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Kybernetika, Vol. 18 (1982), No. 4, 278--289

Persistent URL: http://dml.cz/dmlcz/125379

KYBERNETIKA — VOLUME 18 (1982), NUMBER 4

THE BAYESIAN SEQUENTIAL MODEL WITH THE RANGE - BASED PROBABILITY ESTIMATION*

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The idea of a certain non-typical Bayesian model is presented. The probability estimation based on insufficient statistical data or nonformal experience is a basis for the problem solving procedures in heuristic reasoning. The paper is concerned with some of the properties of *the range-based probability* estimation model in the discrete Bayesian sequential model. We assume that each of hypotheses d_i has the probability p_i, which takes the values from a certain closed interval $\langle a_i; b_i \rangle$. The considerations developed in the paper check the posssibilities of the synthesis of the algorithms in the model.

INTRODUCTION

The statistical data underlying the probability estimation are frequently insufficient, i.e. they are based on the set of events of a too small size. Besides, these data may be associated with errors of various kinds particularly in biological sciences, biomedicine, but also in technology. That is why the quantitative estimation of probability is usually rough and inaccurate; the assumption of a specific real number as a probability value is more often a convention, rather than an exact estimation.

In view of unavailability or the expected low reliability of the data, estimation of probability of the events must be often based on the experience for which a numerical value cannot be ascribed (e.g. on the expert's judgment). Then we try to determine the probability without resorting to any number, for instance using the order relation in the set of the events.

Nevertheless, the probability estimation based on insufficient statistical data or nonformal experience is a basis for the problem solving procedures in heuristic reasoning. This may be used in the computer systems designed to support the heuristic reasoning in decision making, e.g. in medical diagnostics. Thus a serious problem

* This work was supported by Grant 10. 4. 3 of the Institute of Biocybernetics and Biomedical Engineering, Polish Academy of Sciences.

here is the selection of a correct method of probability estimation to obtain possibly the best estimation based on the actually available information. This approach is related to the view that the maximum damage caused by an erroneous decision is usually less serious when inflicted under conditions of a "deliberate" uncertainty than in the case of arbitrary assumed but unverified premises. That is why the concept of the probability estimation with an accuracy to a definite numerical interval is considered. It means that instead of assigning to a probability $P(x)$ a definite real number p simply an interval $\langle a; b\rangle$ is determined so that $p \in \langle a; b\rangle$, i.e. p lies between a and b, neither of these limits being well defined. This concept has been lucidly detailed and discussed in its application for the probability estimation of the diagnostic hypotheses in the study by J. Doroszewski [1].

This paper is concerned with certain properties of *the range-based probability* estimation in the discrete Bayesian sequential process. The aim of these considerations is to show the possible procedure under circumstances when probability can be merely roughly estimated or else it cannot be estimated at all.

The use of this procedure appears to be of interest and advantageous in medical consulting systems based upon numerical estimations as well as upon the medical knowledge and experience. In general the use of the idea may be a helpful aid to obtain higher accuracy of the diagnostic models.

1. GENERAL MODEL. THE REGULAR SYSTEMS

Let us consider a finite set $\boldsymbol{D} = \{d_1, d_2, \ldots, d_n\}$ of events (diagnostic hypotheses) that form a complete system, viz.:

$$(1) \qquad \left(\bigcup_{i=1}^{n} d_i = \xi\right) \wedge \left(\forall_{1 \leqq i \leqq n} \left(i \neq j \Rightarrow d_i \cap d_j = \emptyset\right)\right)$$

where ξ is certain event.

We assume that each of hypotheses d_i has the probability $p_i = p(d_i)$, which takes the values from a certain closed interval $\langle a_i; b_i\rangle$.

We additionally admit that the following conditions are fulfilled:

$$(2) \qquad \text{a)} \quad \forall_{1 \leqq i \leqq n} \left(0 < a_i \leqq b_i < 1\right)$$

$$\text{b)} \quad \left(\sum_{k=1}^{n} a_k \leqq 1\right) \wedge \left(\sum_{k=1}^{n} b_k \geqq 1\right).$$

It follows from conditions (2a) and (2b) that there is at least one sequence of the probability values $p_1, p_2, \ldots, p_n$ such that

$$(3) \qquad p_i \in \langle a_i; b_i\rangle\,, \quad (i = 1, 2, \ldots, n) \quad \text{and} \quad \sum_{i=1}^{n} p_i = 1\,.$$

In a particular case if $\sum_{k=1}^{n} a_k = 1$ or $\sum_{k=1}^{n} b_k = 1$, there is exactly one sequence $p_1, p_2, \ldots, p_n$ of the probability values that fulfils condition (3).

Accordingly, the procedure of assigning specific values to probabilities $p(d_i)$ consists in an arbitrary selection of exactly one number from each interval $\langle a_i; b_i \rangle$, $(i = 1, 2, \ldots, n)$, so as to make $\sum_{i=1}^{n} p_i = 1$ (condition (3)).

Definition 1. For any system of hypotheses that fulfils condition (1) its corresponding sequence of intervals $\langle a_1, b_1 \rangle, \langle a_2; b_2 \rangle, \ldots, \langle a_n; b_n \rangle$ which meets conditions (2), will be referred to as *the basic limiting conditions system* and denoted as

$$\mathbf{S} = \{\langle a_1; b_1 \rangle, \langle a_2; b_2 \rangle, \ldots, \langle a_n; b_n \rangle\}.$$

E.g., for $n = 3$, the basic limiting conditions system could be given thus:

$$0{\cdot}1 \leqq p(d_1) \leqq 0{\cdot}4\,; \quad 0{\cdot}5 \leqq p(d_2) \leqq 0{\cdot}7\,; \quad 0{\cdot}2 \leqq p(d_3) \leqq 0{\cdot}8$$

that is,

(4) $$\mathbf{S} = \{\langle 0{\cdot}1; 0{\cdot}4 \rangle, \langle 0{\cdot}5; 0{\cdot}7 \rangle, \langle 0{\cdot}2; 0{\cdot}8 \rangle\}.$$

It fulfils conditions (1) and (2), and in consequence, condition (3) is met; it can be easily found the presence of an infinite number of possibilities of selecting exactly one p_i number from each of the three intervals so that

$$p_1 + p_2 + p_3 = 1.$$

In the system $\mathbf{S}$ of our example the probability of hypothesis d_3 may assume a value from interval $\langle 0{\cdot}2; 0{\cdot}8 \rangle$. It can be readily seen that if it is selected $p(d_3) < 0{\cdot}4$ the selection of such $p(d_1)$ and $p(d_2)$ values from the remaining two intervals cannot be made to have $p(d_1) + p(d_2) + p(d_3) = 1$. As a result, for $p(d_3) < 0{\cdot}4$ there is no sequence $p(d_1)$, $p(d_2)$, $p(d_3)$ which is a probability distribution function of the hypothesis d_1, d_2, d_3 to be a complete system (1).

Definition 2. Let $\mathbf{S} = \{\langle a_1; b_1 \rangle, \langle _2; b_2 \rangle, \ldots, \langle a_n; b_n \rangle\}$ be a basic limiting conditions system.

System $\mathbf{S}$ will be referred to as *regular* and denoted by $\mathbf{S}^R$, if for any interval $\langle a_i; b_i \rangle$ and for any $p_i \in \langle a_i; b_i \rangle$ there is such a sequence $p_1, \ldots, p_{i-1}, p_{i+1}, \ldots, p_n$ (not containing p_i) that

(5) $$p_i = 1 - \sum_{\substack{k=1 \\ k \neq i}}^{n} p_k$$

which can be written as

$$(\mathbf{S} = \mathbf{S}^R) \Leftrightarrow \underset{i \in \{1,\ldots,n\}}{\forall} \; \underset{p_i \in \langle a_i; b_i \rangle}{\forall} \; \underset{(p_1,\ldots,p_{i-1},p_{i+1},\ldots,p_n)}{\exists}$$
$$((p_k \in \langle a_k; b_k \rangle,\ k = 1, \ldots, i-1, i+1, \ldots, n) \wedge (p_i = 1 - \sum_{\substack{k=1 \\ k \neq i}}^{n} p_k)).$$

Definition 3. Let the basic limiting conditions systems $\mathbf{S}_1$ and $\mathbf{S}_2$ be given

$$\mathbf{S}_1 = \{\langle a_{11}; b_{11}\rangle, \langle a_{12}; b_{12}\rangle, \ldots, \langle a_{1n}; b_{1n}\rangle\} ;$$
$$\mathbf{S}_2 = \{\langle a_{21}; b_{21}\rangle, \langle a_{22}; b_{22}\rangle, \ldots, \langle a_{2n}; b_{2n}\rangle\} .$$

We will say that $\mathbf{S}_1$ *is contained in* $\mathbf{S}_2$, using the notation

$$\mathbf{S}_1 \subset \mathbf{S}_2$$

if for every $i = 1, 2, \ldots, n$, condition

$$\langle a_{1i}; b_{1i}\rangle \subset \langle a_{2i}; b_{2i}\rangle \tag{6}$$

is fulfilled, what can be written in short as:

$$\mathbf{S}_1 \subset \mathbf{S}_2 \Leftrightarrow \underset{1 \leqq i \leqq n}{\forall} \left(\langle a_{1i}; b_{1i}\rangle \subset \langle a_{2i}; b_{2i}\rangle\right).$$

Definition 4. We will say an n-element sequence $\Pi = \{p_1, p_2, \ldots, p_n\}$ is *contained* in $\mathbf{S} = \{\langle a_1; b_1\rangle, \langle a_2; b_2\rangle, \ldots, \langle a_n; b_n\rangle\}$ and write $\Pi[\mathbf{S}]$

if
$$\underset{1 \leqq i \leqq n}{\forall} \left(p_i \in \langle a_i; b_i\rangle\right).$$

Theorem 1. For any basic limiting conditions system $\mathbf{S} = \{\langle a_1; b_1\rangle, \langle a_2; b_2\rangle, \ldots$ $\ldots, \langle a_n; b_n\rangle\}$ there exists at least one regular system $\mathbf{S}^R$ contained in $\mathbf{S}$.

Proof. For every basic limiting conditions system $\mathbf{S}$ there must be fulfilled condition (2)

$$\left(\sum_{k=1}^{n} a_k \leqq 1\right) \wedge \left(\sum_{k=1}^{n} b_k \geqq 1\right)$$

which warrants an existance if at least one $\Pi[\mathbf{S}'] = \{p_1, \ldots, p_n\}$ such that $\mathbf{S}' =$ $= \{\langle p_1, p_1\rangle, \ldots, \langle p_n; p_n\rangle\} \subset \mathbf{S}$ and $\sum_{k=1}^{n} p_k = 1$. According to Definition 2 $\mathbf{S}' = \mathbf{S}^R$ which completes the proof. □

Theorem 2. In the set of all regular systems contained in $\mathbf{S} = \{\langle a_1; b_1\rangle, \langle a_2; b_2\rangle, \ldots$ $\ldots, \langle a_n; b_n\rangle\}$ we have

$$\underset{\mathbf{S}^R_{\max} \subset \mathbf{S}}{\exists} \; \underset{\mathbf{S}^R \subset \mathbf{S}}{\forall} \left(\mathbf{S}^R \subset \mathbf{S}^R_{\max}\right) \tag{7}$$

where

$$\mathbf{S}^R_{\max} = \{\langle a'_1; b'_1\rangle, \ldots, \langle a'_i; b'_i\rangle \ldots \langle a'_n; b'_n\rangle\}$$

and

$$a'_i = \max\left(a_i; 1 - \sum_{\substack{k=1 \\ k \neq i}}^{n} b_k\right)$$

$$b'_i = \min\left(b_i; 1 - \sum_{\substack{k=1 \\ k \neq i}}^{n} a_k\right); \quad (i = 1, 2, \ldots, n). \tag{8}$$

Proof. Condition (7) follows immediately from Theorem 1. Equations (8) may be obtained from properties of the systems of algebraic linear equations. □

Conclusions from Theorem 2:

1^0 in general there exists an infinite number of systems $\mathbf{S}$ which possess a common set $\mathbf{S}^{R}_{\max}$;

2^0 there exists an algorithm such that for any $\mathbf{S}$ its corresponding set $\mathbf{S}^{R}_{\max}$ can be efficiently found;

3°
$$(\mathbf{S} = \mathbf{S}^R) \Rightarrow \underset{1 \leq i \leq n}{\forall} \left(\left(a_i \geqq 1 - \sum_{\substack{k=1 \\ k \neq i}}^{n} b_k \right) \wedge \left(b_i \leqq 1 - \sum_{\substack{k=1 \\ k \neq i}}^{n} a_k \right) \right).$$

The notions of the regular systems $\mathbf{S}^R$ and $\mathbf{S}^{R}_{\max}$ are probabilistic meaningfull. As demonstrated previously, in certain intervals $\langle a_i; b_i \rangle$ such $p(d_i)$ values can exist that the selection of such $p(d_k)$ values from the remaining $n - 1$ intervals to satisfy

$$p(d_i) + \sum_{\substack{k=1 \\ k \neq i}}^{n} p(d_k) = 1$$

becomes unfeasible. This, however, would contradict the fundamental assumption that we consider exclusively the complete system of random events $d_1, d_2, \ldots, d_n$. For that reason we will be concerned in our discussion merely with regular limiting conditions system $\mathbf{S}^R$, which guarantees the preservation of the probabilistic sense of the model advanced.

On the other hand, we have by virtue of Theorem 2 a simple procedure to resolve any $\mathbf{S}$ into $\mathbf{S}^{R}_{\max}$. The pattern of this procedure could run as follows:

for $i := 1$ **step** 1 **until** n **do**

begin

$a'[i] :=$ **if** $a[i] \geqq (1 - \sum_{\substack{k=1 \\ k \neq i}}^{n} b[k])$ **then** $a[i]$ **else** $1 - \sum_{\substack{k=1 \\ k \neq i}}^{n} b[k]$;

$b'[i] :=$ **if** $b[i] \leqq (1 - \sum_{\substack{k=1 \\ k \neq i}}^{n} a[k])$ **then** $b[i]$ **else** $1 - \sum_{\substack{k=1 \\ k \neq i}}^{n} a[k]$;

end

This is evident that the procedure produces $\mathbf{S}^{R}_{\max}$. Its conditional instruction $a'[i]$, $b'[i]$ just entirely realise thesis (8) of the Theorem 2.

Every iteration of the procedure computes single pair $a'[i]$, $b'[i]$ independently of the other $a'[k]$, $b'[k], \ldots$, so that $\mathbf{S}^{R}_{\max}$ may be produced for any order of the intervals within the system $\mathbf{S}$.

From conclusion 3° also a simple procedure follows to verify whether or not the basic limiting conditions system is simultaneously a regular one.

Then $\mathbf{S} = \mathbf{S}^{R}_{\max}$, that is $\underset{1 \leq i \leq n}{\forall} ((a_i = a'_i) \wedge (b_i = b'_i))$.

2. THE BAYESIAN SEQUENTIAL PROCESS

Subsequently, we will examine some of the properties of the Bayesian sequential process in which the distributions of the preliminary probability $p(d_1), p(d_2), \ldots, p(d_n)$, likelihoods $p(s_j \mid d_1)$, $p(s_j \mid d_2), \ldots, p(s_j \mid d_n)$, the posterior probability $p'(d_1)$, $p'(d_2), \ldots, p'(d_n)$ are given with an accuracy to the intervals of the corresponding basic limiting conditions systems.

We will denote

1. $\mathbf{X} = \{\langle a_1; b_1\rangle, \langle a_2; b_2\rangle, \ldots, \langle a_n; b_n\rangle\}$
   – the basic limiting conditions system for the preliminary probability distribution, where

   $$x_i = p(d_i); \quad x_i \in \langle a_i; b_i\rangle, \quad (i = 1, 2, \ldots, n).$$

2. $\mathbf{V}_j = \{\langle a_{1j}; \beta_{1j}\rangle, \langle \alpha_{2j}; \beta_{2j}\rangle, \ldots, \langle \alpha_{nj}; \beta_{nj}\rangle\}$
   – the basic limiting conditions system for the likelihood distribution, where

   $$p(s_j \mid d_i) = v_{ij}, \quad v_{ij} \in \langle \alpha_{ij}; \beta_{ij}\rangle, \quad (j = 1, 2, \ldots, m), \quad (i = 1, 2, \ldots, n)$$

   and $\Sigma = \{s_1, s_2, \ldots, s_m\}$ is a set of events, referred to hereafter symptoms.

3. $\mathbf{Y} = \{\langle \bar{a}_1; \bar{b}_1\rangle, \langle \bar{a}_2; \bar{b}_2\rangle, \ldots, \langle \bar{a}_n; \bar{b}_n\rangle\}$
   – the basic limiting conditions system for the posterior probability distribution, where

   $$y_i = p'(d_i), \quad y_i \in \langle \bar{a}_i; \bar{b}_i\rangle \quad (i = 1, 2, \ldots, n).$$

4. $$x[\mathbf{X}] = \{x_1, x_2, \ldots, x_n\}, \quad \sum_{i=i}^{n} x_i = 1,$$

   $$y[\mathbf{Y}] = \{y_1, y_2, \ldots, y_n\}, \quad \sum_{i=1}^{n} y_i = 1,$$

   $$v[\mathbf{V}_j] = \{v_{1j}, v_{2j}, \ldots, v_{nj}\}, \quad (j = 1, 2, \ldots, m).$$

   where the sequence of elements represents the values chosen from the successive intervals of the corresponding limiting conditions system $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{V}_j$.

Definition 5. We shall say that the Bayes formula *transforms* a system $\mathbf{X}$ in system $\mathbf{Y}$ if a symptom s_j is observed, and write

$$\mathscr{B}(\mathbf{X}; \mathbf{V}_j) \to \mathbf{Y}$$

if

$$\forall_{x[X]} \forall_{v[V_j]} \left(\left\{ y_i : y_i = \frac{x_i v_{ij}}{\sum_{k=1}^{n} x_k v_{kj}} \right\} \Rightarrow y[Y] \right), \quad i = 1, \ldots, n; \quad j = 1, \ldots, m.$$

The process in point is a sequence of transformations (as specified by Definition 5) of the successive limiting systems, starting from an $\mathbf{X}^0$ defined as follows:

1° We consider the sets of hypotheses $D = \{d_1, \ldots, d_n\}$ and symptoms $\Sigma = \{s_1, \ldots, s_m\}$, where card $(D) = n$, card $(\Sigma) = m$.

2° Transformations occur in successive steps $r = 1, 2, \ldots, m$ and in each step one transformation is done.

3° For $r = 1, 2, \ldots, m$ we have

$$\begin{aligned} \mathbf{Y}^1 &= \mathscr{B}(\mathbf{X}^0; \mathbf{V}_j^1)\,; \\ \mathbf{Y}^r &= \mathscr{B}(\mathbf{X}^r; \mathbf{V}_k^r)\,; \\ \mathbf{X}^{r+1} &= \mathbf{Y}^r\,; \quad j, k = 1, 2, \ldots, m\,; \quad j \neq k \end{aligned}$$

that is, the posterior probability distribution limiting system $\mathbf{Y}^r$ becomes the corresponding preliminary probability distribution system in the next step beginning from an $\mathbf{X}^0$.

4° If $s^1, s^2, \ldots, s^r$ constitutes a sequence of symptoms under consideration, then

$$\forall_{1 \leq i,j \leq r} (i \neq j \Rightarrow s^i \neq s^j)$$

which simultaneously results in a restricted sequence length to at most m steps.

Let be given sets $D = \{d_1, \ldots, d_n\}$; $\Sigma = \{s_1, \ldots, s_m\}$ with their corresponding limiting systems

$$\mathbf{X}^0 = \{\langle a_1; b_1 \rangle, \langle a_2; b_2 \rangle, \ldots, \langle a_n; b_n \rangle\}$$

and

$$\begin{aligned} \mathbf{V}_1 &= \{\langle \alpha_{11}; \beta_{11} \rangle, \langle \alpha_{21}; \beta_{21} \rangle, \ldots, \langle \alpha_{n1}; \beta_{n1} \rangle\} \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ \mathbf{V}_m &= \{\langle \alpha_{1m}; \beta_{1m} \rangle, \langle \alpha_{2m}; \beta_{2m} \rangle, \ldots, \langle \alpha_{nm}; \beta_{nm} \rangle\} \end{aligned}$$

We shall examine whether the choice of the order of the symptom sequence (any permutation $s^1, s^2, \ldots, s^r$ of elements of the set Σ) affects the result of the transformation of the limiting system $\mathbf{X}^0$ into the limiting system $\mathbf{Y}^r$ after r steps of the process

$$\mathbf{X}^0, \mathbf{Y}^1, \mathbf{Y}^2, \ldots, \mathbf{Y}^r \tag{9}$$

considered under different preliminary assumptions denoted henceforward by A, B, and C.

A. We assume that

$$\forall_{1 \leq i \leq n} ((a_i = b_i = x_i) \wedge (\sum_{i=1}^{n} x_i = 1)) \tag{A1}$$

and

$$\forall_{1 \leq i \leq n} \forall_{1 \leq j \leq m} (\alpha_{ij} = \beta_{ij} = v_{ij})\,. \tag{A2}$$

This means that we have to deal with a certain preliminary distribution system $x[\mathbf{X}]$ and with a set of the likelihood distributions $v[\mathbf{V}_j]$, $(j = 1, 2, \ldots, m)$, as a consequence of the limitation of intervals $\langle a_i; b_i \rangle$ and $\langle \alpha_{ij}; \beta_{ij} \rangle$ to single points

x_i and v_{ij}. Thus, the sequence $x[\mathbf{X}^0]$ is by assumption a regular limiting conditions system.

Lemma 1. If card $(\Sigma) = m$ and conditions (A1) and (A2) are satisfied, for any of the m symptom sequences, the posterior probability distribution $\mathbf{Y}^m$ in the process (9) are given by expression

$$y_i = \frac{x_i \prod_{r=1}^{m} v_{ij}^r}{\sum_{k=1}^{n} x_k \prod_{r=1}^{m} v_{kj}^r}, \quad j = 1, 2, \ldots, m, \quad i = 1, 2, \ldots, n.$$

Proof. The lemma can be easily proved by induction.

Corollary. For conditions (A1) and (A2) the final distribution is independent of order in the sequence of the symptoms observed.

B. We assume that

(B1) $$\forall_{1 \leqq i \leqq n} \forall_{1 \leqq j \leqq m} ((\alpha_{ij} = \beta_{ij} = v_{ij}) \wedge (v_{ij} \neq 0))$$

and

(B2) $$\mathbf{X}^0 = \{\langle a_1; b_1 \rangle; \langle a_2; b_2 \rangle, \ldots, \langle a_n; b_n \rangle\}$$

is a certain regular limiting system of the preliminary probability distribution.

Now, we are going to seek the limiting system of the posterior probability distribution, as a result of the process (9).

Lemma 2. If card $(\Sigma) = m$, and conditions (B1) and (B2) are satisfied, there exists an effective algorithm for the calculation of the limiting system $\mathbf{Y}^m$ and its corresponding regular system $\mathbf{Y}^{m'} \subset \mathbf{Y}^m$.

Proof. From condition (B2) we have

$$a_i \leqq x_i \leqq b_i; \quad a_i v_{ij}^r \leqq x_i v_{ij}^r \leqq b_i v_{ij}^r; \quad (i = 1, 2, \ldots, n), \quad (j = 1, 2, \ldots, m)$$

after m steps

(10) $$\sum_{\substack{k=1 \\ k \neq i}}^{n} a_k \prod_{r=1}^{m} v_{kj}^r \leqq \sum_{\substack{k=1 \\ k \neq i}}^{n} x_k \prod_{r=1}^{m} v_{kj}^r \leqq \sum_{\substack{k=1 \\ k \neq i}}^{n} b_k \prod_{r=1}^{m} v_{kj}^r$$

and

(11) $$\frac{1}{b_i \prod_{r=1}^{m} v_{ij}^r} \leqq \frac{1}{x_i \prod_{r=1}^{m} v_{ij}^r} \leqq \frac{1}{a_i \prod_{r=1}^{m} v_{ij}^r}; \quad (i = 1, 2, \ldots, n).$$

From (10) and (11) we have

$$1 + \frac{1}{b_i \prod_{r=1}^{m} v_{ij}^r} \sum_{\substack{k=1 \\ k \neq i}}^{n} a_k \prod_{r=1}^{m} v_{kj}^r \leqq 1 + \frac{1}{x_i \prod_{r=1}^{m} v_{ij}^r} \sum_{\substack{k=1 \\ k \neq i}}^{n} x_k \prod_{r=1}^{m} v_{kj}^r \leqq 1 + \frac{1}{a_i \prod_{r=1}^{m} v_{ij}^r} \sum_{\substack{k=1 \\ k \neq i}}^{n} b_k \prod_{r=1}^{m} v_{kj}^r.$$

It is easy to show that

$$1 + \frac{1}{x_i \prod_{r=1}^{m} v_{ij}^r} \sum_{\substack{k=1 \\ k \neq i}}^{n} x_k \prod_{r=1}^{m} v_{kj}^r = \frac{1}{y_i}$$

where y_i was formed in Lemma 1.

Now, after simple transformation we have:

$$\bar{a}_i = \frac{a_i \prod_{r=1}^{m} v_{ij}^r}{a_i \prod_{r=1}^{m} v_{ij}^r + \sum_{\substack{k=1 \\ k \neq i}}^{n} b_k \prod_{r=1}^{m} v_{kj}^r};$$

$$\bar{b}_i = \frac{b_i \prod_{r=1}^{m} v_{ij}^r}{b_i \prod_{r=1}^{m} v_{ij}^r + \sum_{\substack{k=1 \\ k \neq i}}^{n} a_k \prod_{r=1}^{m} v_{kj}^r}; \quad (i = 1, 2, \ldots, n).$$

It is quite easy to prove that for the limiting conditions system $\mathbf{Y}^m = \{\langle \bar{a}_1, \bar{b}_1 \rangle, \ldots \ldots, \langle \bar{a}_n; \bar{b}_n \rangle\}$ conditions (1) and (2) are satisfied, what means that $\mathbf{Y}^m$ is a basic limiting conditions system (Definition 2) and by virtue of Theorem 1 there exists corresponding $\mathbf{Y}^{m'} = \{\langle \bar{a}'_1; \bar{b}'_1 \rangle, \ldots, \langle \bar{a}'_n; \bar{b}'_n \rangle\}$ such that $\mathbf{Y}^{m'} \subset \mathbf{Y}^m$ what ends the proof. □

Corollary. For any set of events (symptoms) $\Sigma = \{s_1, s_2, \ldots, s_m\}$ the order in which they appear in the process (9) has no effect on the final posterior probability limiting conditions system.

C. We assume that

(C1) $\quad \mathbf{V}_j = \{\langle \alpha_{1j}; \beta_{1j} \rangle, \langle \alpha_{2j}; \beta_{2j} \rangle, \ldots, \langle \alpha_{nj}; \beta_{nj} \rangle\}$

and

(C2) $\quad \mathbf{X}^0 = \{\langle a_1; b_1 \rangle, \langle a_2; b_2 \rangle, \ldots, \langle a_n; b_n \rangle\}$.

As previously, $v[\mathbf{V}_j]$ denotes any sequence $v_{1j}, v_{2j}, \ldots, v_{nj}$ of the likelihoods $p(s_j \mid d_i)$, where each element is chosen from the corresponding interval $\langle \alpha_{ij}; \beta_{ij} \rangle$ being the element of system $\mathbf{V}_j$.

Theorem 3. If card $(\Sigma) = m$ and condition (C1) and (C2) are satisfied, the posterior probability limiting conditions system $\mathbf{Y}^m = \{\langle A_1; B_1 \rangle, \ldots, \langle A_n; B_n \rangle\}$ is evaluated for any symptoms sequence by means of expressions:

$$(12) \qquad A_i = \frac{a_i \prod_{r=1}^{m} \alpha_{ij}^r}{a_i \prod_{r=1}^{m} \alpha_{ij}^r + \sum_{\substack{k=1 \\ k \neq i}}^{n} b_k \prod_{r=1}^{m} \beta_{kj}^r}; \qquad B_i = \frac{b_i \prod_{r=1}^{m} \beta_{ij}^r}{b_i \prod_{r=1}^{m} \beta_{ij}^r + \sum_{\substack{k=1 \\ k \neq i}}^{n} a_k \prod_{r=1}^{m} \alpha_{kj}^r}.$$

Proof. From (C1) and (C2) we have:

$$a_i \leqq x_i \leqq b_i\,; \quad a_i\alpha_{ij}^r \leqq x_i v_{ij} \leqq b_i\beta_{ij}^r.$$

Further reasoning follows similarly to that applied in the proof of Lemma 2. □

Conclusion. For any given $\mathbf{X}^0$, $\mathbf{V}_j$, $(j = 1, \ldots, m)$ there exists exactly one posterior probability limiting conditions system $\mathbf{Y}^m$ independently of ordering in the sequence of symptoms $s_1, \ldots, s_m$.

The considerations developed up to this point allow to form the following theorem of practical significance.

Theorem 4. If $\mathbf{X}^0$ satisfies conditions (1), (2), the posterior probability limiting conditions system

$$\mathbf{Y}^m = \{\langle A_1; B_1\rangle, \ldots, \langle A_n; B_n\rangle\}$$

as the result of the process (9) is a regular limiting conditions system.

Proof. From conclusion 3° of Theorem 2 we have necessary and sufficient condition for any limiting conditions system be a regular system in the meaning of Definition 2. It should therefore be verified, if the following statement is true:

$$(13) \qquad \forall_{1 \leqq i \leqq n} \Big(\big(A_i \geqq 1 - \sum_{\substack{k=1\\k \neq i}}^{n} B_k\big) \wedge \big(B_i \leqq 1 - \sum_{\substack{k=1\\k \neq i}}^{n} A_k\big)\Big).$$

We shall examine for this purpose successively the validity of both the terms of conjugation (13), substituting suitable expressions for A_i and B_i from (12)

$$b_k \prod_{r=1}^{m} \beta_{kj}^r + \sum_{\substack{t=1\\t \neq k}}^{n} a_t \prod_{r=1}^{n} \alpha_{tj}^r = b_k \prod_{r=1}^{m} \beta_{kj}^r + a_i \prod_{r=1}^{m} \alpha_{ij}^r + \sum_{\substack{t=1\\t \neq i,k}}^{n} a_t \prod_{r=1}^{m} \alpha_{tj}^r\,;$$

$$a_k \prod_{r=1}^{m} \alpha_{kj}^r + \sum_{\substack{t=1\\t \neq k}}^{n} b_t \prod_{r=1}^{m} \beta_{tj}^r = a_k \prod_{r=1}^{m} \alpha_{kj}^r + b_i \prod_{r=1}^{m} \beta_{ij}^r + \sum_{\substack{t=1\\t \neq k,i}}^{n} b_t \prod_{r=1}^{m} \beta_{tj}^r$$

and the inequality

$$b_k \prod_{r=1}^{m} \beta_{kj}^r + a_i \prod_{r=1}^{m} \alpha_{ij}^r + \sum_{\substack{t=1\\t \neq i,k}}^{n} a_t \prod_{r=1}^{m} \alpha_{tj}^r \leqq$$

$$\leqq b_k \prod_{r=1}^{m} \beta_{kj}^r + a_i \prod_{r=1}^{m} \alpha_{ij}^r + \sum_{\substack{t=1\\t \neq i,k}}^{n} b_t \prod_{r=1}^{m} \beta_{tj}^r.$$

Making use (12) and (13) we obtain

$$A_i + \sum_{\substack{k=1\\k \neq i}}^{n} B_k \geqq \frac{a_i \prod_{r=1}^{m} \alpha_{ij}^r}{a_i \prod_{r=1}^{m} \alpha_{ij}^r + \sum_{\substack{k=1\\k \neq i}}^{n} b_k \prod_{r=1}^{m} \beta_{kj}^r} + \sum_{\substack{k=1\\k \neq i}}^{n} \frac{b_k \prod_{r=1}^{m} \beta_{kj}^r}{a_i \prod_{r=1}^{m} \alpha_{ij}^r + \sum_{\substack{t=1\\t \neq i}}^{n} b_t \prod_{r=1}^{m} \beta_{tj}^r} = 1.$$

Analogously, we demonstrate the validity of the second term of conjugation (13). This completes the proof. □

The property contained in Theorem 4 is significant for the probabilistic sense of the results obtained on any steps of the process (9), where A_i, B_i are defined as in (12).

To complement the treatment of the process (9) properties, one more theorem will be stated.

Theorem 5. If $\mathbf{X}$ is a regular limiting conditions system of the preliminary probabilities and

1° $$\forall_{1 \leqq j \leqq m} \forall_{1 \leqq i \leqq n} ((\alpha_{ij} = \alpha_i) \wedge (\beta_{ij} = \beta_i))$$

2° $$\forall_{1 \leqq i \leqq n} ((\alpha_i \leqq \min_{\substack{1 \leqq k \leqq n \\ k \neq i}} (\beta_k)) \wedge (\beta_i \geqq \max_{\substack{1 \leqq k \leqq n \\ k \neq i}} (\alpha_k)))$$

then

$$\mathbf{X} \subset \mathbf{Y}^m .$$

Proof. According to Definition 3 we ought to check and prove

$$\forall_{1 \leqq i \leqq n} ((a_i - A_i \geqq 0) \wedge (b_i - B_i \leqq 0)). \tag{14}$$

From the assumptions and expressions (12) the following relation has to be shown

$$a_i - \frac{a_i(\alpha_i)^m}{a_i(\alpha_i)^m + \sum\limits_{\substack{k=1 \\ k \neq i}}^{n} b_k(\beta_k)^m} \geqq 0 .$$

After some transformations we have the above condition reduced to

$$\sum_{\substack{k=1 \\ k \neq i}}^{n} b_k(\beta_k)^m - (\alpha_i)^m (1 - a_i) \geqq 0 .$$

Using regularity of system $\mathbf{X}$ and conclusion 3° from Theorem 2

$$\sum_{\substack{k=1 \\ k \neq i}}^{n} b_k(\beta_k)^m - (\alpha_i)^m (1 - a_i) \geqq \sum_{\substack{k=1 \\ k \neq i}}^{n} b_k(\beta_k)^m - (\alpha_i)^m \sum_{\substack{k=1 \\ k \neq i}}^{n} b_k \geqq 0 .$$

The reasoning to prove $b_i - B_i \geqq 0$ follows similarly to this shown above. □

As a conclusion of this theorem we could form the following

Lemma. If $\mathbf{X}$ is a regular limiting conditions system of preliminary probabilities and

$$\forall_{1 \leqq j \leqq m} \forall_{1 \leqq i \leqq n} ((\alpha_{ij} = \alpha) \wedge (\beta_{ij} = \beta))$$

then

$$\mathbf{X} \subset \mathbf{Y}^m .$$

The proof of the lemma is quite simple and follows the same way as in the Theorem 5.

(Received November 21, 1980.)

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