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ON THE STABILITY IN STOCHASTIC PROGRAMMING: THE CASE OF INDIVIDUAL PROBABILITY CONSTRAINTS[1]

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Stochastic programming problems with individual probability constraints belong to a class of optimization problems depending on a random element only through the corresponding probability measure. Consequently, the probability measure can be treated as a parameter in these problems.

The aim of the paper is to investigate the stability of the above mentioned problems with respect to the distribution functions space. The main effort is directed to some special situations in which stability investigation can be reduced (from the mathematical point of view) to one dimensional case. The Kolmogorov metric is employed to specify the stability results and, moreover, the achieved stability results are applied to statistical estimates of the optimal value and the optimal solution.

1. INTRODUCTION

There is not doubt that the stability problem (considered with respect to the probability measures space) is a serious problem of the stochastic programming theory. Namely, any responsible application of empirical estimates, parameter estimates as well as many approximate and numerical methods of solution are based on a possibility to replace the theoretical distribution function by some approximating one. In the literature, a great attention has been already paid to the stability of the stochastic optimization problems (see [1, 5, 7, 9, 13, 21, 22, 23, 24, 26]).

Let (Ω, S, P) be a probability space, $\xi = \xi(\omega) = [\xi_1(\omega), \xi_2(\omega), \ldots, \xi_l(\omega)]$ be an l-dimensional random vector defined on (Ω, S, P), $F(z)$, $F_i(z_i)$, $z = (z_1, \ldots, z_l)$, $i = 1, 2, \ldots, l$, $z \in E_l$ be the joint and the marginal one-dimensional distribution functions corresponding to the random vector $\xi(\omega)$ and to the component $\xi_i(\omega)$, $Z = Z_F \subset E_l$, $Z_i = Z_{F_i} \subset E_1$, $i = 1, \ldots, l$ denote the supports of the probability measures $P_F(\cdot)$, $P_{F_i}(\cdot)$ corresponding to the distribution functions $F(z)$ and $F_i(z_i)$.

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Let, moreover, $g_0(x, z)$, $f_i(x)$, $i = 1, 2, \dots, l$ be real–valued, continuous functions defined on $E_n \times E_l$ and E_n, $X \subset E_n$ be a nonempty set. (E_n, $n \geq 1$ denotes the n–dimensional Euclidean space.)

An optimization problem with a random element, in the objective function and on the right-hand side of the constraints only, can be introduced as the problem:

Find
$$\min\{g_0(x, \xi(\omega)) \,|\, x \in X : f_i(x) \leq \xi_i(\omega), \ i = 1, 2, \dots, l\}. \tag{1}$$
If the solution x has to be determined without knowing the realization of the random vector $\xi(\omega)$, then mostly a deterministic optimization problem is solved instead of the original one with a random element. The new problem can depend on the random element only through the corresponding probability measure. We shall consider it in the form:

Find
$$\varphi(F, \alpha) = \inf\{\mathsf{E}_F g(x, \xi(\omega)) \,|\, x \in X : P_{F_i}\{\omega : f_i(x) \leq \xi_i(\omega)\} \geq \alpha_i, \ i = 1, \dots l\}, \tag{2}$$
where $g(x, z)$ is a real–valued, continuous function defined on $E_n \times E_l$, $\alpha_i \in \langle 0, 1 \rangle$, $i = 1, 2, \dots, l$ are parameters. E_F denotes the operator of mathematical expectation corresponding to $F(\cdot)$.

In the literature, this type of the deterministic optimization problems has been investigated many times (see e.g. [4, 10, 20]). The distribution function $F(\cdot)$ can be considered as a parameter of the problem (2) and, consequently, it is reasonable to investigate the stability with respect to it. In the general case, it can mean to determine for a $\delta > 0$ a subset $\mathcal{F}(F, \delta)$ of the l–dimensional distribution functions space and real–valued functions $m_1(\delta), m_2(\delta)$ defined on E_1 (having the "suitable" properties) such that
$$\begin{array}{lll} G \in \mathcal{F}(F, \delta) & \Rightarrow & |\,\varphi(G, \alpha) - \varphi(F, \alpha)\,| \leq m_1(\delta), \\ G \in \mathcal{F}(F, \delta) & \Rightarrow & \|\,x(G, \alpha) - x(F, \alpha)\,\|^2 \leq m_2(\delta), \end{array} \tag{3}$$
$$x(F, \alpha) = \arg\min\{E_F g(x, \xi(\omega)) \,|\, x \in X : P_{F_i}\{\omega : f_i(x) \leq \xi_i(\omega)\} \geq \alpha_i, i = 1, \dots l\}.$$
($\|\cdot\|$ denotes the Euclidean norm in E_n.)

Of course, the second implication in (3) can be considered only if there exists unique $x(F, \alpha)$ fulfilling the last equation in the relations (3).

The aim of the paper is to deal with special cases in which the stability problem can be reduced (from the mathematical point of view) to the one–dimensional case. In particular, the aim of the paper is to introduce several special cases in which it is possible to determine subsets $\mathcal{F}_i(F_i, \delta_i)$, $\delta_i > 0$, $i = 1, 2, \dots, l$ of one–dimensional distribution functions space and real–valued functions $\overline{m}_1(\overline{\delta}), \overline{m}_2(\overline{\delta})$ defined on E_l (having "suitable" properties) such that
$$\begin{array}{llll} G_i \in \mathcal{F}_i(F_i, \delta_i), & i = 1, 2, \dots, l & \Rightarrow & |\,\varphi(G, \alpha) - \varphi(F, \alpha)\,| \leq \overline{m}_1(\overline{\delta}), \\ G_i \in \mathcal{F}_i(F_i, \delta_i), & i = 1, 2, \dots, l & \Rightarrow & \|\,x(G, \alpha) - x(F, \alpha)\,\|^2 \leq \overline{m}_2(\overline{\delta}). \end{array} \tag{4}$$
$G_i(\cdot)$, $i = 1, \dots, l$ denote the marginal one–dimensional distribution functions corresponding to the l–dimensional distribution function $G(\cdot)$, $\overline{\delta} = (\delta_1, \delta_2, \dots, \delta_l)$.

Furthermore, the Kolmogorov metric will be employed to specify the stability results. The new results (in this direction) will be applied to the statistical estimates of the optimal value and the optimal solution.

2. PROBLEM ANALYSIS

If we define the sets $X_{F_i}(\alpha_i)$, $X_F(\alpha)$, $\alpha_i \in \langle 0, 1\rangle$, $i = 1, \ldots, l$, $\alpha = (\alpha_1, \ldots, \alpha_l)$ by

$$\begin{aligned} X_{F_i}(\alpha_i) &= \{x \in X : P_{F_i}\{\omega : f_i(x) \le \xi_i(\omega)\} \ge \alpha_i\}, && (5) \\ X_F(\alpha) &= \bigcap_{i=1}^{l} X_{F_i}(\alpha_i), && (6) \end{aligned}$$

then we can rewrite the problem (2) as the problem:

Find

$$\varphi(F, \alpha) = \inf\{\mathsf{E}_F g(x, \xi(\omega)) \,|\, x \in X_F(\alpha)\}. \tag{7}$$

If $G(\cdot)$ is an arbitrary l–dimensional distribution function, then according to the triangular inequality we obtain that

$$\begin{aligned} |\varphi(F, \alpha) - \varphi(G, \alpha)| \le & \left| \inf_{x \in X_F(\alpha)} \mathsf{E}_F g(x, \xi(\omega)) - \inf_{x \in X_F(\alpha)} \mathsf{E}_G g(x, \xi(\omega)) \right| \\ + & \left| \inf_{x \in X_F(\alpha)} \mathsf{E}_G g(x, \xi(\omega)) - \inf_{x \in X_G(\alpha)} \mathsf{E}_G g(x, \xi(\omega)) \right|. \end{aligned} \tag{8}$$

Consequently, to investigate the stability of the problem (2) it is appropriate to investigate the stability of the following problems (see also e. g. [14, 16]):

Find

$$\inf_{X_F(\alpha)} g_0(x) \quad \text{with} \quad g_0(x) = \mathsf{E}_G g(x, \xi(\omega)). \tag{9}$$

Find

$$\inf_{X'} \mathsf{E}_F g(x, \xi(\omega)) \quad \text{with} \quad X' = X_F(\alpha). \tag{10}$$

It is easy to see that the stability of the problem (9) depends on the properties of the function $g_0(x)$ and on the "distance between the values" of the multifunctions $X_F(\alpha)$ and $X_G(\alpha)$. Consequently, it seems to be reasonable to investigate

$$\Delta[X_F(\alpha), X_G(\alpha)],$$

where $\Delta[\cdot, \cdot] = \Delta_n[\cdot, \cdot]$ denotes the Hausdorff distance in the space of nonempty, closed subsets of E_n (for the definition see e. g. [25]). To this end we define the multifunctions

$$\overline{\mathcal{K}}_i(z_i) = \{x \in X : f_i(x) \le z_i\}, \quad z_i \in E_1, \quad i = 1, 2, \ldots, l \tag{11}$$

and the quantiles

$$k_{F_i}(\alpha_i) = \sup\{z_i : P_{F_i}\{\omega : z_i \le \xi_i(\omega)\} \ge \alpha_i\}, \quad \alpha_i \in (0, 1), i = 1, \ldots, l. \tag{12}$$

Since it is easy to see that for $i = 1, 2, \ldots, l$,

$$x \in X_{F_i}(\alpha_i) \Longleftrightarrow x \in X \quad \text{and simultaneously} \quad P_{F_i}\{\omega : f_i(x) \le \xi_i(\omega)\} \ge \alpha_i$$
$$\Longleftrightarrow x \in X \quad \text{and simultaneously} \quad f_i(x) \le k_{F_i}(\alpha_i),$$

we can obtain that

$$X_{F_i}(\alpha_i) = \overline{\mathcal{K}}_i(k_{F_i}(\alpha_i)), \quad \alpha_i \in (0, 1), \, i = 1, 2, \ldots, l. \tag{13}$$

If, furthermore, we define $\overline{\mathcal{K}}(z)$, $z = (z_1, z_2, \ldots, z_l)$ by the relation

$$\overline{\mathcal{K}}(z) = \bigcap_{i=1}^{l} \overline{\mathcal{K}}_i(z_i), \tag{14}$$

then

$$X_F(\alpha) = \overline{\mathcal{K}}(k_F(\alpha)), \quad \text{where} \quad k_F(\alpha) = (k_{F_1}(\alpha_1), k_{F_2}(\alpha_2), \ldots, k_{F_l}(\alpha_l)). \tag{15}$$

According to the relation (15) it is easy to see that to investigate the stability of the problem (9) it is suitable to investigate the behaviour of the multifunction $\overline{\mathcal{K}}(\cdot)$. Furthermore, it is easy to see that the assumptions under which

$$\Delta[\overline{\mathcal{K}}(z), \overline{\mathcal{K}}(z')] \le C \, \|z - z'\| \quad \text{in a neighbourhood of the point} \quad k_F(\alpha)$$

(together with the relation (15), the triangular inequality and additional assumptions) imply that

$$\Delta[X_F(\alpha), X_G(\alpha)] \le \sum_{i=1}^{l} C \, |k_{F_i}(\alpha_i) - k_{G_i}(\alpha_i)|. \tag{16}$$

In general, to investigate the stability of the problem (10) it is necessary to find $\mathcal{F}(F, \delta)$ and the functions $m_1(\delta)$, $m_2(\delta)$, $\delta > 0$ fulfilling the relations (3). In this paper we shall try to introduce some special cases for which there exist also $\mathcal{F}_i(F_i, \delta_i)$, $\delta_i > 0$, $i = 1, 2, \ldots, l$, $\overline{m}_1(\overline{\delta})$, $\overline{m}_2(\overline{\delta})$, $\overline{\delta} = (\delta_1, \delta_2, \ldots, \delta_l)$ fulfilling the relations of the type (4).

3. STABILITY RESULTS

Before presenting the first assertions we introduce several systems of the assumptions. Let $\overline{Z}_i \subset E_1$, $i = 1, 2, \ldots l$ be nonempty, convex sets, $\overline{Z} = \prod_{i=1}^{l} \overline{Z}_i$; $\overline{Z}(\varepsilon)$, $\varepsilon > 0$ denote the ε–neighbourhood of the set $\overline{Z}$.

i.1 there exists $\varepsilon > 0$ such that

a) $f_i(x)$, $i = 1, 2, \ldots, l$ are linear functions, $X = E_n$;
without loss of generality, we can consider in this case the constraints in (1) to be in the form of equations,

b) for every $z \in \overline{Z}(\varepsilon)$, $\overline{\mathcal{K}}(z)$ is a nonempty, compact set,

c) if the matrix A of the type $(l \times n)$, $l \leq n$ fulfils for $z \in \overline{Z}(\varepsilon)$ the relation

$$\overline{\mathcal{K}}(z) = \{x \in X : Ax = z\}$$

then all its submatrices of the type $(l \times l)$, $A(1), A(2), \ldots, A(m)$ are nonsingular,

i.2 there exists $\varepsilon > 0$ such that

a) X is a convex, compact set,

b) $f_i(x)$, $i = 1, 2, \ldots, l$ are convex functions on X,

c) for every $z \in \overline{Z}(\varepsilon)$, $\overline{\mathcal{K}}(z)$ is a nonempty set.

i.3 there exist real-valued constants d_1, γ_2, $\varepsilon > 0$ such that

a) if $x \in X$, $z = (z_1, \ldots, z_l)$, $z \in \overline{Z}(\varepsilon)$ fulfil the relations $f_i(x) \leq z_i$, $i = 1, 2, \ldots, l$ and simultaneously $f_j(x) = z_j$ for at least one $j \in \{1, 2, \ldots, l\}$, then there exists a vector $x(0) \in E_n$ (generally depending on x) such that

$$\|x(0)\| = 1, \quad x + dx(0) \in X, \quad f_i(x) - f_i(x + dx(0)) \geq \gamma_2 d$$

for every $d \in (0, d_1)$, $i = 1, 2, \ldots, l$,

b) for every $z \in \overline{Z}(\varepsilon)$, $\overline{\mathcal{K}}(z)$ is a nonempty, compact set.

The introduced systems of the assumptions i.1, 1.2 cover both linear and convex functions on the left-hand side of the constraints in (1). These special cases were investigated in the literature many times (mostly in a connection with parametric linear or quadratic programming, see e.g. [2]). They appear also in the connection with the stochastic programming problems (see e.g. [17]). To justify the system of the assumptions i.3 (in more details) we introduce a simple example. Let $n = 2$, $l = 2$, $X = \langle 1.5, 4 \rangle \times \langle 1.5, 4 \rangle$ and, moreover,

$$f_1(x) = x_1 x_2, \quad f_2(x) = \log(x_1 + x_2), \quad x = (x_1, x_2).$$

It is easy to see that (in this case) the system i.3 is fulfilled, while the systems of the assumptions i.1 and i.2 are not fulfilled.

To introduce assertions concerning stability results we define the constant C by the following relations.

$$C = \min(C_1, C_2, C_3), \tag{17}$$

where

$$\begin{array}{rcll} C_1 & = & l \max\limits_{i,r,s} |a_{ir}(s)| & \text{if the system of the assumptions i.1 is fulfilled,} \\ & & & a_{ir}(s),\ i,\ r = 1,\ 2, \ldots, l \text{ for } s \in \{1,\ 2, \ldots, m\} \\ & & & \text{denote elements of the inverse matrix to } A(s), \\ & = & +\infty & \text{otherwise,} \\ C_2 & = & \dfrac{M_1}{\varepsilon_0} & \text{if the system of the assumptions i.2 is fulfilled,} \\ & & & \varepsilon_0 \in (0,\ \varepsilon),\ M_1 = \sup_{x^1, x^2 \in X} \|x^1 - x^2\|, \\ & = & +\infty & \text{otherwise,} \\ C_3 & = & \dfrac{1}{\gamma_2} & \text{if the system of the assumptions i.3 is fulfilled,} \\ & = & +\infty & \text{otherwise.} \end{array}$$

(In (17) we calculate $\min(c,\ c',\ +\infty) = \min(c,\ c'), \min(c,\ +\infty,\ +\infty) = c$ for every $c,\ c' \in E_1$.)

If we consider a special case of the function $g(x,\ z)$ when

A.1 a) $g(x,\ z) = \overline{g}(x), \quad x \in E_n,\ z \in E_l$, where $\overline{g}(x)$ is a real-valued, Lipschitz function on X with the Lipschitz constant $\overline{L}'$,

then we can already introduce the first assertion.

Proposition 1. Let $\alpha_i \in (0,\ 1)$, $i = 1,\ 2, \ldots, l$. If

1. the assumption A.1a is fulfilled,
2. $G(z)$ is an arbitrary l-dimensional distribution function,
3. $\overline{Z}_i = \langle \min(k_{F_i}(\alpha_i),\ k_{G_i}(\alpha_i)), \max(k_{F_i}(\alpha_i),\ k_{G_i}(\alpha_i)) \rangle$, $i = 1,\ 2, \ldots, l$,
4. at least one of the systems of the assumptions i.1, i.2, i.3 is fulfilled,

then

$$|\varphi(F, \alpha) - \varphi(G, \alpha)| \le C\overline{L}' \sum_{i=1}^{l} |k_{F_i}(\alpha_i) - k_{G_i}(\alpha_i)|.$$

Proof. First, by a little modification of Lemma 1 [18] (see also [2]) we can obtain that

$$\Delta[\overline{\mathcal{K}}(z(1)),\ \overline{\mathcal{K}}(z(2))] \le C\|z(1) - z(2)\| \quad \text{for every } z(1),\ z(2) \in \overline{Z}(\varepsilon) \quad \text{and some } \varepsilon > 0,$$

whenever the assumption 4 is fulfilled. The assertion of Proposition 1 follows from the last inequality, the relation (15), the triangular inequality and the fact that (under the assumptions) $X_F(\alpha)$, $X_G(\alpha)$ are nonempty, compact sets. □

It follows from Proposition 1 that a dependence of the changes of the optimal value (in the case A.1) on the perturbations of the underlying probability measure can be estimated (of course, under some additional assumptions) by the distance of the corresponding one-dimensional quantiles.

To introduce the next assertion we define for $\delta_i > 0$, $i = 1, \ldots, l$ the one-dimensional distribution functions $\underline{F}_{i,\,\delta_i}(z_i)$, $\overline{F}_{i,\,\delta_i}(z_i)$ by the relations

$$\underline{F}_{i,\,\delta_i}(z_i) = F_i(z_i - \delta_i), \quad \overline{F}_{i,\,\delta_i}(z_i) = F_i(z_i + \delta_i), \; z_i \in E_1. \tag{18}$$

We introduce the following assumptions.

ii. X is a convex set and, moreover, $f_i(x)$, $i = 1, 2, \ldots, l$ are quasi convex functions on X,

A.1 b) $\overline{g}(x)$ is a strongly convex function on X with the parameter $\rho > 0$ (for the definition of strongly convex functions see e. g. [19, 28]).

The assumptions A.1b, ii. guarantee just unique $x(F,\,\alpha)$ fulfilling the last equation in the relations (3). In [11] the assumption on strongly convex property is replaced by a little more general assumption on uniformly convex property. These both assumptions give possibility to employ the results on the stability of the optimal value (by a rather simple manner) to the investigation of the stability of the optimal solution. The investigation of the optimal solution set is (generally in optimization problems) rather more complicated (see e. g. [2, 23]).

Proposition 2. Let for $i = 1, 2, \ldots, l$, $\delta_i > 0$, $\alpha_i \in (0,\,1)$ be given, $\overline{Z}_i = (k_{F_i}(\alpha_i) - 2\delta_i,\, k_{F_i}(\alpha_i) + 2\delta_i)$. If

1. the assumption A.1a is fulfilled,
2. at least one of the systems of the assumptions i.1, i.2, i.3 is fulfilled,
3. $G(z)$ is an arbitrary l–dimensional distribution function such that for $i \in \{1, 2, \ldots, l\}$

$$G_i(z_i) \in \langle \underline{F}_{i,\,\delta_i}(z_i),\, \overline{F}_{i,\,\delta_i}(z_i) \rangle, \quad z_i \in (k_{F_i}(\alpha_i) - \delta_i - \varepsilon,\, k_{F_i}(\alpha_i) + \delta_i + \varepsilon),$$

then

$$|\varphi(F, \alpha) - \varphi(G, \alpha)| \leq \overline{L}' C \sum_{i=1}^{l} \delta_i. \tag{19}$$

If, moreover,

4. the assumptions A.1b and ii. are fulfilled,

then also

$$\|x(F,\,\alpha) - x(G,\,\alpha)\|^2 \leq \frac{12}{\rho} \overline{L}' C \sum_{i=1}^{l} \delta_i. \tag{20}$$

The proof of Proposition 2 is given in the Appendix.

To consider another special case of the function $g(x,\,z)$, let $\delta_i > 0$, $i = 1, 2, \ldots, l$. We introduce the new system of assumptions.

A.2 a) $g(x, z) = \sum_{i=1}^{l} g_i(x, z_i), \quad x \in E_n,\ z = (z_1, z_2, \dots, z_l) \in E_l$, where $g_i(x, z_i)$, $i = 1, 2, \dots, l$ are real-valued functions defined on $E_n \times E_1$,

b) for every $x \in X$, $g_i(x, z_i)$, $i = 1, 2, \dots, l$ are Lipschitz functions on $Z_{F_i}(\delta_i)$ with the Lipschitz constants L_i not depending on $x \in X$,

c) for every $z_i \in Z_{F_i}(\delta_i)$, $g_i(x, z_i)$, $i = 1, 2, \dots, l$ are Lipschitz functions on X with the Lipschitz constants L'_i not depending on $z_i \in Z_{F_i}(\delta_i)$,

d) for every $x \in X$, $i \in \{1, 2, \dots, l\}$ there exists a finite $E_{F_i} g_i(x, \xi_i(\omega))$,

e) for every $z_i \in Z_{F_i}(\delta_i)$, $i \in \{1, 2, \dots, l\}$, $g_i(x, z_i)$ is a convex function on E_n and simultaneously there exists $j \in \{1, 2, \dots, l\}$ such that $g_j(x, z_j)$ is a strongly convex function on E_n with a parameter $\rho > 0$.

Proposition 3. Let for $i = 1, 2, \dots, l$, $\delta_i > 0$, $\alpha_i \in (0, 1)$ be given, $\overline{Z}_i = (k_{F_i}(\alpha_i) - 2\delta_i,\ k_{F_i}(\alpha_i) + 2\delta_i)$. If

1. the assumptions A.2a, A.2b, A.2c and A.2d are fulfilled,
2. at least one of the systems of the assumptions i.1, i.2, i.3 is fulfilled,
3. $G(z)$ is an arbitrary l–dimensional distribution function such that

$$G_i(z_i) \in \langle \underline{F}_{i,\delta_i}(z_i),\ \overline{F}_{i,\delta_i}(z_i) \rangle \quad \text{for every } z_i \in E_1,\ i = 1, 2, \dots, l,$$

then

$$|\varphi(F, \alpha) - \varphi(G, \alpha)| \le \sum_{i=1}^{l} \left[L_i + C \sum_{j=1}^{l} L'_j \right] \delta_i. \tag{21}$$

If, moreover,

4. the assumptions A.2e and ii. are fulfilled,

then also

$$\|x(F, \alpha) - x(G, \alpha)\|^2 \le \frac{12}{\rho} \sum_{i=1}^{l} \left[L_i + C \sum_{j=1}^{l} L'_j \right] \delta_i. \tag{22}$$

The proof of Proposition 3 is given in the Appendix.

To deal with the last special case, let $\delta_i > 0$, $i = 1, 2, \dots, l$. We introduce the following system of assumptions.

A.3 a) the components of the random vector $\xi(\omega) = (\xi_1(\omega), \xi_2(\omega), \dots, \xi_l(\omega))$ are stochastically independent,

b) for every $x \in X$, $\delta = \max_i \delta_i$, $g(x, z)$ is a Lipschitz function on $Z_F(\delta)$ with the Lipschitz constant L not depending on $x \in X$,

c) for every $z \in Z_F(\delta)$, $g(x, z)$ is a Lipschitz function on X with the Lipschitz constant L' not depending on $z \in Z_F(\delta)$,

d) for every $x \in X$ there exists a finite $E_F g(x, \xi(\omega))$,

e) for every $z \in Z_F$, $g(x, z)$ is a strongly convex function on E_n with a parameter $\rho > 0$.

Proposition 4. Let for $i = 1, 2, \ldots, l$, $\delta_i > 0$, $\alpha_i \in (0, 1)$ be given, $\overline{Z}_i = (k_{F_i}(\alpha_i) -2\delta_i, k_{F_i}(\alpha_i) + 2\delta_i)$. If

1. the assumptions A.3a, A.3b, A.3c and A.3d are fulfilled,
2. at least one of the systems of assumptions i.1, i.2, i.3 is fulfilled,
3. $G(z)$ is an arbitrary l–dimensional distribution function such that

$$G_i(z_i) \in \langle \underline{F}_{i,\delta_i}(z_i), \overline{F}_{i,\delta_i}(z_i)\rangle \quad \text{for every } z_i \in E_1, \ i = 1, 2, \ldots, l,$$

and simultaneously

$$G(z) = \prod_{i=1}^{l} G_i(z_i), \quad z = (z_1, z_2, \ldots, z_l),$$

then

$$|\varphi(F, \alpha) - \varphi(G, \alpha)| \le [l\,L + C\,L'] \sum_{i=1}^{l} \delta_i. \tag{23}$$

If, moreover,

4. the assumptions ii. and A.3e are fulfilled,

then also

$$\|x(F, \alpha) - x(G, \alpha)\|^2 \le \frac{12}{\rho}\,[l\,L + C\,L'] \sum_{i=1}^{l} \delta_i. \tag{24}$$

The proof of Proposition 4 is given in the Appendix.

4. KOLMOGOROV METRIC AND STABILITY

In the sequel we employ the Kolmogorov metric to specify the stability results. To this end let $\alpha_i \in (0, 1)$, $\delta_i > 0$, $i = 1, \ldots, l$. We define the intervals $\mathcal{Z}_i(\alpha_i, \delta_i)$ by the relations

$$\mathcal{Z}_i(\alpha_i, \delta_i) = \left\langle \max\left(\overline{z}_i^0, k_i(\alpha_i) - 2\delta_i\right), \min\left(\overline{z}_i^1, k_i(\alpha_i) + 2\delta_i\right)\right\rangle, \tag{25}$$

$$\overline{z}_i^0 = \sup\{z_i : F_i(z_i) = 0\}, \quad \overline{z}_i^1 = \inf\{z_i : F_i(z_i) = 1\}$$

(where we calculate $\max(\overline{z}_i^0, z_i) = z_i$ if $\overline{z}_i^0 = -\infty$, $z_i \in E_1$, $\min(\overline{z}_i^1, z_i) = z_i$ if $\overline{z}_i^1 = +\infty$, $z_i \in E_1$, $i = 1, 2, \ldots, l$) and, moreover, we introduce the following system of the assumptions.

B.1 a) for $i = 1, 2, \ldots, l$ the probability measures $P_{F_i}(\cdot)$ are absolutely continuous with respect to the Lebesgue measure in E_1,

b) for $i = 1, 2, \ldots, l$ and an $\varepsilon > 0$ there exist constants $\vartheta_i > 0$ such that

$$h_i(z_i) \ge \vartheta_i \quad \text{for every } z_i \in \mathcal{Z}_i(\alpha_i, \delta_i + \varepsilon),$$

c) for $i = 1, 2, \ldots, l$ there exist $a_i, b_i \in E_1$, $a_i < b_i$, $\vartheta_i > 0$ such that

$$Z_{F_i} = \langle a_i, b_i\rangle, \quad h_i(z_i) \ge \vartheta_i \quad \text{for every } z_i \in Z_{F_i}.$$

($h_i(z_i)$ denotes the probability density corresponding to $F_i(z_i)$, $i = 1, 2, \ldots, l$.)

Lemma 1. Let for $i = 1, 2, \ldots, l$, $\alpha_i \in (0, 1)$, $\delta_i > 0$ be arbitrary. Let, moreover, $G(z)$ be an arbitrary l–dimensional distribution function. If

1. B.1a and B.1b are fulfilled, then for every $i = \{1, 2, \ldots, l\}$

$$|F_i(z_i) - G_i(z_i)| \le \delta_i \vartheta_i, \quad z_i \in \mathcal{Z}_i(\alpha_i, \delta_i + \varepsilon) \quad \text{and simultaneously either}$$
$$Z_{G_i} \subset Z_{F_i}(\delta_i) \ \text{ or } F_{\delta_i}(k_{F_i}(\alpha_i) - \delta_i - \varepsilon) > 0,\ F_{\delta_i}(k_{F_i}(\alpha_i) + \delta_i + \varepsilon) < 1 \Longrightarrow$$
$$\Longrightarrow G_i(z_i) \in \langle \underline{F}_{i,2\delta_i}(z_i), \overline{F}_{i,2\delta_i}(z_i) \rangle, \quad z_i \in (k_{F_i}(\alpha_i) - \delta_i - \varepsilon,\ k_{F_i}(\alpha_i) + \delta_i + \varepsilon),$$

2. B.1a and B.1c are fulfilled, then for every $i = \{1, 2, \ldots, l\}$

$$|F_i(z_i) - G_i(z_i)| \le \delta_i \vartheta_i, \quad z_i \in Z_{F_i}, \ Z_{G_i} \subset Z_{F_i}(\delta_i) \Longrightarrow$$
$$\Longrightarrow G_i(z_i) \in \langle \underline{F}_{i,2\delta_i}(z_i), \overline{F}_{i,2\delta_i}(z_i) \rangle, \quad z_i \in E_1.$$

Proof. First we consider the case 1. Let $i \in \{1, 2, \ldots, l\}$, $z_i \in \langle k_{F_i}(\alpha_i) - \delta_i - \varepsilon, k_{F_i}(\alpha_i) + \delta_i + \varepsilon \rangle$ be arbitrary. Two cases can happen.

a) $z_i \in \langle \overline{z}_i^0 + \delta_i, \overline{z}_i^1 - \delta_i \rangle$,

b) $z_i \notin \langle \overline{z}_i^0 + \delta_i, \overline{z}_i^1 - \delta_i \rangle$.

If the case a) happens, then since (in this case)

$$F_i(z_i - \delta_i) \le F_i(z_i) - \vartheta_i \delta_i \le F_i(z_i) \le F_i(z_i) + \vartheta_i \delta_i \le F_i(z_i + \delta_i)$$

and simultaneously

$$|F_i(z_i) - G_i(z_i)| \le \vartheta_i \delta_i$$

we can see that $G_i(z_i) \in \langle \underline{F}_{i,\delta_i}(z_i), \overline{F}_{i,\delta_i}(z_i) \rangle$ in the case a).

If the case b) happens, then either $F_i(z_i - \delta_i) = 0$ or $F_i(z_i + \delta_i) = 1$. Without loss of generality we can consider only the case $F_i(z_i - \delta_i) = 0$. However, then for $z_i' \in \langle \overline{z}_i^0 - \delta_i, \overline{z}_i^0 + \delta_i \rangle$ we can see that

$$0 = F_i(z_i' - \delta_i) \le F_i(z_i') \le F_i(z_i' + \delta_i) + \vartheta_i \delta_i \le F_i(z_i' + 2\delta_i), \quad |F_i(z_i') - G_i(z_i')| \le \vartheta_i \delta_i$$

and simultaneously

$$G_i(z_i - 2\delta_i) = 0, \quad G_i(\overline{z}_i^0) \le \vartheta_i \delta_i, \quad 2\vartheta_i \delta_i \le F_i(\overline{z}_i^0 + 2\delta_i).$$

Consequently, we can see that also in this case $G_i(z_i) \in \langle \underline{F}_{i,\delta_i}(z_i), \overline{F}_{i,\delta_i}(z_i) \rangle$.

The proof of the Assertion 2 is very similar and consequently it can be omitted. $\square$

Theorem 1. Let for $i = 1, 2, \ldots, l$, $\alpha_i \in (0, 1)$, $\delta_i > 0$ be arbitrary, $\overline{\mathcal{Z}}_i = (k_{F_i}(\alpha_i) - 2\delta_i, k_{F_i}(\alpha_i) + 2\delta_i)$. If

1. the assumptions B.1a and B.1b are fulfilled,
2. $G(z)$ is an arbitrary l-dimensional distribution function such that for $z_i \in \mathcal{Z}_i(\alpha_i, \delta_i + \varepsilon)$, $|F_i(z_i) - G_i(z_i)| \leq \delta_i \vartheta_i$ and simultaneously either $Z_{G_i} \subset Z_{F_i}(\delta_i')$ or $F_i(k_{F_i}(\alpha_i) - \delta_i' - \varepsilon) > 0$, $F_i(k_{F_i}(\alpha_i) + \delta_i' + \varepsilon) < 1$, $\delta_i' = \frac{1}{\vartheta_i} \sup_{\hat{\mathcal{Z}}i(\alpha_i, \delta_i+\varepsilon)} |F_i(z_i) - G_i(z_i)|$, $i = 1, 2, \ldots, l$,
3. the assumptions A.1a is fulfilled,
4. at least one of the systems of the assumptions i.1, i.2, i.3 is fulfilled,

then

$$|\varphi(F, \alpha) - \varphi(G, \alpha)| \leq 2L'C \sum_{i=1}^{l} \frac{1}{\vartheta_i} \sup_{\hat{\mathcal{Z}}_i(\alpha_i, \delta_i+\varepsilon)} |F_i(z_i) - G_i(z_i)|. \tag{26}$$

If, moreover,

5. the assumptions A.1b, and ii. are fulfilled

then also

$$\|x(F, \alpha) - x(G, \alpha)\|^2 \leq \frac{24}{\rho} L'C \sum_{i=1}^{l} \frac{1}{\vartheta_i} \sup_{\hat{\mathcal{Z}}_i(\alpha_i, \delta_i+\varepsilon)} |F_i(z_i) - G_i(z_i)|. \tag{27}$$

Proof. To verify the assertion of Theorem 1 we employ Lemma 1 and we substitute $\delta_i =: 2\delta_i'$, $i \in \{1, \ldots, l\}$ in Proposition 2. $\square$

The assumption 2 of Theorem 1 can seem rather badly understandable. However this complicated form gives possibility to include the cases when $G(\cdot)$ is "closed" to $F(\cdot)$ only in a neighbourhood of the point $k_F(\alpha)$.

Theorem 2. Let for $i = 1, 2, \ldots, l$, $\alpha_i \in (0, 1)$, $\delta_i > 0$ be arbitrary, $\overline{\mathcal{Z}}_i = (k_{F_i}(\alpha_i) - 2\delta_i, k_{F_i}(\alpha_i) + 2\delta_i)$. If

1. the assumptions B.1a and B.1c are fulfilled,
2. $G(z)$ is an arbitrary l-dimensional distribution function such that

   $$\delta_i \vartheta_i \geq \sup\{|F_i(z_i) - G_i(z_i)| : z_i \in Z_{F_i}\}, \ i = 1, 2, \ldots, l,$$

   and simultaneously

   $$Z_{G_i} \subset Z_{F_i}\left(\frac{\sup |F_i(z_i) - G_i(z_i)|}{\vartheta_i}\right), \ i = 1, 2, \ldots, l,$$

3. the assumptions A.2a, A.2b, A.2c, A.2d are fulfilled,
4. at least one of the systems of the assumptions i.1, i.2, i.3 is fulfilled,

then

$$|\varphi(F,\alpha)-\varphi(G,\alpha)| \le 2\sum_{i=1}^{l}\left[L_i + C\sum_{j=1}^{l} L'_j\right]\frac{\sup|F_i(z_i)-G_i(z_i)|}{\vartheta_i}. \tag{28}$$

If, moreover,

5. the assumptions A.2e and ii. are fulfilled,

then also

$$\|x(F,\,\alpha)-x(G,\,\alpha)\|^2 \le \frac{24}{\rho}\sum_{i=1}^{l}[L_i + C\sum_{j=1}^{l} L'_j]\frac{\sup|F_i(z_i)-G_i(z_i)|}{\vartheta_i}. \tag{29}$$

Proof. To verify the assertion of Theorem 2 we employ Lemma 1 and we substitute $\delta_i =: 2\frac{\sup|F_i(z_i)-G_i(z_i)|}{\vartheta_i}$, $i \in \{1,\ldots,l\}$ in Proposition 3. □

Theorem 3. Let for $i = 1,\,2,\ldots,l$, $\alpha_i \in (0,\,1)$, $\delta_i > 0$ be arbitrary, $\overline{Z}_i = (k_{F_i}(\alpha_i) -2\delta_i,\, k_{F_i}(\alpha_i)+2\delta_i)$. If

1. the assumptions B.1a and B.1c are fulfilled,
2. $G(z)$ is an arbitrary l–dimensional distribution function such that

$$\delta_i\vartheta_i \ge \sup\{|F_i(z_i)-G_i(z_i)| : z_i \in Z_{F_i}\},\; i=1,\,2,\ldots,l,$$
$$Z_{G_i} \subset Z_{F_i}\left(\frac{\sup|F_i(z_i)-G_i(z_i)|}{\vartheta_i}\right),\; i=1,\,2,\ldots,l,$$

   and simultaneously

$$G(z) = \prod_{i=1}^{l} G_i(z_i), \quad z = (z_1,\,z_2,\ldots,z_l),$$

3. the assumptions A.3a, A.3b, A.3c and A.3d are fulfilled,
4. at least one of the systems of the assumptions i.1, i.2, i.3 is fulfilled,

then

$$|\varphi(F,\alpha)-\varphi(G,\alpha)| \le 2[l\,L + CL']\sum_{i=1}^{l}\frac{\sup|F_i(z_i)-G_i(z_i)|}{\vartheta_i}. \tag{30}$$

If, moreover,

5. the assumptions A.3e and (ii.) are fulfilled

then also

$$\|x(F,\,\alpha)-x(G,\,\alpha)\|^2 \le \frac{24}{\rho}[l\,L + CL']\sum_{i=1}^{l}\frac{\sup|F_i(z_i)-G_i(z_i)|}{\vartheta_i}. \tag{31}$$

Proof. To verify the assertion of Theorem 3 we employ Lemma 1 and we substitute $\delta_i =: \frac{\sup|F_i(z_i)-G_i(z_i)|}{\vartheta_i}$, $i \in \{1,\ldots,l\}$ in Proposition 4. □

5. APPLICATION TO ESTIMATES

If statistical estimates replace the theoretical distribution functions $F_i(z_i)$, $i = 1, \dots$ $\dots, l$, then it is possible to employ the assertions of Theorems 1, 2 and 3 to investigate the properties of the corresponding estimates of the optimal value and the optimal solution. Evidently, if the case A.1 happens, then the behaviour of these estimates follows from the behaviour of the estimates of the quantiles (see e. g. [6]).

To investigate the cases A.2 and A.3 let $\{\xi_i^k(\omega)\}_{k=-\infty}^{\infty}$, $i = 1, 2, \dots, l$ be sequences of random values defined on (Ω, S, P) such that for every $k = \dots, -1, 0, 1, \dots$ the random value $\xi_i^k(\omega)$ has the same distribution function as the random value $\xi_i(\omega)$. For $i = 1, 2, \dots, l$, $N_i = 1, \dots$ we denote by the symbol $F_i^{N_i}(z_i) = F_i^{N_i}(z_i, \omega)$, $z_i \in E_1$ an arbitrary statistical estimate of $F_i(z_i)$ determined by $\{\xi_i^k(\omega)\}_{k=1}^{N_i}$ and by the symbol $F^{\overline{N}}(z)$, $z \in E_l$ an arbitrary joint l-dimensional distribution function corresponding to the $F_i^{N_i}(z_i)$, $i = 1, 2, \dots, l$. Evidently, under quite general conditions, the theoretical values $\varphi(F, \alpha)$, $x(F, \alpha)$ can be estimated by the values

$$\begin{aligned} \varphi(F^{\overline{N}}, \alpha) &= \inf_{X_{F^{\overline{N}}}(\alpha)} \mathsf{E}_{F^{\overline{N}}} g(x, \xi(\omega)), \\ x(F^{\overline{N}}, \alpha) &= \arg\min\{\mathsf{E}_{F^{\overline{N}}} g(x, \xi(\omega)) | \, x \in X_{F^{\overline{N}}}(\alpha)\}, \end{aligned}$$

where

$$X_{F^{\overline{N}}}(\alpha) = \bigcap_{i=1}^{l} X_{F_i^{N_i}}(\alpha_i), \quad \overline{N} = (N_1, \dots, N_l).$$

Theorem 4. Let for $i = 1, 2, \dots, l$, $\alpha_i \in (0, 1)$, $\delta_i > 0$ be arbitrary, $\overline{Z}_i = (k_{F_i}(\alpha_i) - 2\delta_i, \, k_{F_i}(\alpha_i) + 2\delta_i)$. If

1. either the assumptions 1, 3 and 4 of Theorem 2 or Theorem 3 are fulfilled,
2. for $i \in \{1, 2, \dots, l\}$,

$$P\left\{\omega : Z_{F_i^{N_i}} \subset Z_{F_i}\left(\frac{\sup |F_i(z_i) - F_i^{N_i}(z_i)|}{\vartheta_i}\right)\right\} \to_{N_i \to \infty} 1$$

and simultaneously for every $t > 0$ and a $\nu > 0$

$$P\left\{\omega : (N_i)^{\nu} \sup |F_i(z_i) - F_i^{N_i}(z_i)| > t\right\} \to_{N_i \to \infty} 0,$$

then for every $t > 0$

$$P\left\{\omega : \left(\min_i N_i\right)^{\nu} |\varphi(F, \alpha) - \varphi(F^{\overline{N}}, \alpha)| > t\right\} \to_{\min(N_i) \to \infty} 0. \tag{32}$$

If moreover

3. the corresponding assumption 5 of Theorem 2 or 3 is fulfilled,

then also for every $t > 0$

$$P\left\{\omega : \left(\min_i N_i)^{\nu} \|x(F, \alpha\right) - x(F^{\overline{N}}, \alpha)\|^2 > t\right\} \to_{\min(N_i) \to \infty} 0. \tag{33}$$

Proof. Let, first, the corresponding assumptions of Theorem 2 be fulfilled. It follows from the assumptions and from the elementary properties of the probability measure that (in this case) for every $t > 0$

$$\begin{aligned}
& P\left\{\omega : \left(\min_i N_i\right)^{\nu} |\varphi(F,\alpha) - \varphi(F^{\overline{N}},\alpha)| > t\right\} \\
\leq\ & P\left\{\omega : 2\sum_{i=1}^{l}(N_i)^{\nu}\left[L_i + C\sum_{j=1}^{l} L'_j\right] \sup \tfrac{|F_i^{N_i}(z_i) - F_i(z_i)|}{\vartheta_i} > t\right\} \\
& + P\left\{\omega : Z_{F_i^{N_i}} \not\subset Z_{F_i}\left(\tfrac{\sup|F_i(z_i) - F_i^{N_i}(z_i)|}{\vartheta_i}\right) \quad \text{for at least one } i \in \{1,\ldots,l\}\right\} \\
& + P\left\{\omega : \sup|F_i(z_i) - F_i^{N_i}(z_i)| \geq \vartheta_i\delta_i \quad \text{for at least one } i \in \{1, 2,\ldots,l\}\right\} \\
\leq\ & \sum_{i=1}^{l} P\left\{\omega : (N_i)^{\nu} \sup|F_i^{N_i}(z_i) - F_i(z_i)| > \tfrac{t\vartheta_i}{2l\left[L_i + C\sum_{j=1}^{l} L'_j\right]}\right\} \\
& + P\left\{\omega : Z_{F_i^{N_i}} \not\subset Z_{F_i}\left(\tfrac{\sup|F_i(z_i) - F_i^{N_i}(z_i)|}{\vartheta_i}\right) \quad \text{for at least one } i \in \{1,\ldots,l\}\right\} \\
& + P\left\{\omega : \sup|F_i(z_i) - F_i^{N_i}(z_i)| \geq \vartheta_i\delta_i \quad \text{for at least one } i \in \{1, 2,\ldots,l\}\right\}.
\end{aligned} \tag{34}$$

The first assertion (the relation (32)) of Theorem 4 follows (under the assumptions corresponding to Theorem 2) from the last system of the inequalities and from the assumptions. If the corresponding assumptions of Theorem 3 are fulfilled, then replacing (in the last relations) the constants $L_i + C\sum_{j=1}^{l} L'_j$, $i = 1, 2, \ldots, l$ by the constant $l\,L + CL'$ we can obtain

$$\begin{aligned}
& P\left\{\omega : \left(\min_i N_i\right)^{\nu} |\varphi(F,\alpha) - \varphi(F^{\overline{N}},\alpha)| > t\right\} \\
\leq\ & \sum_{i=1}^{l} P\left\{\omega : (N_i)^{\nu} \sup\left|F_i^{N_i}(z_i) - F_i(z_i)\right| > \tfrac{t\vartheta_i}{2l[lL + CL']}\right\} \\
& + P\left\{\omega : Z_{F_i^{N_i}} \not\subset\ Z_{F_i}\left(\tfrac{\sup|F_i(z_i) - F_i^{N_i}(z_i)|}{\vartheta_i}\right) \quad \text{for at least one } i \in \{1,\ldots,l\}\right\} \\
& + P\left\{\omega : \sup\left|F_i(z_i) - F_i^{N_i}(z_i)\right| \geq \vartheta_i\delta_i \quad \text{for at least one } i \in \{1, 2,\ldots,l\}\right\}.
\end{aligned} \tag{35}$$

Evidently, the assertion (32) (under the corresponding assumptions of Theorem 3) follows from the last inequality.

Replacing, furthermore in the relations (34), (35) $(\min_i N_i)^{\nu}|\varphi(F,\alpha) - \varphi(F^{\overline{N}},\alpha)|$ by $(\min_i N_i)^{\nu}\|x(F,\alpha) - x(F^{\overline{N}},\alpha)\|^2$ and employing the corresponding results of Theorem 2 and Theorem 3 we obtain (by the same technique) the validity of the relation (33). □

Theorem 4 deals with arbitrary statistical estimates $F_i^{N_i}(\cdot)$, $N_i = 1, 2, \ldots$ of the one-dimensional marginal distribution functions $F_i(\cdot)$, $i = 1, 2, \ldots, l$. Furthermore we focus our attention on the case when $F_i^{N_i}(\cdot)$, $N_i = 1, 2, \ldots$ are empirical distribution functions.

The investigation of the convergence rate of empirical estimates was started by the papers [12, 29] in the case of recourse problems and independent random samples. The first result was directed to the optimal value estimates. Satisfactory results (on the estimates of the optimal solution) are due to [30]. The original results were furthermore generalized in [11, 14] and [24]. The article [27] deals with the case of complete integer recourse. The results concerning some types of weakly dependent random samples are presented in [16]. In this paper we continue in this last direction. To this end, first, we recall some types of weakly dependent random sequences [3, 31].

Let $\{\zeta^k(\omega)\}_{k=-\infty}^{+\infty}$, $\zeta^k(\omega) = \zeta^k$, $k = \ldots, -1, 0, 1, \ldots$ be a one-dimensional stationary random sequence defined on (Ω, S, P), $\mathcal{B}(-\infty, a)$ be the σ-algebra given by $\ldots, \zeta^{a-1}, \zeta^a$, $\mathcal{B}(b, +\infty)$ given by $\zeta^b, \zeta^{b+1}, \ldots$, $\mathcal{B}(a, b)$ given by $\zeta^a, \ldots, \zeta^b$, $a < b$, a, b integer. Let, furthermore, $\mathcal{B}^m$, $m \geq 1$ be the Borel σ-algebra of the subsets of E_m.

Definition 1. $\{\zeta^k(\omega)\}_{k=-\infty}^{+\infty}$ is an m-dependent random sequence ($m \geq 2$) if there exists a sequence of independent random values $\{\eta^k(\omega)\}_{k=-\infty}^{+\infty}$ defined on (Ω, S, P) and a $\mathcal{B}^m$ measurable function $f(\cdot)$ defined on E_m such that

$$\zeta^k(\omega) = f(\eta^{k-m+1}(\omega), \ldots, \eta^k(\omega)) \quad \text{for every } k = \ldots, -1, 0, 1, \ldots.$$

Definition 2. Let $\{\zeta^k(\omega)\}_{k=-\infty}^{+\infty}$ be a strongly stationary random sequence. We say that $\{\zeta^k(\omega)\}_{k=-\infty}^{+\infty}$ is an absolutely regular random sequence with $\beta(N)$ if

$$\beta(N) = \sup_k \sup_{A \in \mathcal{B}(N+k, +\infty)} |P(A|\,\mathcal{B}(-\infty, k)) - P(A)| \downarrow 0 \quad (N \to \infty).$$

Definition 3. We say that strongly stationary random sequence $\{\zeta^k(\omega)\}_{k=-\infty}^{+\infty}$ fulfils the condition of Φ-mixing if there exists a real-valued function $\Phi(\cdot)$ defined on the set of natural numbers $\mathcal{N}$ such that

$$|P(B_1 \cap B_2) - P(B_1)P(B_2))| \leq \Phi(N)P(B_1),$$

$$B_1 \in \mathcal{B}(-\infty, u),\ B_2 \in \mathcal{B}(u + N, \infty),\ -\infty < u < +\infty,\ N \geq 1,\ u \text{ an integer}.$$

To recall some auxiliary assertions, let $F^{\zeta}(\cdot)$ denote the distribution function of $\zeta(\omega)$, $F^{\zeta, N}(\cdot)$ an empirical distribution function determined by $\{\zeta^k(\omega)\}_{k=1}^{N}$ and E_{F^ζ}, $\mathsf{E}_{F^{\zeta,N}}$ the corresponding operators of the mathematical expectation.

Lemma 2. ([16] Lemma 2.2.) Let $\{\zeta^k(\omega)\}_{k=-\infty}^{+\infty}$ be an m-dependent random sequence, $m \geq 2$. If $\kappa(z)$ is a $\mathcal{B}^1$ measurable function defined on E_1 such that $|\kappa(z)| \leq M$ ($M > 0$) for $z \in E_1$, then it holds for $t > 0$, $t \in E_1$ that

$$P\left\{\omega : |\mathsf{E}_{F^{\zeta,N}}\kappa(\zeta(\omega)) - \mathsf{E}_{F^\zeta}\kappa(\zeta(\omega))| > t\right\}$$
$$\leq 2r \exp\left\{-\frac{N^2}{m^2(k+1)}\frac{t^2}{2M^2}\right\} + 2(m-r)\exp\left\{-\frac{N^2}{m^2 k}\frac{t^2}{2M^2}\right\},$$

where N, k, r are natural numbers such that $N = mk + r$, $r \in \{0, 1, \ldots, m-1\}$.

Lemma 3. ([16] Lemma 2.4.) Let $\{\zeta^k(\omega)\}_{k=-\infty}^{+\infty}$ be an absolutely regular random sequence with $\beta(N)$. If $\kappa(z)$ is a $\mathcal{B}^1$ measurable function defined on E_1 such that $|\kappa(z)| \leq M \, (M > 0)$ for $z \in E_1$, then it holds for every $v \leq N$, v a natural number, $t > 0$, $t \in E_1$, $N = 1, 2, \ldots$ that
$$P\{\omega : |\mathsf{E}_{F^{\zeta, N}}\kappa(\zeta(\omega)) - \mathsf{E}_{F^{\zeta}}\kappa(\zeta(\omega))| > t\} \leq 2v \exp\left\{-\frac{N}{v}\,\frac{N}{N-1+v}\,\frac{t^2}{2M^2}\right\} + 4N\beta(v).$$

Lemma 4. ([16] Lemma 2.6.) Let $\kappa(z)$, $z \in E_1$ be a $\mathcal{B}^1$ measurable function defined on E_1 such that $|\kappa(z)| \leq M \, (M > 0)$ for $z \in E_1$. If $\{\zeta^k(\omega)\}_{k=-\infty}^{+\infty}$ is a random sequence fulfilling the Φ-mixing condition, then it holds for $t \in E_1$, $t > 0$, $N = 1, 2, \ldots$ that
$$P\{\omega : |\mathsf{E}_{F^{\zeta, N}}\kappa(\zeta(\omega)) - \mathsf{E}_{F^{\zeta}}\kappa(\zeta(\omega))| > t\} \leq \frac{2M^2}{t^2N^2}\left[N + \sum_{k=1}^{N-1}(N-k)\Phi(k)\right].$$

Employing the assertions of Lemmas 2, 3 and 4 and the properties of the one-dimensional distribution functions we can obtain the following auxiliary assertion.

Lemma 5. Let $t > 0$ be arbitrary. If the probability measure $P_{\zeta}(\cdot)$ is absolutely continuous with respect to the Lebesgue measure in E_1 and if $\{\zeta^k(\omega)\}_{k=-\infty}^{+\infty}$ is

1. an m-dependent random sequence, $m \geq 2$, then
$$P\{\omega : \sup|F^{\zeta, N}(z) - F^{\zeta}(z)| > t\}$$
$$\leq \tfrac{3}{t}\left\{2r\exp\left\{-\tfrac{N^2}{m^2(k+1)}\,\tfrac{t^2}{18M^2}\right\} + 2(m-r)\exp\{-\tfrac{N^2}{m^2k}\,\tfrac{t^2}{18M^2}\}\right\},$$
where N, k, r are natural numbers such that $N = mk + r$, $r \in \{0, 1, \ldots, m-1\}$,

2. an absolutely regular random sequence with $\beta(N)$, then for every $v \leq N$, v, N natural numbers it holds that
$$P\{\omega : \sup|F^{\zeta, N}(z) - F^{\zeta}(z)| > t\}$$
$$\leq \tfrac{3}{t}\left\{2v\exp\left\{-\tfrac{N}{v}\,\tfrac{N}{N-1+v}\,\tfrac{t^2}{18M^2}\right\} + 4N\beta(v)\right\},$$

3. a random sequence fulfilling the Φ-mixing condition, then for every N natural number it holds that
$$P\{\omega : \sup|F^{\zeta, N}(z) - F^{\zeta}(z)| > t\} \leq \frac{36M^2}{t^3N^2}\left[N + \sum_{k=1}^{N-1}(N-k)\Phi(k)\right].$$

Theorem 5. Let for $i = 1, 2, \ldots, l$, $\alpha_i \in (0, 1)$, $\delta_i > 0$ be given, $\overline{Z}_i = (k_{F_i}(\alpha_i) - 2\delta_i,\, k_{F_i}(\alpha_i) + 2\delta_i)$. If

1. either the assumptions 1, 3 and 4 of Theorem 2 or Theorem 3 are fulfilled,

2. at least one of the following assumptions is fulfilled (simultaneously) for every $i \in \{1, 2, \ldots, l\}$

a) $\{\xi_i^k(\omega)\}_{k=1}^{+\infty}$ is a sequence of independent random vectors, $0 < \nu < \frac{1}{2}$,

b) $\{\xi_i^k(\omega)\}_{k=-\infty}^{+\infty}$ is an m–dependent random sequence $m \geq 2$, $0 < \nu < \frac{1}{2}$,

c) $\{\xi_i^k(\omega)\}_{k=-\infty}^{+\infty}$ is an absolutely regular random sequence with $\beta(N_i)$, $0 < \nu < \frac{1-\gamma}{2}$, $4(N_i)^{1+n\nu}\beta[N_i^\gamma] \to_{N_i \to \infty} 0$ for a $\gamma \in (0, 1)$,

d) $\{\xi_i^k(\omega)\}_{k=-\infty}^{+\infty}$ is a Φ–mixing random sequence such that

$$\limsup \frac{1}{N_i} \sum_{k=1}^{N_i-1} (N_i - k)\Phi(k) < +\infty, \quad 0 < \nu < \frac{1}{3},$$

3. $F_i^{N_i}(z_i)$, $i = 1, \ldots, l$ is one–dimensional empirical distribution function determined by $\{\xi_i^k(\omega)\}_{k=1}^{N_i}$

then for every $t > 0$

$$P\left\{\omega : (\min N_i)^\nu \, |\varphi(F, \alpha) - \varphi(F^{\overline{N}}, \alpha)| > t\right\} \to_{\min(N_i) \to \infty} 0. \tag{36}$$

If moreover the corresponding assumption 5 of Theorem 2 or 3 is fulfilled, then also for every $t > 0$

$$P\left\{\omega : (\min N_i)^\nu \|x(F, \alpha) - x(F^{\overline{N}}, \alpha)\|^2 > t\right\} \to_{\min(N_i) \to \infty} 0. \tag{37}$$

($[x] = k$ iff $k \leq x < k + 1$, k integer.)

Proof. The proof of Theorem 5 follows from Theorem 2 [8], Theorem 4 and Lemma 5. □

6. CONCLUSION

In the paper the stability of the stochastic programming problems with the individual probability constraints was investigated. In particular the main attention was focused on the special cases in which the regions $\mathcal{F}(F, \delta)$ (fulfilling the relation (3)) can be replaced by several subsets (fulfilling the relation (4)) of the one–dimensional marginal distribution functions space. Employing the Kolmogorov metric the achieved results were applied to the empirical estimates of the optimal value and the optimal solution for some types of weakly dependent random samples.

APPENDIX

The aim of this section is to prove Propositions 2, 3 and 4.

Lemma A.1. Let $\delta_1 > 0$, $\alpha_1 \in (0, 1)$, $\varepsilon > 0$ be arbitrary. If $G_1(z_1)$ is an arbitrary one–dimensional distribution function such that

$$G_1(z_1) \in \langle \underline{F}_{1,\delta_1}(z_1), \overline{F}_{1,\delta_1}(z_1)\rangle \quad \text{for} \quad z_1 \in (k_{F_1}(\alpha_1) - \delta_1 - \varepsilon, \, k_{F_1}(\alpha_1) + \delta_1 + \varepsilon),$$

then

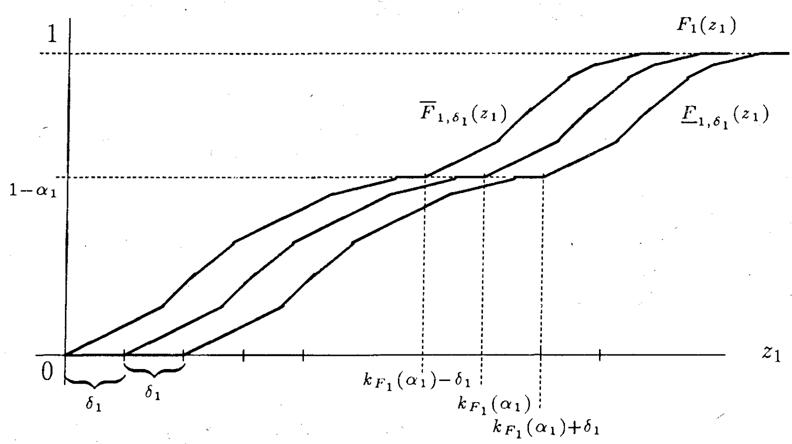
$$|k_{F_1}(\alpha_1) - k_{G_1}(\alpha_1)| \le \delta_1 .$$

P r o o f. Since it follows from the assumptions that

$$G_1(k_{F_1}(\alpha_1) - \delta_1 - \varepsilon') \le F_1(k_{F_1}(\alpha_1)) < G_1(k_{F_1}(\alpha_1) + \delta_1 + \varepsilon')$$

for every $\varepsilon' > 0$, we can see that the assertion of Lemma A.1 holds. □

For a better imagination we present a simple picture.



We recall one well-known assertion that deals with the relationship between the optimal value and the optimal solution.

Lemma A.2. ([19] pp. 54.) Let $K \subset E_n$ be a nonempty, convex set. Further, let $\overline{h}(x)$ be a strongly convex with a parameter $\rho > 0$, continuous function on K. If x_0 is defined by the relation

$$x_0 = \arg\min_{x \in K} \overline{h}(x),$$

then

$$\|x - x_0\|^2 \le \frac{2}{\rho} |\overline{h}(x) - \overline{h}(x_0)| \quad \text{for every } x \in K.$$

P r o o f o f P r o p o s i t i o n 2. Since it follows from the relations (15), (16), Lemma 1 [18] and Lemma A.1 that

$$\Delta[X_F(\alpha), X_G(\alpha)] \le C \sum_{i=1}^{l} \delta_i ,$$

we can (according to the assumption 1) see that the relation (19) holds. Consequently, it remains to prove the second part of the assertion (relation (20)). To this end, first, it holds from the assumptions 2, 4 of Proposition 2 that $X' = \prod_{i=1}^{l} \overline{\mathcal{K}}_i(k_{F_i}(\alpha_i) + \delta_i)$ is a nonempty, compact, convex subset of E_n. Consequently, according to the assumption 4 of Proposition 2 there exists unique $x(X') = \arg\min\{\overline{g}(x) | x \in X'\}$. It follows, successively, from the properties of the Euclidean norm that

$$\|x(F,\alpha) - x(G,\alpha)\|^2 = \|x(F,\alpha) - x(X') + x(X') - x(G,\alpha)\|^2$$
$$= \|x(F,\alpha) - x(X')\|^2 + \|x(X') - x(G,\alpha)\|^2 + 2\langle x(F,\alpha) - x(X') * x(X') - x(G,\alpha)\rangle,$$

and simultaneously

$$\|(x(F,\alpha) - x(X')) - (x(X') - x(G,\alpha))\|^2$$
$$= \|x(F,\alpha) - x(X')\|^2 + \|x(X') - x(G,\alpha))\|^2 - 2\langle x(F,\alpha) - x(X') * x(X') - x(G,\alpha)\rangle,$$

where $\langle \cdot * \cdot \rangle$ denotes the scalar product corresponding to the Euclidean norm in E_n.
Evidently, it follows from the last two relations that

$$\|x(F,\alpha) - x(G,\alpha)\|^2 \le 2\left\{\|x(X') - x(G,\alpha)\|^2 + \|x(X') - x(F,\alpha)\|^2\right\}. \tag{38}$$

It follows from Lemma 1 [18], Lemma A.1 and the relations (6), (14), (15) that $X_F(\alpha)$, $X_G(\alpha)$ are convex sets such that $X_F(\alpha), X_G(\alpha) \subset X'$ and, moreover,

$$\Delta[X_F(\alpha), X'] \le C \sum_{i=1}^{l} \delta_i. \tag{39}$$

Since $X_F(\alpha)$, $X_G(\alpha)$, X' are convex sets employing, moreover, Lemma A.2 and relation (19) we obtain

$$\begin{aligned}
&\|x(G,\alpha) - x(F,\alpha)\|^2 \\
&\le \tfrac{4}{\varrho}\{|\overline{g}(x(X')) - \overline{g}(x(G,\alpha))| + |\overline{g}(x(X')) - \overline{g}(x(F,\alpha))|\} \\
&\le \tfrac{4}{\varrho}\{|\overline{g}(x(G,\alpha)) - \overline{g}(x(F,\alpha))| + |\overline{g}(x(F,\alpha)) - \overline{g}(x(X'))| \\
&\quad + |\overline{g}(x(X')) - \overline{g}(x(F,\alpha))|\} \le \tfrac{12}{\varrho} L' C \sum_{i=1}^{l} \delta_i.
\end{aligned}$$

Evidently, the last system of the inequalities finishes the proof. □

To prove Propositions 3 and 4 we recall the following auxiliary assertion.

Lemma A.3. ([15] Lemma 6.) Let $\delta_1 > 0$, $\varepsilon > 0$ be arbitrary. If

1. $\kappa(z_1)$ is a Lipschitz function on $Z_{F_1}(\delta_1 + \varepsilon)$ with the Lipschitz constant L_κ,
2. there exists a finite $E_{F_1}\kappa(\xi_1(\omega))$,

and if $G_1(z_1)$ is an arbitrary one–dimensional distribution function such that

$$G_1(z_1) \in \langle \underline{F}_{1,\delta_1}(z_1), \overline{F}_{1,\delta_1}(z_1)\rangle \quad \text{for every } z_1 \in E_1,$$

then
$$|\mathsf{E}_{F_1}\kappa(\xi_1(\omega)) - \mathsf{E}_{G_1}\kappa(\xi_1(\omega))| \le L_\kappa \delta_1.$$

Proof of Proposition 3. First, since $X_F(\alpha)$ is a compact set it follows from Lemma A.3 and the assumptions that
$$\left|\inf_{X_F(\alpha)} \mathsf{E}_F g(x, \xi(\omega)) - \inf_{X_F(\alpha)} \mathsf{E}_G g(x, \xi(\omega))\right| \le \sum_{i=1}^{l} \delta_i L_i. \tag{40}$$

According to the fact that $\mathsf{E}_G \sum_{i=1}^{l} g_i(x, \xi_i(\omega))$ is a Lipschitz function on X with the Lipschitz constant $\sum_{i=1}^{l} L'_i$, employing the assertion of Proposition 2, we obtain that
$$\left|\inf_{X_F(\alpha)} \mathsf{E}_G \sum_{i=1}^{l} g_i(x, \xi_i(\omega)) - \inf_{X_G(\alpha)} \mathsf{E}_G \sum_{i=1}^{l} g_i(x, \xi_i(\omega))\right| \le C\left(\sum_{j=1}^{l} L'_j\right) \sum_{i=1}^{l} \delta_i. \tag{41}$$

The validity of the relation (21) follows from the relations (8), (40) and (41). The second part of the assertion (relation (22)) can be proven by the technique employed in the proof of Proposition 2. □

Proof of Proposition 4. First, it follows from the assumptions and from the elementary properties of the integral that
$$\begin{aligned}
&|\mathsf{E}_F g(x, \xi(\omega)) - \mathsf{E}_G g(x, \xi(\omega))| \\
&\le \left| \int_{E_l} g(x, (z_1, z_2, \dots, z_l))\, \mathrm{d}F_1(z_1)\, \mathrm{d}F_2(z_2) \dots \mathrm{d}F_l(z_l) \right. \\
&\quad \left. - \int_{E_l} g(x, (z_1, z_2, \dots, z_l))\, \mathrm{d}G_1(z_1)\, \mathrm{d}F_2(z_2) \dots \mathrm{d}F_l(z_l) \right| \\
&\quad \vdots \\
&\quad + \left| \int_{E_l} g(x, (z_1, z_2, \dots, z_l)) \mathrm{d}G_1(z_1)\, \mathrm{d}G_2(z_2) \dots \mathrm{d}G_{l-1}(z_{l-1}))\, \mathrm{d}F_l(z_l) \right. \\
&\quad \left. - \int_{E_l} g(x, (z_1, z_2, \dots, z_l)) \mathrm{d}G_1(z_1)\, \mathrm{d}G_2(z_2) \dots \mathrm{d}G_{l-1}(z_{l-1}))\, \mathrm{d}G_l(z_l) \right|.
\end{aligned}$$

Moreover, for $i \in \{1, 2, \dots, l\}$, $x \in X$
$$\int_{E_l} g(x, (z_1, z_2, \dots, z_l))\, \mathrm{d}G_1(z_1) \dots \mathrm{d}G_{i-1}(z_{i-1})\, \mathrm{d}F_{i+1}(z_{i+1}) \dots \mathrm{d}F_l(z_l)$$

is a Lipschitz function on $Z_{F_i}(\delta_i)$ with the Lipschitz constant L. Consequently, since $X_F(\alpha)$ is a compact set we obtain (according to Lemma A.3 and the assumptions) that

$$\left| \inf_{X_F(\alpha)} \mathsf{E}_F g(x, \xi(\omega)) - \inf_{X_F(\alpha)} \mathsf{E}_G g(x, \xi(\omega)) \right| \leq lL \sum_{i=1}^{l} \delta_i. \tag{42}$$

Furthermore, it follows from the assumptions that $E_G g(x, \xi(\omega))$ is a Lipschitz function on X with the Lipschitz constant L'. Consequently, employing the first result of Proposition 2 we obtain

$$\left| \inf_{X_F(\alpha)} \mathsf{E}_G g(x, \xi(\omega)) - \inf_{X_G(\alpha)} \mathsf{E}_G g(x, \xi(\omega)) \right| \leq CL' \sum_{i=1}^{l} \delta_i. \tag{43}$$

The first assertion of Proposition 4 follows from the relations (8), (42) and (43). The second part of the assertion can be proven by the technique employed already in the proof of Proposition 2. □

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