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The RKHS methods are used to develop the theory of nonlinear estimation of polynomials in mean value function of a Gaussian random process.

1. INTRODUCTION

The reproducing kernel Hilbert space (RKHS) plays an important part in the investigations of Gaussian stochastic processes. Parzen [8] was the first who used this theory in his studies of linear estimation and prediction problems in the time-series analysis and also non linear problems in [9]. Kallianpur [7] studied the relationship existing between a Gaussian stochastic process and its RKHS. He describes the structure of the estimation space $L_2(\Omega_{X_0}\, \bar{\mathscr{A}}(\Omega_{X_0}), P_0)$ for the stochastic process with its mean value function equal to zero. The main aim is to develop on the basis of Kallianpurś work the theory of the nonlinear estimation of polynomials in $m(\cdot)$, where $m(\cdot)$ is an unknown mean value function of the stochastic process and of polynomials in regression coefficients. Other uses of the RKHS methods in optimal nonlinear design problems are given by Pázman [10].

2. THE STRUCTURE OF THE ESTIMATION SPACE FOR THE MEAN VALUE FUNCTION EQUAL TO ZERO

We present first the main result of Kallianpur [7]. Let $X_0 = \{X_0(t), t \in T\}$ be a real Gaussian stochastic process defined on a probability space $(\Omega_{X_0}, \bar{\mathscr{A}}(\Omega_{X_0}), P_0)$, where $\bar{\mathscr{A}}(\Omega_{X_0})$ is taken to be the completion with respect to P_0 of $\mathscr{A}(\Omega_{X_0})$, the minimal σ-field with respect to which all the random variables $X_0(t)$, $t \in T$, are measurable. It is assumed in [7] that $E[X_0(t)] = 0$ and $R(s, t) = E[X_0(s) . X_0(t)]$ is a known covariance function of the stochastic process X_0. By these assumptions the following assertions are given in [7]. Let $L_2(\Omega_{X_0}, \mathscr{A}(\Omega_{X_0}), P_0)$ be the Hilbert space of square integrable random variables on Ω_{X_0} and let $L_2(X_0)$ be the closed linear subspace of L_2 spanned by the finite, real linear combinations $\sum_{i=1}^{n} c_i X_0(t_i)$.

Then

$$L_2(\Omega_{X_0}, \mathscr{A}(\Omega_{X_0}), P_0) = \sum_{p \geqq 0} \oplus \bar{G}_p \cong \sum_{p \geqq 0} \sigma[\otimes^p L_2(X_0)] \cong$$
$$\cong \sum_{p \geqq 0} \oplus \sigma[\otimes^p H(R)] \cong H(\Gamma_R),$$

where $\cong$ is the symbol for the isometric isomorphism between two Hilbert spaces, $\sigma[\otimes^p H]$ is the symmetric tensor product Hilbert space of a Hilbert space H, $\sum_{p \geqq 0} \otimes H_p$ is the orthogonal direct sum of the Hilbert spaces H_p and $H(R)$ is the RKHS determined by the covariance function R of the random process X_0. $H(\Gamma_R)$ is the RKHS of functions defined on $H(R)$ with its reproducing kernel $\Gamma_R(m, m_0) = \exp . \{\langle m, m_0 \rangle_{H(R)}\}$. The Hilbert spaces $\bar{G}_p$, the linear subspaces of $L_2(\Omega_{X_0}, \mathscr{A}(\Omega_{X_0}), P_0)$ are defined in the following way. Let $G_0 = \{1\}$ be the closed linear subspace spanned by the constant random variable and for $p \geqq 1$ let $\hat{G}_p$ be the linear subspace of all polynomials in $\{\xi_i^0\}_1^\infty$, $\xi_i^0 = \langle X_0, e_i \rangle$; $\{e_i\}_1^\infty$ is a CONS in $H(R)$ and $\{\xi_i^0\}_1^\infty$ is a CONS in $L_2(X_0)$, of a degree not exceeding p. Let G_p be the set of all polynomials in $\hat{G}_p$ orthogonal to every polynomial in $\hat{G}_{p-1}$. Finally, let $\bar{G}_p$ be the closed linear subspace spanned by G_p. Then a random variable $\gamma \in G_p$ if and only if it is of the form

$$\gamma(\omega) = \sum a_{m_1, \ldots, m_r} h_{m_1}(\xi_{\lambda_1}^0(\omega) \ldots h_{m_r}(\xi_{\lambda_r}^0(\omega))$$

for some choice of integers $\lambda_1, \ldots, \lambda_r$; the summation is over $m_i \geqq 0$, r and $\lambda_1, \ldots \ldots, \lambda_r$ being fixed, $h_n(x)$ is the n-th normalized Hermite polynomial $h_n(x) = (-1)^n . (1/\sqrt{n}) \exp\{x^2/2\} . (\mathrm{d}/\mathrm{d}x)^n \exp\{-x^2/2\}$, and the coefficients $a_{m_1, \ldots, m_r}$ satisfy

$$a_{m_1, \ldots, m_r} = 0 \quad \text{if} \quad \sum_{i=1}^{r} m_i \neq p .$$

The following elements of CONS in Hilbert spaces correspond one to another under the isomorphism between the corresponding Hilbert spaces:

$$\prod_1^r h_{n_i}(\xi_{\lambda_i}^0(\cdot)) \leftrightarrow {}^0\xi_{\lambda_1, \ldots, \lambda_r}^{n_1, \ldots, n_r} \leftrightarrow e_{\lambda_1, \ldots, \lambda_r}^{n_1, \ldots, n_r} \leftrightarrow \prod_1^r \frac{\langle \cdot, e_{\lambda_i} \rangle^{n_i}}{\sqrt{n_i!}}$$

and

$$E_0 \left[\prod_{i=1}^{r} h_{n_i}(\xi_{\lambda_i}^0)\right]^2 = \|{}^0\xi_{\lambda_1, \ldots, \lambda_r}^{n_1, \ldots, n_r}\|_{\sigma[\otimes^p L_2(X_0)]}^2 = \|e_{\lambda_1, \ldots, \lambda_r}^{n_1, \ldots, n_r}\|_{\sigma[\otimes^p H(R)]}^2 =$$
$$= \left\| \prod_1^r \frac{\langle \cdot, e^{\lambda_i} \rangle^{n_i}}{\sqrt{(n_i!)}} \right\|_{H(\Gamma_R)}^2 = 1 .$$

For any $U \in L_2(\Omega_{X_0}, \mathscr{A}(\Omega_{X_0}), P_0)$ the expansion

$$U = \sum_{p \geq 0} \sum_{\substack{\sum_1^r n_i = p \\ n_i > 0}} \sum_{\lambda_1, \ldots, \lambda_r} a_{\lambda_1, \ldots, \lambda_r}^{n_1, \ldots, n_r} h_{n_1}(\xi_{\lambda_1}) \ldots h_{n_r}(\xi_{\lambda_r})$$

holds, where

$$\sum_{p \geq 0} \sum_{\substack{\sum_1^r n_i = p \\ n_i > 0}} \sum_{\lambda_1, \ldots, \lambda_r} (a_{\lambda_1, \ldots, \lambda_r}^{n_1, \ldots, n_r})^2 < \infty .$$

The function $\phi(\cdot)$ defined on $H(R)$ by

$$\phi(m) = \sum_{p \geq 0} \sum_{\substack{\sum n_i = p \\ n_i > 0}} \sum_{\lambda_1, \ldots, \lambda_r} a_{\lambda_1, \ldots, \lambda_r}^{n_1, \ldots, n_r} \prod_{i=1}^{r} \frac{\langle m, e_{\lambda_i} \rangle^{n_i}}{\sqrt{(n_i!)}}$$

corresponds to U under the isomorphism between L_2 and $H(\Gamma_R)$.

3. ESTIMATION OF POLYNOMIALS IN MEAN VALUE

Let us assume that we observe a Gaussian random process $X = \{X(t);\ t \in T\}$ of the form

$$X = \{X(t) = X_0(t) + m(t)\ ;\ t \in T\}\ , \tag{3}$$

where $E\,X_0(t) = 0$ and let $R(s, t) = E[X_0(s) \,.\, X_0(t)]$ be the known covariance function of the process X. The mean value function $m(t) = E\,X(t);\ t \in T$, of the stochastic process X will be assumed to belong to $H(R)$ only. Parzen [8] has shown that there exists the transformation $\langle X, \cdot \rangle$:

$$H(R) \xrightarrow{\text{onto}} L_2(X)\,,$$

($L_2(X)$ being now the subspace of $L_2(\Omega_X, \mathscr{A}(\Omega_X), P)$ spanned by the finite, real linear combinations $\sum_{i=1}^{n} c_i X(t_i)$ with $\|U\|^2 = E[U^2]$), with the following properties:

1) $\langle X, R(\cdot, t) \rangle = X(t)\,,\quad t \in T$
2) $E\langle X, g \rangle = \langle m, g \rangle\,,\quad g \in H(R)$
3) $\operatorname{Cov}\{\langle X, g \rangle, \langle X, h \rangle\} = \langle g, h \rangle\,,\quad g, h \in H(R)$.

It is very easy to prove that $U \in L_2(X)$ iff

$$U = \sum_{i=1}^{\infty} a_i \xi_i = \sum_{i=1}^{\infty} a_i (\xi_i^0 + m_i)\,,$$

where $\sum_{i=1}^{\infty} a_i^2 < \infty$,

(4) $$\xi_i = \langle X, e_i \rangle = \langle X^0, e_i \rangle + \langle m, e_i \rangle\,, \quad i = 1, 2, \ldots$$

$\{e_i\}_{i=1}^{\infty}$ being a CONS in $H(R)$. From this $U \in L_2(X)$ iff $U = \langle X^0, g \rangle + \langle m, g \rangle$; $g \in H(R)$. For the random variables ξ_i given by (4) we have:

(5) $$E[h_{n_i}(\xi_i)] = \frac{\langle m, e_i \rangle^{n_i}}{\sqrt{n_i!}}\,.$$

The proof of (5) follows from the next lemma.

Lemma 3.1. Let $h_n(x)$ be the n-th normalized Hermite polynomial in x. Then

$$h_n(x + m) = \sum_{s=0}^{n} \binom{n}{s} \sqrt{\frac{s!}{n!}}\, m^{n-s}\, h_s(x) = \sum_{s=0}^{n} \binom{n}{s}^{1/2} \frac{m^{n-s}}{\sqrt{(n-s)!}}\, h_s(x)$$

for every real constant m.

Proof. Let

$$g_n(x) = \sqrt{(n!)}\, h_n(x) = (-1)^n \exp\{x^2/2\}\, (\mathrm{d}/\mathrm{d}x)^n \exp\{-x^2/2\}\,.$$

Then

$$\begin{aligned} g_n(x + m) &= (-1)^n \exp\{(x + m)^2/2\}\, (\mathrm{d}/\mathrm{d}x)^n \exp\{-(x + m)^2/2\} = \\ &= (-1)^n \exp\{(x^2 + 2xm)/2\}\, (\mathrm{d}/\mathrm{d}x)^n \exp\{(-x^2 + 2xm)/2\} = \\ &= (-1)^n \exp\{(x^2 + 2xm)/2\} \sum_{s=0}^{n} \binom{n}{s} \exp\{-x^2/2\}^{(n-s)} \exp\{-xm\}^{(s)} = \\ &= \sum_{s=0}^{n} \binom{n}{s} m^s\, g_{n-s}(x) \end{aligned}$$

and the lemma is proved.

Corollary 3.1. The following relations are true:

$$E\Big[\prod_{i=1}^{r} h_{n_i}(\xi_{\lambda_i})\Big] = \prod_{i=1}^{r} \frac{\langle m, e_{\lambda_i} \rangle^{n_i}}{\sqrt{n_i!}}\,,$$

$$E[h_{n_i}(\xi_{\lambda_i})\, h_{n_j}(\xi_{\lambda_j})] = \sum_{s=0}^{\min\{n_i, n_j\}} \sqrt{\left[\binom{n_i}{s}\binom{n_j}{s}\right]} \frac{\langle m, e_{\lambda_i} \rangle^{n_i + n_j - 2s}}{\sqrt{[(n_i - s)!\,(n_j - s)!]}} \quad \text{if} \quad \lambda_i = \lambda_j\,,$$

$$E[h_n^2(\xi_{\lambda_i})] = \sum_{s=0}^{n} \binom{n}{s} \frac{\langle m, e_{\lambda_i} \rangle^{2s}}{s!}\,,$$

$$D^2[h_n(\xi_{\lambda_i})] = \sum_{s=0}^{n-1} \binom{n}{s} \frac{\langle m, e_{\lambda_i} \rangle^{2s}}{s!}\,.$$

Proof. They follow from the Lemma 3.1 and from the fact that the random variables $\{\xi_{\lambda_i}\}_{i=1}^r$ are independent.

The Corollary 3.1. can be used in principle to find unbiased estimators of polynomials in $m(t_1), \ldots, m(t_k)$; $t_j \in T$.

Example 3.1. Let us look an unbiased estimate of the polynomial $f_3(m) = m(t_1) \,.\, m(t_2) \,.\, m(t_3)$. We have:

$$f_3(m) = \sqrt{3!} \sum_{\lambda_1} e_{\lambda_1}(t_1)\, e_{\lambda_1}(t_2)\, e_{\lambda_1}(t_3) \frac{\langle m, e_{\lambda_1}\rangle^3}{\sqrt{3!}} +$$

$$+ \sqrt{2!} \sum_{\lambda_1 \neq \lambda_2} e_{\lambda_1}(t_1)\, e_{\lambda_2}(t_2)\, e_{\lambda_2}(t_3) \frac{\langle m. e_{\lambda_1}\rangle}{\sqrt{1!}} \,.\, \frac{\langle m, e_{\lambda_2}\rangle^2}{\sqrt{2!}} +$$

$$+ \sqrt{2!} \sum_{\lambda_1 \neq \lambda_2} e_{\lambda_2}(t_1)\, e_{\lambda_1}(t_2)\, e_{\lambda_2}(t_3) \frac{\langle m, e_{\lambda_1}\rangle}{\sqrt{1!}} \,.\, \frac{\langle m, e_{\lambda_2}\rangle^2}{\sqrt{2!}} +$$

$$+ \sqrt{2!} \sum_{\lambda_1 \neq \lambda_2} e_{\lambda_2}(t_1)\, e_{\lambda_2}(t_2)\, e_{\lambda_1}(t_3) \frac{\langle m, e_{\lambda_1}\rangle}{\sqrt{1!}} \,.\, \frac{\langle m, e_{\lambda_2}\rangle^2}{\sqrt{2!}} +$$

$$+ \sum_{\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \lambda_1} e_{\lambda_1}(t_1)\, e_{\lambda_2}(t_2)\, e_{\lambda_3}(t_3) \frac{\langle m, e_{\lambda_1}\rangle}{\sqrt{1!}} \,.\, \frac{\langle m, e_{\lambda_2}\rangle}{\sqrt{1!}} \,.\, \frac{\langle m, e_{\lambda_3}\rangle}{\sqrt{1!}} \,.$$

After some calculation using (5) we get an unbiased estimate $f_3(X)$ of the polynomial $f_3(\cdot)$:

$$f_3(X) = X(t_1)\, X(t_2)\, X(t_3) - X(t_1)\, R(t_2, t_3) - X(t_2)\, R(t_1, t_3) - \\ - X(t_3)\, R(t_1, t_2) \,.$$

The computation of unbiased estimators by this direct methods is rather complicated and for polynomials of higher degree it is practically impossible to use it. For polynomials of the type $f_n(m) = m(t)^n$ the following lemma is useful.

Lemma 3.3. Let X be an $N(m, \sigma^2)$ distributed random variable with the known dispersion σ^2. Let $g_n(x) = \sqrt{n!}\, h_n(x)$. Then $f_n(X) = \sigma^n g_n(X/\sigma)$ is an unbiased estimate of the polynomial $f_n(m) = m^n$.

Proof. For the random variable X we have: $X = X_0 + m$, where X_0 is the $N(0, \sigma^2)$ – distributed random variable. Following the proof of Lemma 3.1 we may state the relation

$$g_n\left(\frac{x + m}{\sigma}\right) = \sum_{s=0}^{n} \binom{n}{s} \left(\frac{m}{\sigma}\right)^{n-s} g_s\left(\frac{x}{\sigma}\right)$$

from which we have:

$$E\left[g_n\left(\frac{X}{\sigma}\right) | X \sim N(m, \sigma^2)\right] = E\left[g_n\left(\frac{X_0 + m}{\sigma}\right) | X_0 \sim N(0, \sigma^2)\right] = \left[\frac{m}{\sigma}\right]^n,$$

because

$$\frac{1}{\sqrt{(2\pi\sigma)}} \int g_s\left(\frac{x}{\sigma}\right) \exp\{-x^2/2\sigma^2\}\, \mathrm{d}x = \begin{cases} 0 & \text{if} \quad s \geqq 1 \\ 1 & \text{if} \quad s = 0 \end{cases}$$

according to the orthogonality of the Hermite polynomials and the lemma is proved.

Example 3.2. Since $g_6(x) = x^6 - 15x^4 + 45x^2 - 15$, then

$$f_6(X) = X(t)^6 - 15\, X(t)^4\, R(t, t) + 45\, X(t)^2\, R(t, t)^2 - 15\, R(t, t)^3 =$$

$$= R(t, t)^3 \,.\, g_6\left(\frac{X}{R(t, t)^{1/2}}\right)$$

is an unbiased estimate of the polynomial $f_6(m) = m(t)^6$.

Now let us have a look at the coefficients of the polynomial g_6 which may be written in the form $g_6(x) = x^6 - a_2x^4 + a_4x^2 - a_6$. The coefficients a_2, a_4, a_6 can be interpreted in the following way:

$$a_2 = 15 = \binom{6}{4}\binom{2}{2}, \quad a_4 = 45 = \frac{\binom{6}{2}\binom{4}{2}\binom{2}{2}}{2!}, \quad a_6 = 15 = \frac{\binom{6}{2}\binom{4}{2}\binom{2}{2}}{3!};$$

a_2 is equal to the number of the products of the type $X(t_{i_1})\, X(t_{i_2})\, X(t_{i_3})\, X(t_{i_4})\, R(t_{i_5}, t_{i_6})$, a_4 is the number of the distinct products of the type $X(t_{i_1})\, X(t_{i_2})\, R(t_{i_3}, t_{i_4})\,.$ $.\, R(t_{i_5}, t_{i_6})$ and a_6 is the number of the distinct products of the type $R(t_{i_1}, t_{i_2})\, R(t_{i_3}, t_{i_4})\, R(t_{i_5}, t_{i_6})$, where in all three cases $t_{i_j} \in T$, $j = 1, 2, \ldots, 6$ and $t_{i_j} \neq t_{i_k}$ for $j \neq k$. We set up the hypothesis (without proof):

$$f_6(X) = X(t_1) \ldots X(t_6) - \sum X(t_{i_1})\, X(t_{i_2})\, X(t_{i_3})\, X(t_{i_4})\, R(t_{i_5}, t_{i_6}) +$$
$$+ \sum X(t_{i_1})\, X(t_{i_2})\, R(t_{i_3}, t_{i_4})\, R(t_{i_5}, t_{i_6}) - \sum R(t_{i_1}, t_{i_2})\, R(t_{i_3}, t_{i_4})\, R(t_{i_5}, t_{i_6})$$

where the first sum has 15, the second 45 and the third 15 members, is an unbiased estimate of the function $f_6(m) = m(t_1) \ldots m(t_6)$; $t_i \in T$ for $i = 1, \ldots, 6$.

Example 3.3. Since $g_4(x) = x^4 - 6x^2 + 3$, according to the hypothesis (stated above for $n = 6$)

$$f_4(X) = X(t_1)\, X(t_2)\, X(t_3)\, X(t_4) - X(t_1)\, X(t_2)\, R(t_3.\, t_4) -$$
$$- X(t_1)\, X(t_3)\, R(t_2, t_4) - X(t_1)\, X(t_4)\, R(t_2, t_3) - X(t_2)\, X(t_3)\, R(t_1.\, t_4) -$$
$$- X(t_2)\, X(t_4)\, R(t_1, t_3) - X(t_3)\, X(t_4)\, R(t_1, t_2) +$$
$$+ R(t_1, t_2)\, R(t_3, t_4) + R(t_1, t_3)\, R(t_2, t_4) + R(t_1, t_4)\, R(t_2, t_3)$$

 will be an unbiased estimate of $f_4(m) = m(t_1)\, m(t_2)\, m(t_3)\, m(t_4)$. It is not difficult to verify that f_4 is unbiased using the fact that for a Gaussian process of the type (3) we have

$$E[X(t_1)\, X(t_2)\, X(t_3)\, X(t_4)] = R(t_1, t_2)\, R(t_3, t_4) +$$
$$+ R(t_1, t_3)\, R(t_2, t_4) + R(t_1, t_4)\, R(t_2, t_3) + m(t_1)\, m(t_2)\, R(t_3, t_4) +$$
$$+ m(t_1)\, m(t_3)\, R(t_2, t_4) + m(t_1)\, m(t_4)\, R(t_2, t_3) +$$
$$+ m(t_2)\, m(t_3)\, R(t_1, t_4) + m(t_2)\, m(t_4)\, R(t_1, t_3) + m(t_3)\, m(t_4)\, R(t_1, t_2) +$$
$$+ m(t_1)\, m(t_2)\, m(t_3)\, m(t_4),$$
$$E[X(t_i)\, X(t_j)] = R(t_i, t_j) + m(t_i)\, m(t_j).$$

In the general case *we believe that an unbiased estimate of the function* $f(\cdot)$ *of the type* $f(m) = \prod_{i=1}^{n} \langle f_i, m\rangle_{H(R)}, f_i \in H(R)$ *may be found with the help of the polynomial* $g_n(x) = \sqrt{(n!)}\, h_n(x)$ – *where* $h_n(x)$ *is the n-th normalized Hermite polynomial* – proceeding by the method used in Example 3.2.

Remark. In this connection we refer to the Example 3.1, having in mind that $g_3(x) = x^3 - 3x$.

4. ESTIMATION OF POLYNOMIALS IN REGRESSION COEFFICIENTS

Now the regression model for the mean value function of the process X will be assumed. Let

$$m_\theta(t) = \sum_{i=1}^{q} \theta_i\, m_i(t)\,; \quad m_i \in H(R)\,; \quad i = 1, \ldots, q \tag{6}$$

be the known linearly independent functions and $\theta = (\theta_1, \ldots, \theta_q) \in E_q$ let be an unknown parameter. Then

$$\langle m_\theta, m_{\theta'}\rangle_{H(R)} = \sum_{i,j=1}^{q} \theta_i\, \theta'_j \langle m_i, m_j\rangle_{H(R)} = (\mathbf{F}\theta, \theta')_{E_q} = \langle \theta, \theta'\rangle_{H'},$$

where the matrix

$$\mathbf{F} = \|\mathbf{F}_{ij}\| = \|\langle m_i, m_j\rangle_{H(R)}\|$$

and $H' = H(\mathbf{F}^{-1})$. According to (5) and (6)

$$E_\theta[h_{n_i}(\xi_i)] = \frac{\left[\sum_{j=1}^{q} \theta_j \langle m_j, e_i\rangle\right]^{n_i}}{\sqrt{n_i!}}$$

and the function $f(\cdot)$ of the form

$$f(m_\theta) = \sum_{\sum n_i = p} \sum_{\lambda_1,\ldots,\lambda_r} a^{n_1,\ldots,n_r}_{\lambda_1,\ldots,\lambda_r} \prod_{i=1}^{r} \frac{[\sum_{j=1}^{q} \theta_j \langle m_{i_j}, e_{\lambda_i} \rangle]^{n_i}}{\sqrt{n_i!}}\,, \quad p \geqq 0\,,$$

has an unbiased estimate. If the functions $m_i(\cdot)$, $i = 1, \ldots, q$, are orthonormal as the members of $H(R)$, we may choose the CONS $\{e_i\}_{i=1}^{\infty}$ in $H(R)$ in such a way that $e_j = m_j$; $j = 1, \ldots, q$. In this case

$$f(X) = \sqrt{(n_1! \ldots n_q!)}\, h_{n_1}(\xi_1) \ldots h_{n_q}(\xi_q)$$

is the unbiased estimate of the function $f(\theta) = \sum_{i=1}^{q} \theta_i^{n_i}$.

Denote $\mathbf{m} = (m_1, \ldots, m_q)'$. Let $m_1, \ldots, m_q$ be not orthogonal in $H(R)$. Then by orthogonalizing them we get the orthonormal system $\{g_i\}_{i=1}^{q}$ given by

$$g_i = \sum_{k=1}^{q} \mathbf{B}_{ik} m_k = (\mathbf{Bm})_i\,,$$

where for the matrix $\mathbf{B}$ we have $\mathbf{BFB}^T = \mathbf{I}$ (from the condition $\langle g_i, g_j \rangle_{H(R)} = \delta_{ij}$). Next we have:

$$\langle m_\theta, g_i \rangle^{n_i} = \langle m_\theta, (\mathbf{Bm})_i \rangle^{n_i}_{H(R)} = (\mathbf{BF}\theta)_i^{n_i}$$

and

$$\sqrt{(n!)}\, h_{n_i}(\langle X, g_i \rangle) = \sqrt{(n!)}\, h_{n_i}((\mathbf{B}\langle X, \mathbf{m} \rangle)_i)$$

is the unbiased estimate of the function

$$f(\theta) = (\mathbf{BF}\theta)_i^{n_i} = \vartheta_i^{n_i}\,,$$

where we have set

$$\vartheta = \mathbf{BF}\theta\,. \tag{7}$$

From the relation (7) we may give unbiased estimators for polynomials in $\theta_1, \ldots, \theta_q$. We begin with an example again.

Example 4.1. Let us look for an unbiased estimate of the function $f_2(\theta) = \theta_i \theta_j$. From (7) we have:

$$\theta = (\mathbf{BF})^{-1}\, \vartheta = \mathbf{C}\vartheta\,; \quad \mathbf{C} = \mathbf{F}^{-1}\mathbf{B}^{-1}\,.$$

Then

$$f_2(\theta) = \Big(\sum_{k=1}^{q} \mathbf{C}_{ik} \vartheta_k\Big)\Big(\sum_{l=1}^{q} \mathbf{C}_{jl} \vartheta_l\Big) = \sqrt{(2!)} \sum_{k=1}^{q} \mathbf{C}_{ik} \mathbf{C}_{jk} \frac{\vartheta_k^2}{\sqrt{2!}} + \sum_{k \neq l} \mathbf{C}_{ik} \mathbf{C}_{jl} \vartheta_k \vartheta_l\,.$$

$f_2(X)$ – the unbiased estimate for $f_2(\theta)$ is obtained from the last relationship replacing $\vartheta_i^{n_i}/\sqrt{n_i!}$ by its unbiased estimates. Carrying this out we get

$$f_2(X) = \sum_{k=1}^{q} \mathbf{C}_{ik}\mathbf{C}_{jk} \sqrt{(2!)}\, h_2((\mathbf{B}\langle X, \mathbf{m}\rangle)_k) +$$

$$+ \sum_{k \neq l} \mathbf{C}_{ik}\mathbf{C}_{jl}\, h_1((\mathbf{B}\langle X, \mathbf{m}\rangle_k)\, h_1((\mathbf{B}\langle X, \mathbf{m}\rangle)_l) =$$

$$\sum_{k,l=1}^{q} \mathbf{C}_{ik}\mathbf{C}_{jk}(\mathbf{B}\langle X, \mathbf{m}\rangle)_k\, (\mathbf{B}\langle X, \mathbf{m}\rangle)_l - \sum_{k=1}^{q} \mathbf{C}_{ik}\mathbf{C}_{jk} =$$

$$= (\mathbf{F}^{-1}\langle X, \mathbf{m}\rangle)_i\, (\mathbf{F}^{-1}\langle X, \mathbf{m}\rangle)_j - \mathbf{F}_{ij}^{-1}\,.$$

Especially for $f_2(\theta) = \theta_i^2$ we get

$$f_2(X) = (\mathbf{F}^{-1}\langle X, \mathbf{m}\rangle)_i^2 - F_{ii}^{-1}$$

and the connection with the polynomial $g_2(x) = x^2 - 1$ can be seen again.

By the direct evaluation of the mean value of the function

$$f_3(X) = (\mathbf{F}^{-1}\langle X, \mathbf{m}\rangle)_i\, (\mathbf{F}^{-1}\langle X, \mathbf{m}\rangle)_j\, (\mathbf{F}^{-1}\langle X, \mathbf{m}\rangle)_k -$$

$$- (\mathbf{F}^{-1}\langle X, \mathbf{m}\rangle)_i\, \mathbf{F}_{jk}^{-1} - (\mathbf{F}^{-1}\langle X, \mathbf{m}\rangle)_j\, \mathbf{F}_{ik}^{-1} - (\mathbf{F}^{-1}\langle X, \mathbf{m}\rangle)_k\, \mathbf{F}_{ij}^{-1}$$

it is possible to verify that $f_3(X)$ is the unbiased estimate of the function $f_3(\theta) = \theta_i\theta_j\theta_k$.

By analogy with the hypothesis stated in the Section 3 *we believe that an unbiased estimate of the function* $f(\theta) = \theta_{i_1}, \ldots, \theta_{i_n}, \theta_{i_j} \in \{\theta_1, \ldots, \theta_q\}$, $j = 1, \ldots, n$, *can be found with the help of Hermite polynomials, according to relationship stated in Section* 3 for polynomials in $m(t_i)$, $t_i \in T$, $i = 1, \ldots, n$, in which $X(t_{i_j})$ have to be replaced by $(\mathbf{F}^{-1}\langle X, \mathbf{m}\rangle)_{i_j}$ and instead of $R(t_{i_j}, t_{i_k})$ it is necessary to write $\mathbf{F}_{i_j i_k}^{-1}$.

Remarks: 1. It is well known (see [8]) that $\mathbf{F}^{-1}\langle X, \mathbf{m}\rangle = \hat{\theta}$ is the MVULE of θ and $\mathbf{F}_{ij}^{-1} = \mathrm{Cov}\,\{\hat{\theta}_i, \hat{\theta}_j\}$.

2. In the our case observing the vector $X = (X(t_1), \ldots, X(t_n)$ only, the estimate $\hat{\theta} = \mathbf{F}^{-1}\langle X, \mathbf{m}\rangle$ can be written in the form

$$\hat{\theta} = (\mathbf{M}\mathbf{R}^{-1}\mathbf{M}')^{-1}\, \mathbf{M}\mathbf{R}^{-1}X,$$

where

$$\mathbf{M} = \begin{pmatrix} m_1(t_1) \ldots m_1(t_n) \\ \ldots\ldots\ldots\ldots\ldots \\ m_q(t_1) \ldots m_q(t_n) \end{pmatrix}$$

is the design matrix and $\mathbf{R} = \|\mathbf{R}_{ij}\| = \|R(t_i, t_j)\|$ is the covariance matrix of the random vector X.

To compute the variance of estimates of polynomials in mean value of a Gaussian stochastic process the work of Parzen [8] on minimum variance unbiased estimation will be used. In this section we review Parzenś work and then apply his results to our problems.

Let $\mathscr{P} = \{P_\theta;\ \theta \in \Theta\}$ be a set of probability measures defined on a measurable space $(X, \mathscr{B})$. Let

$$\varrho_{\theta_0}(\theta) = \frac{\mathrm{d}P_\theta}{\mathrm{d}P_{\theta_0}}$$

exist for all θ, $\theta_0 \in \Theta$ and let $E_{\theta_0}[\varrho^2_{\theta_0}(\theta)] < \infty$. Then the Hilbert space $L^2(\varrho_{\theta_0}) = L^2\{\varrho_{\theta_0}(\theta);\ \theta \in \Theta\}$ with the norm $\|U\|^2 = E_{\theta_0}[U^2]$ and the RKHS $H(G_{\theta_0})$ with the reproducing kernel

$$G_{\theta_0}(\theta, \theta') = E_{\theta_0}[\varrho_{\theta_0}(\theta) \cdot \varrho_{\theta_0}(\theta')] \text{ ;}$$

$\theta, \theta' \in \Theta$ are isometrically isomorphic. We shall denote by $\langle \varrho_{\theta_0}, f\rangle$ the random variable corresponding of $f \in H(G_{\theta_0})$ by this isomorphism. Parzen proved the following theorem.

Theorem 5.1. A function $f : \Theta \to E^1$ is estimable at θ_0 (i.e. there exists a random variable U defined on $(\Omega_X, \mathscr{A}(\Omega_X))$ such that $E_\theta[U] = f(\theta)$, $\theta \in \Theta$ and $E_{\theta_0}[U - f(\theta_0)]^2 < \infty$) iff $f(\cdot) \in H(G_{\theta_0})$. The MVUE, say $U^*_{\theta_0}$ of an estimable function $f(\cdot)$ is given by $U^*_{\theta_0} = \langle \varrho_{\theta_0}, f\rangle$ and has variance

$$\mathrm{Var}_{\theta_0}[U^*_{\theta_0}] = \|f\|^2_{H(G_{\theta_0})} = f^2(\theta_0) \,.$$

Now we apply this theorem to the estimation of polynomials in mean values.

It is well known, see [6], that a mean value function $m(\cdot)$ and a covariance function $R(\cdot, \cdot)$ of a Gaussian random process determine the Gaussian probability measure P_m on the space $(\Omega_X, \mathscr{A}(\Omega_X))$ and measures P_m and P_{m_0} are equivalent iff $(m - m_0) \in H(R)$, in which case

$$\varrho_{m_0}(m) = \frac{\mathrm{d}P_m}{\mathrm{d}P_{m_0}} = \exp\{\langle X, m - m_0\rangle - \tfrac{1}{2}(\|m\|^2_{H(R)} - \|m_0\|^2_{H(R)})\} \cdot$$

($\langle X, f\rangle$ denotes the random variable corresponding to the $f \in H(R)$ by the isometric isomorphism existing between $L^2\{X(t);\ t \in T\}$ with $\|U\|^2 = E_0[U^2]$ and $H(R)$, see [5].)

It is no difficult to show that

$$G_{m_0}(m, m') = E_{m_0}[\varrho_{m_0}(m) \cdot \varrho_{m_0}(m')] = \exp\{\langle m - m_0, m' - m_0\rangle_{H(R)}\} \,,$$
$$m_0, m, m' \in H(R) \,.$$

According to Theorem 5.1 a function $f(m)$, $m \in H(R)$, is estimable at the $m_0 \in H(R)$ iff $f \in H(G_{m_0})$. For $m_0 = 0$ we have $G_0(\cdot, \cdot) = \Gamma_R(\cdot, \cdot)$. The space $H(\Gamma_R)$ was characterized in the Section 2. Now we give some other characterization of this space: $H(\Gamma_R)$ contains functions of the type

$$f(m) = \sum_{p \geqq 0} \langle g_p, m \otimes \ldots \otimes m \rangle_{\otimes^p H(R)}, \quad m \in H(R),$$

where $g_p \in \otimes^p H(R)$, $p \geqq 0$ are such, that

$$\sum_{p \geqq 0} p! \, \|\hat{g}_p\|^2_{\otimes^p H(R)} = \|f\|^2_{H(\Gamma_R)} < \infty,$$

where $\hat{g}_p$ is the projection of $g_p \in \otimes^p H(R)$ on the subspace $\sigma[\otimes^p H(R)]$: $\hat{g}_p = \mathscr{P}_{\sigma[\otimes^p H(R)]} g_p$. Really, let

$$f_p(m) = \sum_{\substack{\sum_{i=1}^r n_i = p \\ n_i > 0}} \sum_{\lambda_1, \ldots, \lambda_r} a^{n_1, \ldots, n_r}_{\lambda_1, \ldots, \lambda_r} \prod_{i=1}^r \frac{\langle m, e_{\lambda i} \rangle^{n_i}}{\sqrt{n_i!}}$$

and let

$$g_p = \sum_{\substack{\sum_{i=1}^r n_i = p \\ n_i > 0}} \sum_{\lambda_1, \ldots, \lambda_r} \frac{1}{(n_1! \ldots n_r!)^{1/2}} \cdot a^{n_1, \ldots, n_r}_{\lambda 1, \ldots, \lambda_r} \cdot \sigma[\otimes^{n_1} e_{\lambda_1} \ldots \otimes^{n_r} e_{\lambda_r}],$$

then

$$f_p(m) = \langle g_p, m \otimes \ldots \otimes m \rangle_{\otimes^p H(R)}, \tag{9}$$

$$\|f_p\|^2_{H(\Gamma_R)} = p! \, \|\hat{g}_p\|^2_{\otimes^p H(R)}. \tag{10}$$

Conversely, any function of the type (9) can be written in the form (8) with

$$a^{n_1, \ldots, n_r}_{\lambda_1, \ldots, \lambda_r} = \frac{p!}{(n_1! \ldots n_r!)^{1/2}} \langle \hat{g}_p, \otimes^{n_1} e_{\lambda_1} \ldots \otimes^{n_r} e_{\lambda_r} \rangle.$$

Now we are able to prove the following lemma.

Lemma 5.1. Let

$$f_p(m) = \langle g_p, m \otimes \ldots \otimes m \rangle_{\otimes^p H(R)}, \quad m \in H(R), \quad p \geqq 0.$$

Then $f_p(\cdot) \in H(G_k)$ for every $k \in H(R)$ and

$$\|f_p\|^2_{H(G_k)} = \sum_{i=0}^p \binom{p}{i}^2 i! \, \|\langle \hat{g}_p, \otimes^{p-i} k \rangle_{\otimes^{p-i} H(R)}\|^2_{\otimes^i H(R)}. \tag{11}$$

Proof. First the following relation will be proved by induction:

$$f_p(m) = \sum_{i=0}^{p} \binom{p}{i} \langle \hat{g}_p, \otimes^{p-i} k \otimes^i m - k \rangle_{\otimes^p H(R)}$$

for any $k \in H(R)$ and any $p \geqq 0$. For $p = 0$ and 1 the equality holds trivially. Let it holds for $p - 1$. Then

$$f_p(m) = \langle \hat{g}_p, m \otimes \ldots \otimes m \rangle_{\otimes^p H(R)} =$$

$$\langle \langle \hat{g}_p, m \otimes \ldots \otimes m \rangle_{\otimes^{p-1} H(R)}, m - k \rangle_{H(R)} +$$

$$+ \langle \langle \hat{g}_p, m \otimes \ldots \otimes m \rangle_{\otimes^{p-1} H(R)}, k \rangle_{H(R)} = \sum_{i=0}^{p} \binom{p}{i} \langle \hat{g}_p, \otimes^{p-i} k \otimes^i m - k \rangle_{\otimes^p H(R)} .$$

According to this equality for any $p \geqq 0$, $q \geqq 0$ we have

$$\langle f_p, f_q \rangle_{H(G_k)} = \sum_{i=0}^{p} \sum_{j=0}^{q} \binom{p}{i} \binom{q}{j} \langle \langle \langle \hat{g}_p, \otimes^{p-i} k \rangle_{\otimes^{p-i} H(R)} \otimes^i \cdot \rangle_{\otimes^i H(R)} ,$$

$$\langle \langle \hat{g}_q, \otimes^{q-j} k \rangle_{\otimes^{q-j} H(R)}, \otimes^j \cdot \rangle_{\otimes^j H(R)} \rangle_{H(\Gamma_R)} =$$

$$= \sum_{i=0}^{\min\{p,q\}} \binom{p}{i} \binom{q}{i} i! \, \langle \langle \hat{g}_p, \otimes^{p-i} k \rangle_{\otimes^{p-i} H(R)}, \langle \hat{g}_q, \otimes^{q-i} k \rangle_{\otimes^{q-i} H(R)} \rangle_{\otimes^i H(R)} .$$

According to Theorem 5.1 and Lemma 5.1 every function

$$f(m) = \sum_{p=0}^{n} f_p(m) = \sum_{p=0}^{n} \langle g_p, m \otimes \ldots \otimes m \rangle_{\otimes^p H(R)} , \quad m \in H(R) ,$$

is estimable at any $k \in H(R)$ and for the variance of the best unbiassed estimate $f(X)$ we have:

$$\mathrm{Var}_k [f(X)] = \|f\|^2_{H(G_k)} - f^2(k) =$$

$$= \sum_{p=0}^{n} \sum_{q=0}^{n} \sum_{i=1}^{\min\{p,q\}} \binom{p}{i} \binom{q}{i} i! \, \langle \langle \hat{g}_p, \otimes^{p-i} k \rangle_{\otimes^{p-i} H(R)}, \langle \hat{g}_q, \otimes^{q-i} k \rangle_{\otimes^{q-i} H(R)} \rangle_{\otimes^i H(R)} .$$

Example 5.1. Let

$$f_2(m) = \langle g_2, m \otimes m \rangle_{\otimes^2 H(R)}, \; m \in H(R), \; g_2 \in \otimes^2 H(R) .$$

Let

$$\{g_{ij} = \langle g_2, e_i \otimes e_j \rangle_{\otimes^2 H(R)}\}_{i,j=1}^{\infty} .$$

Then

$$f_2(X) = \sum_{i,j} g_{ij}(\langle X, e_i \rangle \langle X, e_j \rangle - \delta_{ij}) \in L_2(\Omega_X, \mathscr{A}(\Omega_X), P_k)$$

for every $k \in H(R)$ (see [2], p. 741) and

$$E_k[f_2(X)] = f_2(k)\,, \quad k \in H(R)\,.$$

Using (11) we get

$$\mathrm{Var}_k\,[f_2(X)] = 2!\,\|\hat{g}_2\|^2_{\otimes^2 H(R)} + 4\|\langle \hat{g}_2, k\rangle_{H(R)}\|^2_{H(R)}\,. \tag{12}$$

The function $g_2 \in \otimes^2 H(R)$ defines on $H(R)$ an operator, say A given by $(Ah)(t) = \langle g_2(\cdot, t), h(\cdot)\rangle_{H(R)}$, $t \in T$, $h \in H(R)$. For (12) we then have other expression:

$$\mathrm{Var}_k\,[f_2(X)] = \frac{\|A + A^*\|^2_{H-S}}{2} + \|Ak + A^*k\|^2_{H(R)}\,,$$

where

$$\|A\|^2_{H-S} = \sum_{i,j} g^2_{ij} = \|g\|^2_{\otimes^2 H(R)}$$

is the Hilbert-Schmidt norm of the operator A and A^* is the adjoint of A. Especially for $g_2 = h_1 \otimes h_2$; $h_i \in H(R)$, $i = 1, 2$,

$$f_2(X) = \langle X, h_1\rangle\langle X, h_2\rangle - \langle h_1, h_2\rangle\,,$$

$$\mathrm{Var}_k\,[f_2(X)] = \|h_1\|^2 \,.\, \|h_2\|^2 + \langle h_1, h_2\rangle^2 + \|\langle h_2, k\rangle\, h_1 + \langle h_1, k\rangle\, h_2\|^2_{H(R)}\,.$$

For $h_i = R(\cdot, t_i)$, $t_i \in T$, $i = 1, 2$, we get $f_2(m) = m(t_1)\, m(t_2)$, $m \in H(R)$.

Example 5.2. If $f_p(m) = \langle g, m\rangle^p$, $m \in H(R)$ then for the variance of the best unbiassed estimate $f_p(X)$, which can be found following the method described in Section 3, we get from (11):

$$\mathrm{Var}_k\,[f_p(X)] = \sum_{i=1}^{p} \binom{p}{i}^2 \,.\, i!\,\langle g, k\rangle^{2(p-i)}\,\|g\|^{2i}\,.$$

Especially for $p = 3$, $f_3(X) = \langle X, g\rangle^3 - 3\langle X, g\rangle \,.\, \|g\|^2$ and

$$\mathrm{Var}_k\,[f_3(X)] = 6 \,.\, \|g\|^6 + 18 \,.\, \|g\|^4 \,.\, \langle g, k\rangle^2 + 9 \,.\, \|g\|^2\,\langle g, k\rangle^4\,.$$

Functions of the type $f(m)$, $m \in H(R)$ were considered till now. But many times it is known that the unknown mean value function $m(\cdot)$ of the process X belongs to a subspace M of $H(R)$. The following example shows that in the later case, the unbiassed estimate $f(X)$ of the function $f(m)$, $m \in M$, minimizing the variance of estimation at the given $m_i \in M$ depends on m_i.

Let M be finite, $M = \{m_1, \ldots, m_q\}$. Then for any $f(m)$, $m \in M$, and any $m_i \in M$ we have:

$$f(\cdot) = (f(m_1), \ldots, f(m_q))\,\mathbf{G}^{-1}_{m_i}\begin{pmatrix} G_{m_i}(\cdot, m_1) \\ \cdots\cdots\cdots \\ G_{m_i}(\cdot, m_q)\end{pmatrix},$$

where

$$\mathbf{G}_{m_i} = \begin{pmatrix} G_{m_i}(m_1, m_1) \ldots G_{m_i}(m_1, m_q) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ G_{m_i}(m_q, m_1) \ldots G_{m_i}(m_q, m_q) \end{pmatrix}$$

and the minimum-variance unbiassed estimate of $f(m)$, $m \in M$, at $m_i \in M$ is given by

$$f_{m_i}(X) = (f(m_1), \ldots, f(m_q))\, \mathbf{G}_{m_i}^{-1} \begin{pmatrix} e^{\langle X, m_1 - m_i\rangle - 1/2(\|m_1\|^2 - \|m_i\|^2)} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ e^{\langle X, m_q - m_i\rangle - 1/2(\|m_q\|^2 - \|m_i\|^2)} \end{pmatrix}.$$

The following lemma enables us to prove a theorem on the existence of the best polynomial estimate.

Lemma 5.2. Let $f_p \in \otimes^p H(R)$ and let $k \in M$, M being a subspace of $H(R)$. Then

$$\mathscr{P}_{\sigma[\otimes^j M]} \langle \hat{f}_p, \otimes^{p-j} k\rangle_{\otimes^{p-j}H(R)} = \langle \mathscr{P}_{[\otimes^p M]} f_p, \otimes^{p-j} k\rangle_{\otimes^{p-j}H(R)}$$

for every $j = 0, 1, \ldots, p$.

Proof. Because both these elements belong to $\sigma[\otimes^j M]$ it is enough to prove that for every $h \in \sigma[\otimes^j M]$ the equality

$$\langle \mathscr{P}_{\sigma[\otimes^j M]} \langle \hat{f}_p, \otimes^{p-j} k\rangle_{\otimes^{p-j}H(R)}, h\rangle = \langle\langle \mathscr{P}_{\sigma[\otimes^p M]} f_p, \otimes^{p-j} k\rangle, h\rangle_{\otimes^j H(R)}$$

holds. But

$$\langle \mathscr{P}_{\sigma[\otimes^j M]} \langle \hat{f}_p, \otimes^{p-j} k\rangle_{\otimes^{p-j}H(R)} - \langle \mathscr{P}_{\sigma[\otimes^p M]} f_p, \otimes^{p-j} k\rangle, h\rangle_{\otimes^j H(R)} =$$
$$= \langle f_p, \mathscr{P}_{\sigma[\otimes^p H(R)]} \otimes^{p-j} k \otimes h\rangle_{\otimes^p H(R)} - \langle f_p, \mathscr{P}_{\sigma[\otimes^p M]} \otimes^{p-j} k \otimes h\rangle_{\otimes^p H(R)} = 0\,,$$

because $\otimes^{p-j} k \otimes h \in \otimes^p M$.

Theorem 5.2. Let $f(m) = \sum_{p=0}^{n} f_p(m) = \sum_{p=0}^{n} \langle g_p, \otimes^p m\rangle_{\otimes^p H(R)}$, $m \in M$, $g_p \in \otimes^p M$, $p = 0, 1, \ldots, n$. Then $f(X) = \sum_{p=0}^{n} f_p(X)$ is the best polynomial estimate of the function $f(m)$, $m \in M$, at every $k \in M$.

Proof. Let

$$M_f = \{h(\cdot) : h(m) = \sum_{p=0}^{n} \langle l_p, \otimes^p m\rangle,\ l_p \in \otimes^p H(R)\,,\ \mathscr{P}_{\sigma[\otimes^p M]} l_p =$$
$$= \hat{g}_p,\ p = 0, 1, \ldots, n\}\,.$$

Then using the Lemma 5.2 we get for any $h \in M_f$:

$$\operatorname{Var}_k [h(X)] = \sum_{p=0}^{n} \sum_{q=0}^{n} \sum_{i=1}^{\min\{p,q\}} \binom{p}{i} \binom{q}{i} i! \langle\langle \hat{l}_p, \otimes^{p-i} k \rangle, \langle \hat{l}_q, \otimes^{q-i} k \rangle\rangle_{\otimes^i H(R)} =$$

$$= \sum_{p=0}^{n} \sum_{q=0}^{n} \sum_{i=1}^{\min\{p,q\}} \binom{p}{i} \binom{q}{i} i! [\langle\langle \mathscr{P}_{\sigma[\otimes^p M]} l_p, \otimes^{p-i} k \rangle, \langle \mathscr{P}_{\sigma[\otimes^p M]} l_q, \otimes^{q-i} k \rangle\rangle_{\otimes^i H(R)} +$$

$$+ \langle\langle \hat{l}_p - \mathscr{P}_{\sigma[\otimes^p M]} l_p, \otimes^{p-i} k \rangle, \langle \hat{l}_p - \mathscr{P}_{\sigma[\otimes^p M]} l_q, \otimes^{p-i} k \rangle\rangle_{\otimes^i H(R)}] =$$

$$= \operatorname{Var}_k [f(X)] + \operatorname{Var}_k [(h - f)(X)] \geqq \operatorname{Var}_k [f(X)]$$

for every $k \in M$.

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