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Kybernetika, Vol. 21 (1985), No. 4, 287--297

Persistent URL: http://dml.cz/dmlcz/125455

KYBERNETIKA — VOLUME 21 (1985), NUMBER 4

OPTIMAL DISCRETE APPROXIMATION OF CONTINUOUS LINEAR OPERATORS APPLICABLE TO CONTROL PROBLEMS

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This article is devoted to some problems of optimal discrete approximation of continuous linear operators. It presents continuation of our work [18] and brings a more general theorem of approximation (Theorem 2.1), some examples of the form of cardinal spline-functions (in Sec. 3) and an example of solution of a simple optimum control problem (Sec. 4), on which we try to compare the current and new optimal formulae for derivatives. Some necessary facts and notations are introduced in Section 1. Our results are based on a theory which was originally developed for purposes of numerical quadratures and cubatures and for the interpolation [11], [13], [14], [15]. Bibliographical notes see e.g. in [7], [10]. In the following we deal mostly with approximation of derivatives.

1. THEORETICAL BACKGROUND

Let H be a real Hilbert space with the scalar product $\langle \cdot, \cdot \rangle$ and the norm $\|\cdot\|$. Let $\langle h, \cdot \rangle$ and $\langle \varphi_1, \cdot \rangle, \ldots, \langle \varphi_n, \cdot \rangle$ be some bounded linear functionals on H. We shall approximate h by the expression $\sum_{i=1}^{n} c_i \varphi_i$ with c_i real in such a way that the norm of error functional will be minimized:

$$(1.1) \qquad \left\| h - \sum_{i=1}^{n} c_i^0 \varphi_i \right\| = \min_{c} \left\| h - \sum_{i=1}^{n} c_i \varphi_i \right\|, \quad \text{where} \quad c \in \mathbb{R}^n .$$

The coordinates of $\mathbf{c}^0 = (c_1^0, \ldots, c_n^0)$ are called the optimal coefficients. The elements φ_i, $i = 1, \ldots, n$ are supposed to be linearly independent. Then they form a *natural* basis of a linear n-dimensional subspace $S \subset H$. It is easy to prove the existence of the unique set of linearly independent elements $\sigma_j \in S$, $j = 1, \ldots, n$, for which the following conditions hold

$$(1.2) \qquad \langle \varphi_i, \sigma_j \rangle = \delta_{ij}; \quad i, j = 1, \ldots, n .$$

Here δ_{ij} is Kronecker's symbol. We call this set $\{\sigma_1, \ldots, \sigma_n\}$ the *cardinal basis* of S. Our following concept is based on

Theorem 1.1. The real numbers $c_1^0, \ldots, c_n^0$ are optimal coefficients in the approximation of bounded linear functional $\langle h, \cdot \rangle$ on H by means of $\langle \sum_{i=1}^{n} c_i \varphi_i, \cdot \rangle$, iff

$$c_j^0 = \langle h, \sigma_j \rangle, \quad j = 1, \ldots, n, \tag{1.3}$$

where $\sigma_1, \ldots, \sigma_n$ are given as the solution of system (1.2). For $h \in S$ (1.1) is equal to zero.

Proof. The nearest element in S to the given $h \in H$ is the orthogonal projection of h to S. Therefore $h - \sum_{i=1}^{n} c_i^0 \varphi_i$ is orthogonal to all elements σ_j of the cardinal basis

$$\langle \sigma_j, h - \sum_{i=1}^{n} c_i^0 \varphi_i \rangle = 0, \quad j = 1, \ldots, n,$$

which, using the property (1.2), immediately leads to (1.3). □

2. APPROXIMATION OF DERIVATIVES ON SOBOLEV SPACE

We shall investigate the optimal coefficients of formulae, which approach the linear bounded operators on Sobolev spaces $H^q(a, b)$ with integer $q \geqq 2$. The elements of $H^q(a, b)$ are absolutely continuous functions on $[a, b]$ with quadratic integrable generalized (in sense of theory of distributions) derivatives of order q. The scalar product and the corresponding norm are introduced in the following manner (see, e.g. [3] or [15], Chapter XIV, § 4). Let be given q arbitrary points

$$-\infty < a < x_1 < \ldots < x_q < b < +\infty. \tag{2.1}$$

and put

$$\langle f, g \rangle_H = \int_a^b f^{(q)}(x)\, g^{(q)}(x)\, dx + \sum_{i=1}^{q} f(x_i)\, g(x_i), \tag{2.2}$$

$$\|f\|_H^2 = \langle f, f \rangle_H. \tag{2.3}$$

This norm is equivalent to the usual norm

$$\|f\|^2 = \sum_{i=0}^{q} \int_a^b [f^{(i)}(x)]^2\, dx. \tag{2.4}$$

The main advantage of the norm given by (2.2)–(2.3) is connected with the fact that we obtain the elements of the linear subspace S in a very simple form. Namely we will show that the functionals of the form

$$\langle \varphi_{p,y}, f \rangle_H = f^{(p)}(y), \quad a < y < b, \quad 0 \leqq p < q$$

are generated by spline functions, i.e. $\varphi_{p,y}$ are piecewise polynomials. Introducing another norm, e.g. (2.4), this will not be true.

The following inequalities are well known as the Sobolev imbedding theorem:

$$\|f^{(p)}\|_{C(a,b)} \leqq K \|f\|_H, \quad p = 0, 1, \ldots, q - 1, \tag{2.5}$$

where $q > 1$ and K does not depend on $f \in H^q(a, b)$. Therefore the values of function and its derivatives of order less then q in given interior points of $]a, b[$ are linear bounded functionals on $H^q(a, b)$ and so there is reasonable to find the optimal approximations for them using Theorem 1.1. We will find the approximative formulae for evaluation of some derivative using the values of function and its derivatives of lower order in the points of a given net. Generally, let be given the points

$$
\begin{aligned}
&a < x_{01} < \ldots < x_{0n_0} < b \\
&a < x_{11} < \ldots < x_{1n_1} < b \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&a < x_{p1} < \ldots < x_{pn_p} < b \,,
\end{aligned} \tag{2.6}
$$

where $0 \leqq p \leqq q - 1$, $q > 1$. Moreover, suppose that $n_0 \geqq q$ and that the points (2.1) are some arbitrary, but fixed of $x_{01}, \ldots, x_{0n_0}$. We introduce the bounded linear functionals on $H^q(a, b)$, for which

$$
\begin{aligned}
&\langle \varphi_{kl}, f \rangle_H = f^{(k)}(x_{kl}) \,, \quad \forall f \in H^q(a, b) \,, \\
&k = 0, \ldots, p \,; \quad l = 1, \ldots, n_k \,.
\end{aligned} \tag{2.7}
$$

Those functionals are linearly independent. Indeed, in $H^q(a, b)$ there evidently exist elements ψ_{kl} such that $\psi_{kl}^{(k)}(x_{kl}) = 1$ and $\psi_{st}^{(s)}(x_{st}) = 0$ for all remainding ordered pairs $(s, t) \neq (k, l)$. Let us suppose

$$\sum_{s,t} \lambda_{st} \varphi_{st} = 0 \,,$$

where not all $n_0 + n_1 + \ldots + n_p$ constants λ_{st} are equal to zero. Multiplying the left side scalarly in $H^q(a, b)$ by ψ_{kl}, we obtain

$$\langle \psi_{kl}, \sum_{s,t} \lambda_{st} \varphi_{st} \rangle_H = \lambda_{kl} \,,$$

therefore all $\lambda_{kl} = 0$ and this contradiction proves the linear independency of φ_{kl}. We denote by S the linear subspace spanned on all φ_{kl} defined by (2.7). This subspace S is, as we have just prooved, n-dimensional, where

$$n = n_0 + n_1 + \ldots + n_p \,. \tag{2.8}$$

The set of all elements φ_{kl} is the natural basis of S. The cardinal basis in S consists of elements σ_{kl} with the following properties:

$$\langle \varphi_{kl}, \sigma_{st} \rangle = \delta_{ks} \delta_{lt} \,, \tag{2.9}$$

where

$$
\begin{aligned}
k, s &= 0, 1, \ldots, p \,, \\
l &= 1, \ldots, n_k \,, \\
t &= 1, \ldots, n_s \,.
\end{aligned}
$$

The system of n equations (2.9), where n is given by (2.8), has exactly one solution of the form

$$\sigma_{st} = \sum_{u,v} \lambda_{stuv} \varphi_{uv} \tag{2.10}$$

where λ_{stuv} are real constants. This fact follows from linear independency of the elements φ_{uv}, as the matrix of linear system (2.9) after substitution from (2.10) is the Gramm matrix $[\langle\varphi_{kl}, \varphi_{st}\rangle]_{k,l,s,t}$, which is necessarily regular. The elements φ_{kl} with properties (2.9) and generally the elements of the linear subspace S can be found using the following

Theorem 2.1. Let the points (2.1) be given, let $y \in \left]a, b\right[$ and let p be an integer such that $0 \leqq p < q$, $q > 1$. Then the unique element $\psi \in H^q(a, b)$, such that

$$\langle\psi, f\rangle_H = f^{(p)}(y)\,, \quad \forall f \in H^q\,, \tag{2.11}$$

is the solution of the following boundary value problem:

$$\psi^{(2q)}(x) = 0\,, \quad x \in \left]a, b\right[- \bigcup_{i=1}^{q} \{x_i\} - \{y\}\,, \tag{2.12}$$

$$(-1)^{q-p}[\psi^{(2q-p-1)}(y_+) - \psi^{(2q-p-1)}(y_-)] = \begin{cases} 1 - \psi(y)\,, & \text{if } p = 0 \\ \text{and } y = x_i & \text{for some} \\ i = 1, \ldots, q\,, \\ 1 & \text{otherwise} \end{cases} \tag{2.13}$$

$$(-1)^{q+1}[\psi^{(2q-1)}(x_{i+}) - \psi^{(2q-1)}(x_{i-})] = \psi(x_i)\,, \quad i = 1, \ldots, q\,, \quad x_i \neq y\,, \tag{2.14}$$

$$\psi^{(q)}(a) = \psi^{(q)}(b) = \psi^{(q+1)}(a) = \psi^{(q+1)}(b) = \ldots = \psi^{(2q-1)}(a) = \psi^{(2q-1)}(b) = 0\,, \tag{2.15}$$

$$\psi(\cdot), \ldots, \psi^{(2q-p-2)}(\cdot) \quad \text{are continuous on} \quad [a, b]\,. \tag{2.16}$$

Denoting by $(z)_+ = \max(0, z)$, $z \in \mathbb{R}^1$ we can express the function $\psi(\cdot)$ in the form

$$\psi(x) = \eta_{q-1}(x) + \frac{(-1)^{q-p}}{(2q-p-1)!}(x - y)_+^{2q-p-1} + \sum_{i=1}^{q} \frac{(-1)^{q-1}\psi(x_i)}{(2q-1)!}(x - x_i)_+^{2q-1}\,, \tag{2.17}$$

where $\eta_{q-1}(\cdot)$ is a suitable polynomial of order less than q.

Proof. According to the Sobolev imbedding theorem (2.5) for an arbitrary $y \in \left]a, b\right[$ and $0 \leqq p < q$, $q \geqq 2$ the linear functional (2.11) is bounded. The existence and uniqueness of ψ in (2.11) follows from the theorem by F. Riesz.

On the other hand it is easy to verify that the solution of boundary value problem (2.12) ÷ (2.16) can be found in the form (2.17). Notice that the unknown q coefficients of the polynomial η_{q-1} and q values $\psi(x_i)$, $i = 1, \ldots, q$ are determined from the equations $\psi^{(q)}(b) = \psi^{(q+1)}(b) = \ldots = \psi^{(2q-1)}(b) = 0$ (q independent equations) and from the substitution of x_i, $i = 1, \ldots, q$ into (2.17) (also q independent equations). Let $\psi(\cdot)$. be the function (2.17). Using integration by parts and denoting $a = x_0$,

$b = x_{q+1}$, we express the scalar product as follows:

$$
(2.18) \qquad \langle \psi, f \rangle_H = \int_{x_0}^{x_{q+1}} \psi^{(q)}(x)\, f^{(q)}(x)\, \mathrm{d}x + \sum_{i=1}^{q} \psi(x_i)\, f(x_i) =
$$

$$
\psi^{(q)}(x)\, f^{(q-1)}(x)\Big|_{x_0}^{x_{q+1}} - \psi^{(q+1)}(x)\, f^{(q-2)}(x)\Big|_{x_0}^{x_{q+1}} + \ldots
$$

$$
\ldots + (-1)^{q-p-1}\, \psi^{(2q-p-1)}(x)\, f^{(p)}(x)\Big|_{x_0}^{-y} + (-1)^{q-p-1}\, \psi^{(2q-p-1)}(x)\,.
$$

$$
.\, f^{(p)}(x)\Big|_{y+}^{x_{q+1}} + \ldots + \sum_{i=0}^{q} (-1)^{q-1}\, \psi^{(2q-1)}(x)\, f(x)\Big|_{x_i+}^{x_{i+1}} +
$$

$$
+ (-1)^q \int_{x_0}^{x_{q+1}} \psi^{(2q)}(x)\, f(x)\, \mathrm{d}x + \sum_{i=1}^{q} \psi(x_i)\, f(x_i)\,.
$$

Using properties (2.12) ÷ (2.16) the equality (2.11) can be verified. □

Remark. From the previous theorem it follows that the elements of linear subspace S are spline-functions of order equal or less than $2q - 1$. Moreover, in the intervals $]a, \min(x_1, y)[$ and $]\max(x_q, y), b[$ they are equal to polynomials of order $r \leqq q - 1$. For $x \in\,]a, \min(x_1, y)[$ this assertion follows directly from (2.17) and for $x \in\,]\max(x_q, y), b[$ from the conditions (2.15) in b.

Example. Let be $a = 0$, $b = 1$, $p = 1$, $q = 2$ and $x_1 = \frac{1}{4}$, $x_2 = \frac{3}{4}$, $y = \frac{1}{2}$. In this case using expression (2.17) there is

$$
(2.19) \quad \psi(x) = a_1 x + a_0 - \tfrac{1}{2}(x - \tfrac{1}{2})_+^2 - \tfrac{1}{6}[\psi(\tfrac{1}{4})\,(x - \tfrac{1}{4})_+^3 + \psi(\tfrac{3}{4})\,(x - \tfrac{3}{4})_+^3]\,.
$$

Denoting by $\mathbf{1}(\cdot)$ the unit jump and by $\delta(\cdot)$ the Dirac delta distribution, we express

$$
\psi''(x) = -\mathbf{1}(x - \tfrac{1}{2}) - \psi(\tfrac{1}{4})\,(x - \tfrac{1}{4})_+ - \psi(\tfrac{3}{4})\,(x - \tfrac{3}{4})_+\,,
$$

$$
\psi'''(x) = -\delta(x - \tfrac{1}{2}) - \psi(\tfrac{1}{4})\,\mathbf{1}(x - \tfrac{1}{4}) - \psi(\tfrac{3}{4})\,\mathbf{1}(x - \tfrac{3}{4})\,,
$$

and applying conditions (2.15) at $b = 1$, we obtain the solution of equations $\psi''(1) = \psi'''(1) = 0$ as $\psi(\frac{1}{4}) = -2$, $\psi(\frac{3}{4}) = 2$. Now it is possible to determine the coefficients a_0, a_1 using the substitution $x = \frac{1}{4}$ and $x = \frac{3}{4}$ into (2.19):

$$
-2 = \tfrac{1}{4}a_1 + a_0\,,
$$

$$
2 = \tfrac{3}{4}a_1 + a_0 + \tfrac{1}{2}(\tfrac{3}{4} - \tfrac{1}{2})^2 - \tfrac{1}{6}\{2(\tfrac{3}{4} - \tfrac{1}{4})^3\}\,.
$$

Solving this system, we obtain

$$
(2.20) \qquad \psi(x) = \frac{383}{48}\, x - \frac{767}{192} - \tfrac{1}{2}(x - \tfrac{1}{2})_+^2 + \tfrac{1}{3}[(x - \tfrac{1}{4})_+^3 - (x - \tfrac{3}{4})_+^3]\,.
$$

There is not difficult to verify that for $x \in\,]0, \frac{1}{4}[$ and for $x \in\,]\frac{3}{4}, 1[$ the function (2.20) its equal to a polynomial of order $q - 1 = 1$.

Using Theorem 2.1 it is possible in every concrete case to evaluate the corresponding elements of the natural basis. As their linear combinations have obligatory equal structure we can modify this method for computation of elements of cardinal basis. According to the previous remark we shall call this elements cardinal splines.

Let be $0 \leqq p < s < q$ and let $y \in]a, b[$. From Theorems 1.1 and 2.1 we can compute the optimal approximation of derivative $f^{(s)}(y)$ for arbitrary $y \in]a, b[$, namely

$$f^{(s)}(y) \doteq \sum_{i=0}^{p} \sum_{j=1}^{n_i} \sigma_{ij}^{(s)}(y) f^{(i)}(x_{ij}) . \tag{2.21}$$

Analogously the optimal quadrature formula for $f \in H^q(a, b)$ is of the form

$$\int_a^b f(x)\,\mathrm{d}x \doteq \sum_{i=0}^{p} \sum_{j=1}^{n_i} \left(\int_a^b \sigma_{ij}(x)\, dx \right) f^{(i)}(x_{ij}) . \tag{2.22}$$

3. CARDINAL SPLINES FOR EQUIDISTANT PARTITION

Since now we shall consider the simplest case $p = 0$ and

$$\langle \varphi_i, f \rangle_H = f(x_i) , \quad i = 1, \ldots, n , \quad \forall f \in H^q(a, b) , \tag{3.1}$$

where $n \geqq q > 1$ and

$$x_i = a + \frac{i}{n+1} (b - a) , \quad i = 0, 1, \ldots, n + 1 . \tag{3.2}$$

Especially, $x_0 = a$, $x_{n+1} = b$. S is the linear subspace of $H^q(a, b)$ spanned on $\{\varphi_1, \ldots, \varphi_n\}$.

From Theorem 2.1 it follows

Proposition 3.1. The cardinal splines $\sigma_i \in S$, where S is the linear subspace in $H^q(a, b)$ consisting of elements φ_i, $i = 1, \ldots, n$, with properties (3.1), are the functions of the form

$$\sigma_i(x) = \begin{cases} \sum_{j=0}^{q-1} a_{ij} x^j + \sum_{j=1}^{n-1} \lambda_{ij} (x - x_j)_+^{2q-1} , & x \in [x_0, x_n[\\ \sum_{j=0}^{q-1} b_{ij} x^j , & x \in [x_n, x_{n+1}] \end{cases} \tag{3.3}$$

with the properties

$$\sigma_i(x_j) = \delta_{ij} , \quad i, j = 1, \ldots, n . \tag{3.4}$$

There is $\sigma_i \in C^{(2q-2)}(a, b)$ and the $2q + n - 1$ coefficients $a_{10}, \ldots, a_{iq}, \lambda_{i1}, \ldots, \lambda_{i,n-1}$, $b_{i0}, \ldots, b_{iq}$ are the (unique) solution of the linear system of independent n equations (3.4) and $2q - 1$ equations

$$\sigma_i^{(k)}(x_n -) = \sigma_i^{(k)}(x_n +) , \quad k = 0, 1, \ldots, 2q - 2 . \tag{3.5}$$

Remark. The assertion about the independency of equations (3.4), (3.5) can be verified ([2]) also directly, if we analyse the matrix of the corresponding linear system. We present here some numerical results, based on Proposition 3.1. In our examples we suppose the interval $[a, b]$ transformed to the special form $[0, (n + 1)h]$ In the tables there are values of coefficients in the expression (3.3).

Example 3.1. $n = 3,\ q = 2$

i	a_{i0}	a_{i1}	λ_{i1}	λ_{i2}	b_{i0}	b_{i1}
1	$\frac{9}{4}$	$-\frac{5}{4h}$	$\frac{1}{4h^3}$	$-\frac{2}{4h^3}$	$-\frac{3}{4}$	$\frac{1}{4h}$
2	$-\frac{3}{-2}$	$\frac{3}{2h}$	$-\frac{1}{2h^3}$	$\frac{2}{2h^3}$	$\frac{9}{2}$	$-\frac{3}{2h}$
3	$\frac{1}{4}$	$-\frac{1}{4h}$	$\frac{1}{4h^3}$	$-\frac{2}{4h^3}$	$-\frac{11}{4}$	$\frac{5}{4h}$

Example 3.2. $n = 4,\ q = 2$

i	a_{i0}	a_{i1}	λ_{i1}	λ_{i2}	λ_{i3}	b_{i0}	b_{i1}
1	$\frac{84}{15}$	$-\frac{19}{15h}$	$\frac{4}{15h^3}$	$-\frac{9}{15h^3}$	$\frac{6}{15h^3}$	$\frac{4}{15}$	$-\frac{1}{15h}$
2	$\frac{8}{5}$	$\frac{8}{5h}$	$-\frac{3}{5h^3}$	$\frac{8}{5h^3}$	$-\frac{7}{5h^3}$	$-\frac{8}{5}$	$\frac{2}{5h}$
3	$\frac{2}{5}$	$-\frac{2}{5h}$	$\frac{2}{5h^3}$	$-\frac{7}{5h^3}$	$\frac{8}{5h^3}$	$\frac{32}{5}$	$-\frac{8}{5h}$
4	$-\frac{1}{15}$	$\frac{1}{15h}$	$-\frac{1}{15h^3}$	$\frac{6}{15h^3}$	$-\frac{9}{15h^3}$	$-\frac{61}{15}$	$\frac{19}{15h}$

Example 3.3. $n = 5,\ q = 2$

i	a_{i0}	a_{i1}	λ_{i1}	λ_{i2}	λ_{i3}	λ_{i4}	b_{i0}	b_{i1}
1	$\frac{127}{56}$	$-\frac{71}{56h}$	$\frac{15}{56h^3}$	$-\frac{34}{56h^3}$	$\frac{24}{56h^3}$	$-\frac{6}{56h^3}$	$-\frac{5}{56}$	$\frac{1}{56h}$
2	$-\frac{45}{28}$	$\frac{45}{28h}$	$-\frac{17}{28h^3}$	$\frac{46}{28h^3}$	$-\frac{44}{28h^3}$	$\frac{18}{28h^3}$	$\frac{15}{28}$	$-\frac{3}{28h}$
3	$\frac{3}{7}$	$-\frac{3}{7h}$	$\frac{3}{7h^3}$	$-\frac{11}{7h^3}$	$\frac{16}{7h^3}$	$-\frac{11}{7h^3}$	$-\frac{15}{7}$	$\frac{3}{7h}$
4	$-\frac{3}{28}$	$\frac{3}{28h}$	$-\frac{3}{28h^3}$	$\frac{18}{28h^3}$	$-\frac{44}{28h^3}$	$\frac{46}{28h^3}$	$\frac{225}{28}$	$-\frac{45}{28h}$
5	$\frac{1}{56}$	$-\frac{1}{56h}$	$\frac{1}{56h^3}$	$-\frac{6}{56h^3}$	$\frac{24}{56h^3}$	$-\frac{34}{56h^3}$	$-\frac{355}{56}$	$\frac{71}{56h}$

Example 3.4. $n = 4,\ q = 3$

i	a_{i0}	a_{i1}	a_{i2}	λ_{i1}	λ_{i2}	λ_{i3}	b_{i0}	b_{i1}	b_{i2}
1	$\frac{448}{132}$	$-\frac{409}{132h}$	$\frac{93}{132h^2}$	$-\frac{2}{132h^5}$	$\frac{6}{132h^5}$	$-\frac{6}{132h^5}$	$-\frac{332}{132}$	$\frac{191}{132h}$	$-\frac{27}{132h^2}$
2	$-\frac{187}{44}$	$\frac{255}{44h}$	$\frac{-71}{44h^2}$	$\frac{2}{44h^5}$	$-\frac{6}{44h^5}$	$\frac{6}{44h^5}$	$\frac{596}{44}$	$-\frac{345}{44h}$	$\frac{49}{44h^2}$
3	$\frac{96}{44}$	$-\frac{145}{44h}$	$\frac{49}{44h^2}$	$-\frac{2}{44h^5}$	$\frac{6}{44h^5}$	$-\frac{6}{44h^5}$	$-\frac{684}{44}$	$\frac{445}{44h}$	$-\frac{71}{44h^2}$
4	$-\frac{52}{132}$	$\frac{79}{132h}$	$-\frac{27}{132h^2}$	$\frac{2}{132h^5}$	$-\frac{6}{132h^5}$	$\frac{6}{132h^5}$	$\frac{728}{132}$	$-\frac{521}{132h}$	$\frac{93}{132h^2}$

This concrete cardinal bases can be used as in (2.21), (2.22) for determining of optimal formulae for different linear operators.

4. EXAMPLE OF NUMERICAL SOLUTION OF OPTIMUM CONTROL PROBLEM

Let us minimize the integral cost function

$$\mathscr{I}(u) = \frac{1}{2}\int_0^T (x^2(t) + u^2(t))\,\mathrm{d}t\,,$$

where $T > 0$ is given and

$$\dot{x}(t) = -a\,x(t) + u(t)\,, \quad a > 0\,,$$
$$x(0) = x_0\,.$$

Using the Pontrjagin principle, the Hamilton function is

$$\mathscr{H}(\psi, x, u) = \tfrac{1}{2}\psi_0(x^2 + u^2) + \psi_1(-ax + u)\,,$$

where we can put $\psi_0 = -1$.

The optimal control is therefore

$$u^*(t) = \psi_1(t)\,.$$

The function $\psi_1(t)$ is the solution of the following boundary value problem:

$$\begin{aligned} \dot{x}(t) &= -a\,x(t) + u(t)\,, & x(0) &= x_0\,,\\ \dot{\psi}_1(t) &= x(t) + a\,\psi_1(t)\,, & \psi_1(T) &= 0\,, \end{aligned}$$

which solved can be analytically and there is

$$\psi_1(t) = x_0\,\frac{\mathrm{e}^{\lambda t} - \mathrm{e}^{\lambda(2T-t)}}{\lambda - a + (\lambda + a)\,\mathrm{e}^{2\lambda T}}\,, \quad \lambda = \sqrt{(a^2 + 1)}$$

This boundary value problem was discretized by means of the following formulae:

(1) Optimal formula for $f \in H^2$, using the values of $f(\cdot)$ in 3 points with $h = x_{i+1} - x_i$, $i = 0, 1, \ldots, n + 1$

$$f'(x_j) \doteq \frac{1}{4h}\left[-5f(x_j) + 6f(x_{j+1}) - f(x_{j+2})\right], \quad j = 1, \ldots, n - 1,$$

$$f'(x_j) \doteq \frac{1}{2h}\left[f(x_{j+1}) - f(x_{j-1})\right], \quad j = 1, \ldots, n,$$

$$f'(x_j) \doteq \frac{1}{4h}\left[f(x_{j-2}) - 6f(x_{j-1}) + 5f(x_j)\right], \quad j = 2, \ldots, n.$$

This formula was derived using the cardinal splines from Example 3.1. It is precise for polynomials of order 1.

(2) Classical three-points formula derived using the Lagrange interpolation, precise for polynomials of order 2

$$f'(x_j) \doteq \frac{1}{2h}\left[-3f(x_j) + 4f(x_{j+1}) - f(x_{j+2})\right], \quad j = 1, \ldots, n - 1,$$

$$f'(x_j) \doteq \frac{1}{2h}\left[f(x_{j+1}) - f(x_{j-1})\right], \quad j = 1, \ldots, n,$$

$$f'(x_j) \doteq \frac{1}{2h}\left[f(x_{j-2}) - 4f(x_{j-1}) + 3f(x_j)\right], \quad j = 2, \ldots, n.$$

(3) The common Euler method.

The results of computation (due by Z. Beran in [2]) for $T = 1$, $a = 1$, $x_0 = 1$ with the step $h = 0, 2$ on $[0, 1]$ are given in the following table, where $e(\cdot)$ is the error in comparison with the analytic solution. For $u_i^* = \psi_{1i}$ we used the first, for x_i the third formula in (1) and (2). As the conditions for x_i are given at the initial point and those for u_i^* at the final point of the interval, this choice seems to be the simplest one, although evidently not the only possible way of application of discretization formulas.

(Received April 13, 1984.)

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	Optimal formula				Lagrange's formula				Euler's method			
t_i	x_i	$e(x_i)$	u_i	$e(u_i)$	x_i	$e(x_i)$	u_i	$e(u_i)$	x_i	$e(x_i)$	u_i	$e(u_i)$
0	1	0	−0·406	−0·020	1	0	−0·359	0·027	1	0	−0·334	0·052
0·2	0·766	$6{\cdot}9 \,.\, 10^{-3}$	−0·293	−0·016	0·707	−0·052	−0·246	0·031	0·733	−0·026	−0·234	0·043
0·4	0·592	0·0124	−0·202	−0·012	0·509	−0·071	−0·163	0·027	0·539	−0·040	−0·158	0·032
0·6	0·463	0·016	−0·127	$-7{\cdot}8 \,.\, 10^{-3}$	0·367	−0·080	−0·099	0·019	0·400	−0·046	−0·097	0·022
0·8	0·368	0·018	−0·061	$-3{\cdot}6 \,.\, 10^{-3}$	0·267	−0·084	−0·048	$8{\cdot}9 \,.\, 10^{-3}$	0·301	−0·049	−0·046	0·011
1	0·302	0·019	—	—	0·206	−0·076	—	—	0·231	−0·050	—	—

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