches shows that they work with similar methods in describing the plant, in designing the controller, and in evaluating the properties of the closed-loop system. For instance, many design procedures use the small gain theorem to check the stability of the closed-loop original system and evaluate the difference between the trajectories of the closed-loop model and the closed-loop original system by means of norm bounds. Nevertheless, theory is lacking a systems theoretic framework that unifies and generalizes these different approaches.

This *general background of the design of robust multivariable controllers* will be presented in this paper. Being applicable in the time and frequency domains and to both continuous and discrete-time systems it includes many known approaches as special cases and points out their methodical similarities (Section 6). This enables us to find the main sources of conservatism of the known robustness tests and to show a way towards better methods for evaluating the robustness properties.

At the same time, the following approach overcomes some severe restrictions of all the known methods. First, the model uncertainties are described by means of multi-input multi-output 'comparison systems' (Definition 1). Hence, in all calculations concerning the model uncertainties the character of the system as a multivariable dynamical system is preserved. At second, this approach is not restricted to linear original systems, because the error bound can be determined even for non-linear or time varying systems. As the third advantage, the structure of the plant model is more general than that used in the known approaches (Fig. 1).

In the next two sections the problems of modelling uncertain control systems and evaluating the I/O-behaviour of the closed-loop system will be solved. As a result, a general design principle is described in Section 4. Then the consequences for the design of robust controllers in the time and frequency domains, and general features of the controller design in the presence of unknown-but-bounded uncertainties are derived.

Notations

$\mathbb{C}$ and $\mathbb{R}$ denote the fields of complex and real numbers, respectively. $\mathbb{R}_+$ is the set of all non-negative real numbers, $\mathbb{C}_+$ the set of the complex numbers with non-negative real parts. All operators are defined between two extended normed function spaces L_e of appropriate dimensions containing functions of the types

$$f(p):\ \mathbb{T} \rightarrow \mathbb{C}^n \quad \text{with} \quad \mathbb{T} \subseteq \mathbb{C}_+$$

or

$$f(t):\ \mathbb{T} \rightarrow \mathbb{R}^n \quad \text{with} \quad \mathbb{T} \subseteq \mathbb{R}_+\,,$$

respectively (for details see [11]). The bars $|\cdot|$ signify that all elements of the vector or matrix are replaced by their absolute values. The relations $\leqq$, $>$ etc. apply for all elements of the vectors or matrices and for all arguments.

2. THE MODEL OF THE INCOMPLETELY KNOWN PLANT

To describe the the incompletely known original system (OS) a model is used that consists of two subsystems (SS) (Fig. 1(a)). SS 1 is completely known and described by some operator equations

$$\begin{aligned} y &= \boldsymbol{S}_{yu}\boldsymbol{u} + \boldsymbol{S}_{ys}\boldsymbol{s} \\ z &= \boldsymbol{S}_{zu}\boldsymbol{u} + \boldsymbol{S}_{zs}\boldsymbol{s} \end{aligned} \tag{1}$$

$\boldsymbol{u}$, $\boldsymbol{y}$, $\boldsymbol{s}$ and $\boldsymbol{z}$ are the control input, control output, interconnection input, and interconnection output, respectively. They are elements of extended normed function spaces L_e of appropriate dimensions with the same $\mathbb{T}$. The choice of $\mathbb{T}(\mathbb{T} \subseteq \mathbb{C}_+$, $\mathbb{T} \subseteq \mathbb{R}_+)$ determines whether the system is continuous or discrete-time and whether it is described in the time or in the frequency domain. In this and the next sections, this general model is used to derive general results, which are specified for frequency or time domain considerations in Section 5.

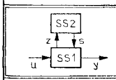
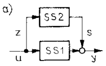
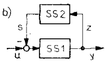
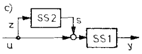
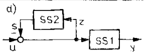
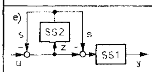
	Decomposition of the OS $y = Su$	Description of SS2: $s = S_2 z$ Description of SS1:	$\Sigma = \{S = \hat{S} + dS: \lvert s_2 \rvert \leq V_2$ and $dS = \dots\}$
	$S = \hat{S} + S_{ys} S_2 (I - S_{zs} S_2)^{-1} S_{zu}$	$y = \hat{S}u + S_{ys} s$ $z = S_{zu} u + S_{zs} s$	$dS = S_{ys} S_2 (I - S_{zs} S_2)^{-1} S_{zu}$
a) 	$S = \hat{S} + S_2$	$y = \hat{S}u + s$ $z = u$	$dS = S_2$
b) 	$S = \hat{S}(I + S_2 \hat{S})^{-1}$	$y = \hat{S}u - \hat{S}s$ $z = \hat{S}u - \hat{S}s$	$dS = -\hat{S} S_2 (I + \hat{S} S_2)^{-1} \hat{S}$
c) 	$S = \hat{S}(I + S_2)$	$y = \hat{S}u + \hat{S}s$ $z = u$	$dS = \hat{S} S_2$
d) 	$S = \hat{S}(I + S_2)^{-1}$	$y = \hat{S}u - \hat{S}s$ $z = u - s$	$dS = -\hat{S} S_2 (I + S_2)^{-1}$
e) 	$S = \hat{S}(I - S_2)(I + S_2)^{-1}$	$y = \hat{S}u - 2\hat{S}s$ $z = u - s$	$dS = -2\hat{S} S_2 (I + S_2)^{-1}$

Fig. 1. General and special structures of the model.

Ignoring the influence of SS 2 ($s \equiv 0$), SS 1 gives an approximation of the I/O-behaviour of the OS

$$y^{\wedge} = S_{yu} u . \tag{2}$$

SS 2 describes all the incompletely known properties or those properties of the OS, which are to be neglected in the controller design. In principle, it could be described by some I/O-relation

$$s = S_2 z , \tag{3}$$

but the operator S_2 is not exactly known. Therefore, it is only assumed that an auxiliary system

$$r_2 = V_2 w \tag{4}$$

is known that majorizes the I/O-behaviour of (3)

$$r_2 = V_2 |z| \geqq |s| . \tag{5}$$

Definition 1. A linear system (4) is called a *comparison system* (CS) of the OS (3), if the inequality (5) holds for all inputs $z(\cdot)$.

The CSs are linear positive systems [35], i.e.

$$V_2 w \geqq 0 \quad \text{for all} \quad w \geqq 0 . \tag{6}$$

Eqns. (1), (5) represent a set-theoretic model, in which the uncertainties are dealt with in the sense of unknown-but-bounded uncertainties. They describe the set Σ of all plants with input $\boldsymbol{u}$ and output $\boldsymbol{y}$ that can be decomposed into SS 1 (1) and some SS 2 of the form (3) satisfying the inequality (5). Although both SSs may be non-linear, SS 1 is usually a linear system.

The model (1), (5) has several important properties, which should be emphasized with respect to modelling incompletely known systems:

(1) Since $\boldsymbol{V}_2$ is the operator of a MIMO system, the character of the OS as a multivariable dynamical system is preserved even in the considerations of the effects of the model uncertainties.

(2) Parametrical as well as structural uncertainties are dealt with in a unified approach. They are interpreted as I/O-properties of the system (3) and described in the way shown in eqn. (5). Therefore, they need not be interpreted as e.g. intervals of system parameters.

(3) The model uncertainties are described by upper bounds. This corresponds to the engineers' kind of thinking. The determination of these bounds necessitates merely a distinction between 'possible' and 'impossible' model errors. No additional information such as multidimensional distribution densities is required.

(4) The model structure offers a considerable freedom in the choice of the signals $\boldsymbol{s}$ and $\boldsymbol{z}$. The OS can be decomposed such that SS 2 comprises only the incompletely known properties and has the least possible dimensions. This contrasts with nearly all known approaches to robust control, e.g. [7], [10], [17], [19] where the approximate model and the error model must be in parallel (Fig. 1(a)). Besides the use of MIMO systems on the right-hand side of eqn. (5), the freedom in the choice of the model structure enables us to describe the model uncertainties in a highly structured way.

3. I/O-BEHAVIOUR OF THE CLOSED-LOOP SYSTEM

Assume that a linear feedback controller

$$\boldsymbol{u} = \boldsymbol{S}_{ry}\boldsymbol{y} + \boldsymbol{S}_{rv}\boldsymbol{v} \tag{7}$$

is given, where $\boldsymbol{v}$ denotes the command signal. Then the I/O-behaviour of the closed-loop OS (1), (3), (7) can be analyzed as follows. Combining eqns. (1) and (7) provided that the operator $(\boldsymbol{I} - \boldsymbol{S}_{ry}\boldsymbol{S}_{yu})$ is invertible we have

$$\begin{aligned} \boldsymbol{y} &= \boldsymbol{S}_{yu}(\boldsymbol{I} - \boldsymbol{S}_{ry}\boldsymbol{S}_{yu})^{-1}\,\boldsymbol{S}_{rv}\boldsymbol{v} + \boldsymbol{S}_{yu}(\boldsymbol{I} - \boldsymbol{S}_{ry}\boldsymbol{S}_{yu})^{-1}\,\boldsymbol{S}_{ry}\boldsymbol{S}_{ys}\boldsymbol{s} \\ \boldsymbol{z} &= \boldsymbol{S}_{zu}(\boldsymbol{I} - \boldsymbol{S}_{ry}\boldsymbol{S}_{yu})^{-1}\,\boldsymbol{S}_{rv}\boldsymbol{v} + (\boldsymbol{S}_{zs} + \boldsymbol{S}_{zu}(\boldsymbol{I} - \boldsymbol{S}_{ry}\boldsymbol{S}_{yu})^{-1}\,\boldsymbol{S}_{ry}\boldsymbol{S}_{ys})\,\boldsymbol{s}\,. \end{aligned} \tag{8}$$

These equations will be abbreviated by

$$\begin{aligned} \boldsymbol{y} &= \bar{\boldsymbol{S}}_{yv}\boldsymbol{v} + \bar{\boldsymbol{S}}_{ys}\boldsymbol{s}\,. \\ \boldsymbol{z} &= \bar{\boldsymbol{S}}_{zv}\boldsymbol{v} + \bar{\boldsymbol{S}}_{zs}\boldsymbol{s}\,. \end{aligned} \tag{9}$$

For $s \equiv \mathbf{0}$ eqn. (9) represents an approximation of the closed-loop system

$$y^{\wedge} = \bar{S}_{yv} v \,. \tag{10}$$

The approximation error $y - y^{\wedge}$ is described by (5), (9) and (10). Eqns. (9) and (10) yield

$$\begin{aligned} y - y^{\wedge} &= \bar{S}_{ys} s \\ z &= \bar{S}_{zv} v + \bar{S}_{zs} s \,. \end{aligned} \tag{11}$$

Using eqn. (11) as 'original system', CSs can be derived (for details see Section 5). According to Definition 1 these CSs satisfy the inequalities

$$\begin{aligned} r_y &= V_{ys}|s| \geqq |y - y^{\wedge}| \\ r_z &= V_{zv}|v| + V_{zs}|s| \geqq |z| \,. \end{aligned} \tag{12}$$

With (5) and (12) we get

$$|z| \leqq V_{zv}|v| + V_{zs}V_2|z| \,, \tag{13}$$

since eqn. (6) holds for all CSs. For a further transformation of (13) we use

Lemma 1. Assume that V is a bounded positive operator, i.e. eqn. (6) and

$$\|V\| \leqq M < \infty$$

hold. If the spectral radius $\varrho(V)$ of the operator V satisfies the inequality $\varrho(V) < 1$ then $(I - V)^{-1}$ exists and the inverse operator is a bounded positive operator. This lemma follows directly from

$$(\lambda I - V)^{-1} = \sum_{k=0}^{\infty} \frac{V^k}{\lambda^{k+1}} \,,$$

which holds for $|\lambda| > \varrho(V)$ and is used here for positive operators V and $\lambda = 1$.

Accordingly, if

$$\varrho(V_{zs}V_2) < 1 \tag{14}$$

then

$$|z| \leqq (I - V_{zs}V_2)^{-1} V_{zv}|v| \,.$$

With (5) and (12) we get

$$|y - y^{\wedge}| \leqq V_{ys}V_2(I - V_{zs}V_2)^{-1} V_{zv}|v| \,. \tag{15}$$

Theorem 1. Consider the open-loop OS of the form (1), (3), which is described by the approximate model (4) and the uncertainty bound (5). For a given controller (7) the closed-loop OS (1), (3), (7) can be approximated by eqn. (10). If V_{zs} and V_2 are bounded and satisfy eqn. (14) then an upper bound of the approximation error is given by eqn. (15).

The eqns. (10) and (15) describe an envelope of the trajectory of the closed-loop OS. For every given command input v the shape of this band is given by the output $y^{\wedge}$ of the closed-loop approximate model (10), and the width is determined by the output of system (15) (Fig. 2 and 4).

The closed-loop OS is stable in the sense of I/O-stability [38] if the closed-loop approximate model (10) as well as the system (15) are stable. Hence we have

Theorem 2. The closed-loop OS (1), (3), (7) is I/O-stable, if all the operators in eqns. (4), (10) and (12) are bounded and if the inequality (14) is satisfied.

Hence the inequality (15) holds for all closed-loop OS whose stability is proved by Theorem 2.

Proof of Theorem 2. For the original closed-loop system (1), (3), (7) the inequality

$$|y| \leqq |y^\wedge| + |y - y^\wedge| \leqq (|\bar{S}_{yv}| + V_{ys}V_2(I - V_{zs}V_2)^{-1} V_{zv}) |v|$$

holds. Hence

$$\|y\| \leqq \| |\bar{S}_{yv}| \| + \|V_{ys}\| \cdot \|V_2\| \cdot \|(I - V_{zs}V_2)^{-1}\| \cdot \|V_{zv}\| \cdot \|v\| \leqq k\|v\| ,$$

where $\|y\|$ and $\|v\|$ denote the norm in the function space L_e and $\|V\|$ etc. is the induced operator norm. There is a $k < \infty$ such that this inequality holds because all operators are assumed to be bounded and eqn. (14) ensures the boundedness of $(I - V_{zs}V_2)^{-1}$ (cf. Lemma 1). Hence the closed-loop OS is I/O-stable. □

4. THE DESIGN OF ROBUST CONTROLLERS

4.1. The design problem

Definition 2. A *robust multivariable controller* is a linear time invariant feedback controller (7) that satisfies the given design requirements in connection with the original system (OS) with certainty, although the given model (1) describes the plant with severe uncertainties (3).

Since each element of the set Σ explained in Section 3 may be the representation of the OS, the design problem can be formulated as follows:

Given an incompletely known system described by an approximate model (1) and an error bound (5) such that the OS is known to be an element of the set Σ. Find a linear regulator (5) such that the following design requirements are met:

(I) The closed-loop system σ_c consisting of some plant $\sigma \in \Sigma$ and the controller (7) is stable.

(II) Asymptotic regulation occurs for σ_c according to a given class of command inputs and disturbances.

(III) The dynamical I/O-behaviour of the closed-loop system σ_c is well suited according to given specifications (such as bounds of the settling time and overshoot of the step responses, demands of certain degree of non-interaction, etc.).

(IV) The controller is robust enough to tolerate the model uncertainties, i.e. the requirements (I)–(III) are satisfied for all closed-loop systems σ_c consisting of some plant $\sigma \in \Sigma$ and the given controller (7).

4.2. Design strategy

The results of the preceding sections lead to the following design strategy:

(1) Find a model of the form (1), (5) for the incompletely known OS.

(2) Find a controller (7) such that the closed-loop approximate model (10) satisfies the design requirements (I)–(III).

(3) Check the robustness requirement (IV) by means of the stability condition of Theorem 2 and the envelope of the I/O-behaviour of the closed-loop OS given by the eqns. (10) and (15).

Note that in this design strategy the intricate problem of choosing the controller with respect to a set Σ of systems rather than a single plant is removed to the usual servomechanism problem (I)–(III) for a given plant (2). The result is a feedback controller (7) whose structure and parameters can be designed by means of the well known procedures, which are based on the assumption an exact model be available. Then, the robustness of the controller is investigated in a separate step according to the rules described in the next section.

4.3. Robustness of the controller

Assuming that the controller (7) is designed such that the closed-loop model (10) satisfies the design specifications (I)–(III), the robustness requirement (IV) can be checked by means of the following corollaries of Theorems 1 and 2:

- If the error model (4) is stable and the inequality (14) is satisfied, then the closed-loop OS is stable.
- If the controller (7) includes an internal model of the command and disturbance signals described in requirement (II) and if the stability of the closed-loop OS is proved, then asymptotic regulation occurs at least in a linear and some kind of non-linear closed-loop OS.
- If all trajectories possible within the tolerance bands given by (10) and (15) for specified command inputs $\boldsymbol{v}$ satisfy the requirements (III), then these requirements are fulfilled by the closed-loop OS.

The second statement must be restricted to linear and some classes of non-linear OSs, because the Internal Model Principle has been derived for linear systems [8] and extended for certain non-linear OSs or special controller structures.

5. DESIGN PROCEDURES IN THE TIME AND FREQUENCY DOMAINS

The design strategy presented in Section 4.2 can be used in the time and frequency domains for continuous as well as discrete-time systems. This will be shown for continuous systems, while the discrete-time counterpart is straightforward.

5.1. Frequency domain design

In the frequency domain all the given equations hold with $\mathbb{T} = \mathbb{C}_+$. If as usually SS 1 is linear, the operators S_{ij} and and V_{ij} must be replaced by transfer function matrices $S_{ij}(p)$ and $V_{ij}(p)$. The CSs (12) are given by

(16) $$V_{ys}(p) = |\bar{S}_{ys}(p)| , \quad V_{zv}(p) = |\bar{S}_{zv}(p)| , \quad V_{zs}(p) = |\bar{S}_{zs}(p)|$$

[23]. These matrices and the matrix V_2 in (19) have non-negative real elements depending on the complex variable $p \in \mathbb{C}_+$ (cf. eqn. (6)). Applying eqns. (8) and (16) the inequality (14) reads as

(17) $$\max_p \lambda_{\max}[|S_{zs} + S_{zu}(I - S_{ry}S_{yu})^{-1} S_{ry}S_{ys}| V_2] < 1 , \quad \text{where } p \in \mathbb{C}_+ ,$$

where $\lambda_{\max}(\cdot)$ represents the Perron-root (maximum eigenvalue) of the indicated non-negative matrix [5]. If all transfer function matrices are rational, eqn. (17) must only be proved for $p \in \mathscr{D}$, where $\mathscr{D}$ denotes the Nyquist contour.

Corollary 1. Consider the OS described by the linear approximate model

(18) $$\begin{aligned} y(p) &= S_{yu}(p)\, u(p) + S_{ys}(p)\, s(p) \\ z(p) &= S_{zu}(p)\, u(p) + S_{zs}(p)\, s(p) \end{aligned}$$

and the uncertainty bound

(19) $$|s(p)| \leqq V_2(p)\, |z(p)| .$$

With a given controller

(20) $$u(p) = S_{ry}(p)\, y(p) + S_{rv}(p)\, v(p)$$

the closed-loop OS can be approximated by

(21) $$y^{\wedge}(p) = S_{yu}(I - S_{ry}S_{yu})^{-1} S_{rv} v .$$

If the matrix norm of $V_2(p)$, $V_{ys}(p)$, $V_{zv}(p)$ and $V_{zs}(p)$ is bounded for all $p \in \mathbb{C}_+$ and eqn. (17) is satisfied, then the closed-loop OS is stable and an upper bound of the approximation error is given by

(22) $$|y(p) - y^{\wedge}(p)| \leqq V_{ys}V_2(I - V_{zs}V_2)^{-1} V_{zv}|v(p)| .$$

Eqns. (21) and (22) describe a linear approximation of the possibly non-linear closed-loop OS (1), (3), (7) and an error bound, respectively (Fig. 2).

This corollary extends the results derived in many papers on robust control in the frequency domain to non-linear OSs (1), (3), multidimensional error bounds (19), and the general model structure of Fig. 1. To demonstrate this, the well known results of the singular value approach are stated as specifications of our results. First consider the modelling stage. Fig. 3 shows that the model uncertainties can be described in a more structured way by multidimensional bounds than by a bound of the singular value (for example: $S_2 = (s_{11}, s_{12})$). Second, consider the stability condition. If the parallel model (Fig. 1(a)), a control error-actuated controller and model bounds in the form of norm inequalities are used, Corollary 1 states: The closed-loop OS is stable, if all transfer function matrices of eqns. (19), (22) represent stable systems and if

$$\|(I - S_{ry}S_{yu})^{-1} S_{ry}\|_M < \frac{1}{v_2(p)} \tag{23}$$

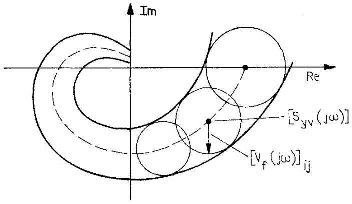


Fig. 2. Approximation of the element ij of the transfer function matrix of the closed-loop system by means of the corresponding elements in the right-hand sides S_{yv} and V_f of the eqns. (21) and (22).

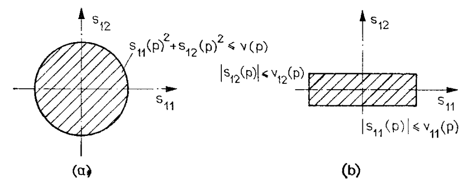


Fig. 3. Set of matrices $S_2 = (s_{11}\ s_{12})$ for SS 2 with dim $z = 2$, dim $s = 1$.
(a) $\|S_2(p)\| \leq v_2(p)$
(b) $|S_2(p)| \leq V_2(p)$

holds for all $p \in \mathbb{C}_+$ with $v_2(p)$ as upper bound of the matrix norm of the model error. Then the I/O-behaviour of the closed-loop OS can be approximated by

$$y^{\wedge}(p) = \bar{S}_{yv} v . \tag{24}$$

$$\|y(p) - y^{\wedge}(p)\| \leq \|\bar{S}_{ys}\|_M v_2(1 - \|\bar{S}_{zs}\|_M v_2)^{-1} \| \bar{S}_{zv}\|_M \|v\| . \tag{25}$$

$\|\cdot\|$ denotes a vector norm and $\|\cdot\|_M$ the induced matrix norm, which is used without respect to $p \in \mathbb{C}_+$. The stability condition (23) corresponds directly to those given e.g. in [3], [7], [9], [10], [13] and other papers concerning the singular value approach. Eqn. (25) extends these results to quantitative considerations of the robustness of the I/O-behaviour of the closed-loop OS.

As a further remark it should be mentioned that the result of Corollary 1 is closely related to the stability criterion that is being used in the generalized Nyquist Array methods for designing multivariable controllers. In these methods the OS is assumed to be exactly known, but only the diagonal elements of the transfer function matrix $G(p)$ of the plant

$$y(p) = \boldsymbol{G}(p)\, \boldsymbol{u}(p)$$

are used when designing the controller $\boldsymbol{u} = -\operatorname{diag} f_i(p)\,(\boldsymbol{y}(p) - \boldsymbol{v}(p))$. This can be interpreted as decomposing the plant into

$$\text{SS 1} \qquad \boldsymbol{y} = \operatorname{diag} g_{ii}(p) \,.\, \boldsymbol{u}(p) + \boldsymbol{s}(p)$$

$$z(p) = \boldsymbol{u}(p)$$

$$\text{SS 2} \qquad \boldsymbol{s}(p) = (\boldsymbol{G}(p) - \operatorname{diag} g_{ii}(p))\, z(p)\,.$$

After designing the controller elements $f_i(p)$ independently by means of $g_{ii}(p)$ ignoring the cross couplings, the stability of the closed-loop OS is proved. In this case the stability condition of Corollary 1 leads to

$$\lambda_{\max}\left(\left|\frac{f_i(p)}{1 - f_i(p)\, g_{ii}(p)}\right| \cdot \left| \boldsymbol{G}(p) - \operatorname{diag} g_{ii}(p)\right|\right) < 1$$

which coincides with the M-matrix condition presented in [2]. Eqn. (22) provides an *explicit* description of an upper bound of the uncertainty of the closed-loop system. It can be shown that eqn. (22) leads to the smallest values that could be found for this bound by means of the implicit characterization given in [2]. This demonstrate that, in principle, the Nyquist Array method uses the design strategy of Section 4.2, where the model uncertainties occur because of the neglection of the cross couplings of the plant during the design step.

5.2. Time domain design

In the time domain the linear operators $\boldsymbol{S}_{ij}$ and $\boldsymbol{V}_{ij}$ must be replaced by the convolution operators $\boldsymbol{S}_{ij}(t) *$ or $\boldsymbol{V}_{ij}(t) *$, respectively, where the star $*$ denotes the convolution operation

$$\boldsymbol{S}(t) * \boldsymbol{u}(t) = \int_{-\infty}^{+\infty} \boldsymbol{S}(t - \tau)\, \boldsymbol{u}(\tau)\, \mathrm{d}\tau\,.$$

Since most of the time domain design procedures start from state equations rather than convolution integrals, the approximate model and the controller should have the form

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u} + \boldsymbol{E}\boldsymbol{s}$$

$$(26) \qquad \boldsymbol{y} = \boldsymbol{C}_y\boldsymbol{x} + \boldsymbol{D}_y\boldsymbol{u} + \boldsymbol{F}_y\boldsymbol{s}$$

$$\boldsymbol{z} = \boldsymbol{C}_z\boldsymbol{x} + \boldsymbol{D}_z\boldsymbol{u} + \boldsymbol{F}_z\boldsymbol{s}$$

and

$$(27) \qquad \dot{\boldsymbol{x}}_r = \boldsymbol{A}_r\boldsymbol{x}_r + \boldsymbol{B}_y\boldsymbol{y} + \boldsymbol{B}_v\boldsymbol{v}$$

$$\boldsymbol{u} = \boldsymbol{K}_2\boldsymbol{x}_r + \boldsymbol{K}_1\boldsymbol{y} + \boldsymbol{K}_0\boldsymbol{v}\,.$$

Then the closed-loop model (26), (27) is described by

$$\dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}v + \bar{E}s$$
(28) $$y = \bar{C}_y\bar{x} + \bar{D}_y v + \bar{F}_y s$$
$$z = \bar{C}_z\bar{x} + \bar{D}_z v + \bar{F}_z s\,,$$

where $\bar{x} = (x', x_r')'$ and the matrices of eqn. (28) follow from those of eqns. (26) and (27). To use Theorems 1 and 2 eqn. (28) must be transformed into the I/O-model

(29) $$y = \bar{S}_{yv} * v + \bar{S}_{ys} * s$$
$$z = \bar{S}_{zv} * v + \bar{S}_{zs} * s$$

with

(30) $$\bar{S}_{yv}(t) = \bar{D}_y\,\delta(t) + \bar{C}_y \exp(\bar{A}t)\,\bar{B}$$

etc. Then the CSs (12) can be determined according to

(31) $$V_{ys}(t) = |\bar{S}_{ys}(t)| \quad \text{etc}\,.$$

The stability conditions of Theorem 2 requires that all impulse response matrices represent stable systems and that eqn. (14) holds. For the time domain representation of the positive operator V

$$\varrho(V) \leqq \lambda_{\max}\left[\int_0^\infty V(t)\,\mathrm{d}t\right]$$

holds [23]. Hence eqn. (14) is satisfied if

(32) $$\lambda_{\max}\left(\int_0^\infty |S_{zs}(t)|\,\mathrm{d}t \int_0^\infty V_2(t)\,\mathrm{d}t\right) < 1\,.$$

Corollary 2. Consider the OS described by the approximate model (26) and the uncertainty bound

(33) $$|z(t)| \leqq V_2 * |s(t)|\,.$$

Assume that a controller (27) has been designed. Then the closed-loop OS can be approximated by

(34) $$y^\wedge(t) = \bar{S}_{yv} * v\,.$$

If $V_2(t)$, $V_{ys}(t)$, $V_{zv}(t)$ and $V_{zs}(t)$ are bounded, i.e.

$$\int_0^\infty V_i(t)\,\mathrm{d}t \leqq M_i < \infty\,,$$

and the inequality (32) is satisfied, then the closed-loop OS is stable and an upper bound of the approximation error is given by

(35) $$|y(t) - y^\wedge(t)| \leqq |\bar{S}_{ys}(t)| * V_2(t) * \bar{V}(t) * |\bar{S}_{zv}(t)| * |v(t)|$$
$$\bar{V}(t) = \delta(t)\,I + |S_{zs}(t)| * V_2(t) * \bar{V}(t)\,.$$

This result is illustrated by the example of Fig. 4, which is taken from [26]. If the only information about the OS is represented by a band that includes the step responses of the OS then the middle of this band can be used as approximation of the step response of the OS while the distance between the approximate model and the border of the band describes an upper bound of the approximation error.

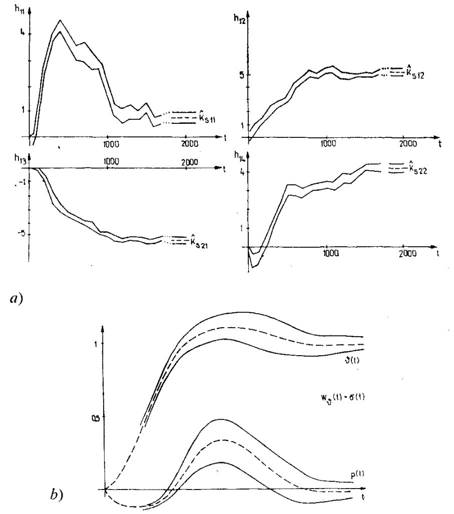


Fig. 4. Tolerance bands of the step responses of the plant (boiler) and the closed-loop system.
(a) Bands received by measuring the step responses of the boiler in several experiments
(b) Bands of the closed-loop system received from eqns. (34), (35) for $y_1(t)$ and $y_2(t)$ if $v_1(t) = \sigma(t)$ and $v_2 = 0$.

Using this model of the plant and a multivariable I-controller eqns. (34), (35) yields a tolerance band which includes the step response of the closed-loop OS. The dashed lines are given by eqn. (34) whereas the distance between the dashed and the solid lines is given by eqn. (35).

As in the frequency domain, several known approaches can be proved to be specifications of this result. For examples the stability conditions published in [3], [12], [17], [27], [28] are similar to that in eqn. (32) but weaker, because bounds of the norm of the impulse response matrices of the closed-loop model are used rather than bounds for the absolute values of each element (cf. eqn. (31)).

Additionally, it can be shown that a lot of procedures for designing decentralized controllers use, in principle, the way described in Section 4.2. As the control stations are designed for the isolated subsystems it must be checked whether the ignored couplings within the plant will be tolerated by these controllers. Although starting from Lyapunov functions of the closed-loop subsystems these procedures use only bounds of these Lyapunov functions in the stability test. This can be interpreted as using single-input single-output CS of first order. Hence the robustness tests of the methods presented e.g. in [6], [17], [19], [37], [39] use the results of Corollary 2 (for details see [23]).

6. GENERAL FEATURES OF THE DESIGN OF ROBUST CONTROLLERS

From our general approach the similarities of a lot of existing design procedures and important generalizations become obvious:

- The model of the incompletely known plant consists of two SSs the first being the approximate model and the second representing the model error. Both SSs can be interconnected in an arbitrary way (Fig. 1).
- The basis for describing the model errors as unknown-but-bounded uncertainties is represented by Definition 1 of the CS which possesses the majorization property (5). This general formulation extends the applicability of this error estimate to non-linear and time varying OSs and to multi-input multi-output CSs.
- All the approaches use the design procedure described in Section 4.2, in which the problems of modelling the incompletely known plant, designing the controller, and evaluating the robustness of the controller are solved in different steps one after another.
- In the considerations of the uncertainties in the closed loop some conservatism occur because of the following two sources:

1. The model errors of the plant are described by some linear model (4) with property (5), which is necessarily a positive system (cf. eqn. (6)). Hence, the difference between the upper bound r_2 on the left-hand side of eqn. (5) and $|s|$ cannot be made arbitrarily small, even if SS 2 in eqn. (3) is completely known. For this reason, the model of the plant cannot be set up in such a way that it reflects all the information about the plant, which are actually available.

2. The closed-loop system is analysed by means of the CS (12) rather than the complete description of the system (11), which would be available. Therefore, the stability condition (14) is of 'small-gain type' and the error bound (15) is too broad, i.e. there need not exist a plant (1), (3) satisfying eqn. (5) such that eqn. (15) or eqns. (22) and (35) holds for the closed-loop system with the equality sign at least for one instant of time t or frequency p.

The general considerations in Section 3 show that these are *the only two sources* of conservatism in the evaluation of the robustness of the controller.

- The choice of the controller must be a compromise between two aims:

- The controller must yield a well suited 'nominal trajectory' $y^{\wedge}$.
- The controller must be chosen so as to satisfy the stability condition (14) and to get a sufficiently small tolerance (15).

For great model errors these aims are contradictory, i.e., the tolerance band cannot be made, by appropriately choosing the controller parameters, to have a shape such that all trajectories within this band satisfy the design specifications. Hence during the design it becomes obvious whether the quantitatively given model uncertainties are small enough in relation to the given design specifications to be tolerable by linear controllers (see e.g. [19]).

The methodical similarities do *not* mean that the results in the time and frequency domains correspond via the Laplace transform. But they show that the procedures in both domains use the same systems theoretic framework. Hence the results of this paper can be used to 'transform' design principles from the time domain into the frequency domain and vice versa.

For instance, with these similarities in mind we can unify the design methods for decentralized and structurally constrained centralized controllers, which possess the properties of connective stability or integrity, respectively. Until now, decentralized controllers are mostly designed in the time domain [37] while centralized controllers with integrity properties are obtained by frequency domain methods [2]. However, interpreting the properties of connective stability and integrity as robsutness properties of the regulator with respect to sensor and actuator failures, both the design problems can be formulated in a unified way and solved by means of the method presented in Section 4.2. In this way, generalisations of the known methods are obtained for designing decentralized controllers in the time domain (cf. Section 5.2) and centralized controllers in the frequency domain (cf. Section 5.1). Moreover, the general design principle yields new methods for the design of decentralized controllers in the frequency domain and centralized controllers in the time domain. This unification of the design tools is described in more detail in [24].

At second, the elaboration of the system theoretic background for the design of robust controllers point out the two sources of conservatism of all the different design procedures that use this way of solution. To overcome this conservatism, the new design principles must be based on methods for describing the model uncertainties in a more structured way than that given by the use of upper bounds (5) as well as on ways of analysing the 'error model' (11) directly rather than by means of the CS (12). While the former would yield a smaller difference between the model and the real plant, the latter would lead to necessary and sufficient conditions for the stability and the smallest possible error bound (15) for the trajectories of the closed-loop OS. Unfortunately, the price for these improvements is a large computing effort even for simple models. This has been shown in the very recent papers [14], [25], in which a necessary and sufficient condition for robust stability and an algorithm for calculating the smallest tolerance band of the step response of the closed-loop system is described, respectively.

7. CONCLUSIONS

A general approach for designing robust multivariable controllers has been presented which unifies and generalizes many well known procedures. Using the general model structure of Fig. 1 and the definition of the model error bound by means of a majorization property, Theorems 1 and 2 present the most general results concerning the robustness of multivariable feedback controllers in the presence of unknown-but-bounded uncertainties. On this basis, important methodical similarities of the known design procedures are elaborated. As a direct consequence, design methods can be improved and transformed from the time into the frequency domain and from continuous to discrete-time systems and vice versa. Moreover, the two sources of conservatism of the known methods for analysing the robustness are exhibited. They show that conservatism can only be reduced if the model uncertainties are described in a more structured way than only by upper bounds, and if all the information included in this model is actually used in the robustness analysis.

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REFERENCES

[1] J. Ackermann: Parameter space design of robust control systems. IEEE Trans. Automat. Control *AC-25* (1980), 6, 1058–1072.

[2] M. Araki and O. I. Nwokah: Bounds for closed-loop transfer functions of multivariable systems. IEEE Trans. Automat. Control *AC-20* (1975), 5, 666–670.

[3] K. J. Åström: Robustness of design method based on assignment of poles and zeros. IEEE Trans. Automat. Control *AC-25* (1980), 3, 588–591.

[4] A. Y. Balakrishnan: Applied Functional Analysis. Springer-Verlag, New York 1976.

[5] A. Berman and P. R. Plemmons: Nonnegative Matrices in the Mathematical Sciences. Academic Press, New York 1973.

[6] J. Bernussou and J. C. Geromel: Stability approach to robust control for interconnected systems. In: Handbook of Large Scale Systems Engineering Applications (M. G. Singh and A. Titli, eds.), North-Holland, Amsterdam 1979.

[7] D. Carlccui and M. Vallauri: Feedback control of linear multivariable systems with uncertain description in the frequency domain. Internat. J. Control *33* (1981), 5, 803–812.

[8] E. J. Davison: The robust control of a servomechanism problem for time-invariant multivariable systems. IEEE Trans. Automat. Control *AC-21* (1976), 1, 25–34.

[9] J. C. Doyle: Achievable performance in multivariable feedback systems. IEEE Conf. Decision Control, Houston 1979, 250–251.

[10] J. C. Doyle and G. Stein: Multivariable feedback design: concept for a classical/modern synthesis. IEEE Trans. Automat. Control *AC-26* (1981), 1, 4–16.

[11] C. A. Desoer and M. Vidyasagar: Feedback System. I/O-properties. Academic Press, New York 1975.

[12] R. A. El-Attar and M. Vidyasagar: Subsystem simplification in large-scale systems analysis. IEEE Trans. Automat. Control *AC-24* (1979) 2, 321–323.

[13] E. A. Freeman: Stability of linear constant multivariable systems. Contraction mapping approach. Proc. Inst. Electr. Engrs. *120* (1973), 5, 379–384.

[14] B. Kouvaritakis and H. Latchman: Necessary and sufficient stability criterion for systems with structured uncertainties: the major principal direction alignment property. Internat. J. Control (to appear).

[15] H. G. Kwatny and K. C. Kalnitzky: On alternative methodologies for the design of robust linear multivariable regulators. IEEE Trans. Automat. Control *AC-23* (1978), 5, 930—933.

[16] J. Lunze: Übersicht über die Verfahren zum Entwurf dezentraler Regler für lineare Systeme. Messen Steuern Regeln *23* (1980), 6, 315—322.

[17] J. Lunze: Ein Verfahren zum Entwurf von Mehrgrößenreglern für unvollständig bekannte Regelstrecken. Messen Steuern Regeln *25* (1981), 8, 428—432.

[18] J. Lunze: Notwendige Modellkenntnisse zum Entwurf robuster Mehrgrößenregler mit I-Charakter. Messen Steuern Regeln *25* (1982), 11, 608—612.

[19] J. Lunze: The design of robust feedback controllers for partly unknown systems by optimal control procedures. Internat. J. Control *36* (1982), 4, 611—630.

[20] J. Lunze: Entwurf robuster Mehrgrößenregler bei beschränkter Modellunsicherheit. ZKI-Informationen Heft 4, Berlin: Akademie der Wissenschaften der DDR 1982 (available from the author).

[21] J. Lunze: Eine Entwurfsstrategie für die dezentrale Regelung zur Störbekämpfung in gekoppelten Systemen. 27. Internat. Wiss. Kolloquium, Ilmenau (DDR), Part 1, 111—114.

[22] J. Lunze: The design of robust feedback controllers in the time domain. Internat. J. Control *39* (1984), 6, 1243—1260.

[23] J. Lunze: Robust Multivariable Feedback Control. ZKI-Informationen No. 3, Berlin: Akademie der Wissenschaften der DDR 1984 (available from the author).

[24] J. Lunze: Ein Kriterium zur Überprüfung der Integrität dezentraler Mehrgrößerenregler. Messen Steuern Regeln *28* (1985), 6, 257—260.

[25] J. Lunze und E. Zscheile: Analyse der Stabilität und des Übergangsverhaltens linearer Systeme mit beschränkter Parameterunsicherheit. Messen Steuern Regeln *28* (1985), 8, 365—369.

[26] J. Lunze: Determination of robust multivariable I-controllers by means of experiments and simulation. System Anal. Model. Simul. *2* (1985), 3, 227—249.

[27] D. H. Owens and A. Chotai: Robust control of unknown discrete multivariable systems. IEEE Trans. Automat. Control *AC-27* (1982), 1, 188—190.

[28] D. H. Owens and A. Chotai: Controller design for unknown multivariable systems using monotone modelling errors. Proc. Inst. Electr. Engrs. *D-129* (1982), 3, 57—69.

[29] D. H. Owens and A. Chotai: On eigenvalues eigenvectors and singular values in robust stability analysis. Internat. J. Control *40* (1984), 2, 285—296.

[30] R. V. Patel and N. Munro: Multivariable System Theory and Design. Pergamon Press, Oxford 1982.

[31] M. G. Safonov: Frequency domain design of multivariable control systems for insensitivity to large plant modeling errors. IEEE Conf. Decision Control, Fort Lauderdale 1979, 247—249.

[32] M. G. Safonov: Robustness of Multivariable Feedback Systems. MIT Press, Cambridge, Mass. 1980.

[33] M. G. Safonov: Stability margins of diagonally perturbed multivariable feedback systems. IEEE Conf. Decision Control, San Diego 1981, 1472—1478.

[34] M. G. Safonov and M. Athans: Gain and phase margin for multiloop LQG-regulators. IEEE Trans. Automat. Control. *AC-22* (1977), 2, 173—179.

[35] J. Schröder: Lineare Operatoren mit positiver Inversen. Arch. Rat. Mech. Anal. *8* (1961), 408—434.

[36] M. E. Sezer and D. D. Šiljak: Robustness of suboptimal control: gain and phase margin. IEEE Conf. Decision Control, Albuquerque 1980, 878—884.

[37] D. D. Šiljak: Large Scale Dynamic Systems. North-Holland, New York 1978.
[38] M. Vidyasagar: Nonlinear Systems Analysis. Prentice Hall, Englewood Cliffs 1978.
[39] J. Weissenberger: Tolerance of decentrally optimal controllers to nonlinearity and coupling. 12. Allerton Conf., Illinois 1974.

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