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Synthesis of Time-Optimal Control for Linear Systems and the Minimal-Time Lyapunoff Function

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The minimal-time control to the origin is considered. It is shown that for a certain class of problems the minimal-time function, if it exists, completely determines the performance of a system. The intent here is to demonstrate that the existence of the minimal-time function guarantees asymptotic stability, and it is a Lyapunoff function. The minimal-time control problem is associated with reachability set. The main result here, for the problem solved, leads to the determination of the reachability set and the construction of the switch curve.

1. INTRODUCTION

It is well known that the gradient of the minimal-time function can be used in place of the adjoint response in synthesising the optimal control.

Here we shall discuss the time-optimal control problem in a manner suggested by Lee and Markus [1, pp. 100]. We shall associate our problem with reachability set, and show that for certain class of problems the minimal-time function, if it exists, completely determines the performance of a system.

The class of problems are those in which the system can be represented by an idealisation which may not be controllable. It may happen that the controllability space of the actual system is limited to a certain region of the state space. Then we can always restrict the actual system to this region of the state space, since there is always a stable control law in this region that will steer the system to the origin in an optimal fashion. The existence of the minimal time function guarantees asymptotic stability of such systems, and it is a Lyapunoff function in this region of the state space.

The minimal-time control problem as presented here leads to the determination of the reachability set and the construction of the switch curve.

Consider the linear dynamical system in R^n,

$$\dot{x} = Ax + Bu \tag{1}$$

with *initial data* $x(0) = x_0$, x is the *state* of the system, and the controls $u(t) \in \Omega \subset \subset R^m$ are restricted by $|u_i| \leqq 1$. The well-known solution of (1) for an *admissible control* $u(t) \in \Omega$ is given by

$$x(t, u) = X(t)\, x_0 + X(t) \int_0^t X^{-1}(\tau)\, B\, u(\tau)\, \mathrm{d}\tau$$

where $X(t) = \mathrm{e}^{At}$ *is the fundamental matrix solution of the homogeneous system* $\dot{x}(t) = A\, x(t)$, *and* $X(0) = I$ *is the identity matrix.*

Further, consider the change of coordinates defined by

$$x(t, u) = X(t)\, y(t, u) = \mathrm{e}^{At}\, y(t, u)$$

with

$$x(0, u) = X(0)\, y(0, u) = y(0, u)\,.$$

Then an *equivalent system* to (1) is

$$\dot{y}(t, u) = Y(t)\, u(t) \tag{2}$$

where $Y(t) = \mathrm{e}^{-At}B$. Thus $y(t; u)$ is the solution of (2) satisfying $y(0; u) = x(0; u)$.

The *reachability set* is the set of all initial states from which it is possible by *admissible controls* to go to the origin in time t and is defined as

$$\mathscr{R}(t) = \left\{ \int_0^t \mathrm{e}^{-A\tau} B\, u(\tau)\, \mathrm{d}\tau : |u_i| \leqq 1,\ u \in \Omega_{\langle 0, t \rangle} \right\}.$$

The set of states that can be reached from the origin in time t is given by $\mathrm{e}^{At}\, \mathscr{R}(t)$. Thus the *reachable set* is

$$\mathscr{R} = \cup \{\mathscr{R}(t) : t \geqq 0\}\,.$$

It has been established [2, pp. 46] that $\mathscr{R}(t)$ is *compact, convex, symmetric* about the origin and satisfies the inclusion relations

$$\mathscr{R}(\tau) \subset \mathscr{R}(t)\,, \quad 0 \leqq \tau \leqq t\,.$$

Let $\operatorname{Int} \mathscr{R}(t)$ and $\partial \mathscr{R}(t)$ denote the *interior* and *boundary* of $\mathscr{R}(t)$ respectively. If (1) is controllable, and $\mathscr{R}$ is open then

$$\mathscr{R}(\tau) \subset \operatorname{Int} \mathscr{R}(t)\,, \quad 0 \leqq \tau < t$$

 obviously,

$$x \in \partial\mathscr{R}(t) .$$

Consider the case where the controls u_i are not restricted by any preassigned bound, then we obtained the controllability space

$$\mathscr{E}(t) = \left\{ \int_0^t e^{-A\tau} B\, u(\tau)\, d\tau : u \in \Omega_{\langle 0, t \rangle} \right\} ,$$

$$\mathscr{E} = \cup\{\mathscr{E}(t) : t \geqq 0\}$$

which are linear subspaces of R^n.

Remark 1. From the above we see that the geometry of the set of reachability $\mathscr{R}(t)$ does not depend on the initial state x_0, except for the location of this set in R^n. For autonomous linear system only the difference $t - t_0$ is important and t_0 is usually taken as zero.

In all that follows we shall be concerned with positive time solutions. Thus reaching the origin in minimal time corresponds to $x \in \mathscr{R}(t)$. Let $T(x)$ be the minimal time for steering x to the origin. Then,

$$T(x) > 0 \quad \text{if} \quad x \neq 0 ,$$

$$T(x) = 0 \quad \text{if} \quad x = 0 .$$

The minimal-time function $T(x)$ is given [1, pp. 145] by

$$T(x) = \inf\{t \geqq 0 : x \in \mathscr{R}(t)\} .$$

Obviously, $0 \leqq T \leqq +\infty$ with $T(x) < +\infty$ if and only if $x \in \mathscr{R}(t)$. From the maximal principle we deduce that

$$\max\{\langle -\operatorname{grad} T(x), Ax + Bu\rangle : u \in \Omega, |u_i| \leqq 1\} = 1$$

provided that the gradient of $T(x)$ exists almost everywhere in $\mathscr{R}$ [1, pp. 146]. Then the optimal feedback control is given by

$$U(x) = -\operatorname{sgn}(B^T \operatorname{grad} T(x)) .$$

Lemma 1. *Let $\mathscr{E}$ be a unique linear controllability space in R^n for*

(S) $$\dot{x} = Ax + Bu .$$

Then there exist coordinates $x = \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \end{pmatrix}$ *in* R^n [1, pp. 99] *such that* $\bar{x}_2 = 0$ *in* $\mathscr{E}$

and (S) *can be written as*

$$\dot{\bar{x}}_1 = A_{11}\bar{x}_1 + A_{12}\bar{x}_2 + B_1 u\,,$$
$$\dot{\bar{x}}_2 = A_{22}\bar{x}_2$$

where

$$\bar{x}_1 = \begin{pmatrix} \bar{x}^1 \\ \vdots \\ \bar{x}^k \end{pmatrix} \quad and \quad \bar{x}_2 = \begin{pmatrix} \bar{x}^{k+1} \\ \vdots \\ \bar{x}_n \end{pmatrix}.$$

The

$$\dim \mathscr{E} = \operatorname{rank}\left[B, AB, A^2B, \ldots, A^{n-1}B\right].$$

If (S) *is restricted to* $\mathscr{E}$ *we obtain*

$$(\mathrm{S_c}) \qquad \dot{y} = A_{11}y + B_1 u\,, \quad \bar{x}_2 = 0\,.$$

Obviously $\mathscr{R}(t) = \mathscr{R}_1(t) \times \{0\}$ *and* $\mathscr{R}(t) \subset \mathscr{E}$. *The minimal-time function* $T(\bar{x}_1) = T(y)$ *if* $\bar{x}_2 = 0$, *it is finite-valued on* $\mathscr{R}_1(t)$ *and continuous,* $T(x) = \infty$ *whenever* $x_2 \neq 0$.

Proof. The (S) is controllable on $\mathscr{E}$, and $\mathscr{E}$ is an invariant subspace of R^n. Also the rank of the controllability matrix under linear equivalence is n. Thus, the

$$\dim \mathscr{E} = \operatorname{rank}\left[B_1, A_{11}B_1, \ldots, A_{11}^{k-1}B_1\right] = \operatorname{rank}\left[B, AB, \ldots, A^{n-1}B\right] = n\,.$$

The second part of the lemma is self-evidence. Q. E. D.

3. THE MINIMAL-TIME FUNCTION AS A LYAPUNOFF FUNCTION

Theorem 1. *The minimal-time function* $T : \mathscr{E} \to R^1 \cup \{+\infty\}$ *is continuous, with* $\mathscr{R}$ *open in the linear space* $\mathscr{E} \subset R^n$.

Proof. Assume that $T(x)$ is defined and finite-valued on $\mathscr{R}(t)$, and $x_n \to x$ with $T(x_n) \leqq t$. Then $x_n \in \mathscr{R}(t)$ and so $x \in \mathscr{R}(t) \subset \mathscr{R}$ from closedness. Q. E. D.

Remark 2. Since the origin is the only critical point of $\mathscr{E}$ it is immediate from lemma 1 that we can always find an optimal asymptotically stable solution $y(t)$ of (S) through $x_0 \in \mathscr{R}(t)$. We shall verify this in the proof of the next theorem by showing that the minimal-time function for $(\mathrm{S_c})$ is a Lyapunoff function.

Theorem 2. *Consider the linear autonomous system in* R^n,

$$(\mathrm{S}) \qquad \dot{x} = Ax + Bu$$

with controls $u(t)$ *constrained to the unit cube.*

Suppose there exists a unique linear controllability space in R^n with appropriate coordinates $x = \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \end{pmatrix}$, $\bar{x}_2 = 0$ *and* (S) *is precisely*

$$\begin{pmatrix} \dot{\bar{x}}_1 \\ \dot{\bar{x}}_2 \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix} \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \end{pmatrix} + \begin{pmatrix} B_1 \\ 0 \end{pmatrix} u .$$

Assume that (S) *is restricted to* $\mathscr{E}$ *so that*

$$(\mathrm{S_c}) \qquad \dot{y} = A_{11} y + B_1 u , \quad \bar{x}_2 = 0 .$$

For every $x_0 \in \mathscr{R}(t)$ *and* $t \geqq 0$ *let* $y(t)$ *be the unique time optimal solution of* (S) *with initial data* $y_0 = x_0$. *Then the minimal-time function* $T(y(t))$ *is a Lyapunoff function.*

Proof. The minimal-time function

$$\xi = T(x_0) \quad \text{for} \quad 0 \leqq t < \xi$$

if and only if $x_0 \in \partial\mathscr{R}(t)$ (boundary relative to $\mathscr{E}$), and $y(t)$ is the value at t of the time optimal solution beginning at x_0, then

$$T(y(t)) = \xi - t ,$$

that is,

$$T(y(t)) = T(x_0) - t$$

and

$$\frac{\mathrm{d}T(y(t))}{\mathrm{d}t} = -1 .$$

Hence the minimal-time function for (S) is a Lyapunoff function for $(\mathrm{S_c})$, and decreases monotonically along the optimal response, $T(y(t)) = 0$ if and only if $y(t) = 0$. Q. E. D.

4. EXAMPLE 1: CONTROL OF A HARMONIC OSCILLATOR

Consider the harmonic oscillator

$$\ddot{x} + x = u , \quad |u| \leqq 1 .$$

An equivalent system of first-order equations is

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u .$$

It is easily verified that the eigenvalues of A are

$$\lambda_1 = \mathrm{j} , \quad \lambda_2 = -\mathrm{j} .$$

The open loop system is neutrally stable. That is, it is on the border line between stability and instability [3, pp. 18]. Since it is a normal system, it is controllable. Hence there exists time-optimal control in the controllability space $\mathscr{E}$ that will steer the system to the origin. In this case,

$$X(t) = e^{At} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$$

and

$$Y(t) = e^{-At}B = \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} \quad \text{for} \quad u = +1 .$$

Thus, the equivalent system is (see (2))

$$\text{(3)} \qquad \dot{y}_1(t) = -\sin t ,$$

$$\text{(4)} \qquad \dot{y}_2(t) = \cos t .$$

We confine ourselves to the interval $\pi < t < 0$. Integrate (3) and (4) from 0 to t to obtain

$$y_1(t) = \cos t - 1 , \quad \pi < t < 0 ,$$

$$y_2(t) = \sin t .$$

Obviously,

$$Y(t) = e^{-At}B = \begin{pmatrix} \sin t \\ -\cos t \end{pmatrix} \quad \text{for} \quad u = -1$$

and

$$\text{(5)} \qquad \dot{y}_1(t) = \sin t ,$$

$$\text{(6)} \qquad \dot{y}_2(t) = -\cos t .$$

Integrating (5) and (6) from 0 to t we obtain

$$y_1(t) = -\cos t + 1 , \quad -\pi < t < 0 ,$$

$$y_2(t) \doteq \sin t .$$

Now we need to express our result in terms of x coordinates. Thus under change of coordinates we have

$$\text{(7)} \qquad x = e^{At}y$$

or

$$\text{(8)} \qquad x_i(t) = \sum_{k=1}^{n} q_{ik}\, y_k(t) , \quad q_{ik} = [e^{At}]_{ik} , \quad i = 1, \ldots, n .$$

Obviously, when $u = +1$ we obtain

$$x_1(t) = 1 - \cos t \quad \text{on} \quad -\pi < t < 0\,, \tag{9}$$
$$x_2(t) = \sin t$$

and when $u = -1$ we obtain

$$x_1(t) = \cos t - 1 \quad \text{on} \quad \pi < t < 0\,,$$
$$x_2(t) = -\sin t\,.$$

Clearly, the set of states which can be forced to the origin in no more than π seconds by the control $u = +1$ is defined as

$$\nu_+ = \{(x_1, x_2) : (x_1 - 1)^2 + x_2^2 = 1;\ x_2 < 0,\ x_1 > 0\}\,,$$

and the set of states which can be forced to the origin in no more than π seconds by the control $u = -1$ is defined as

$$\nu_- = \{(x_1, x_2) : (x_1 + 1)^2 + x_2^2 = 1;\ x_2 > 0,\ x_1 < 0\}\,.$$

The ν switch curve is given by

$$\nu = \begin{cases} \nu_+ \cup \nu_- & \text{if } |x_1| \leqq 2\,, \\ x_2 = 0 & \text{if } |x_1| > 2\,. \end{cases}$$

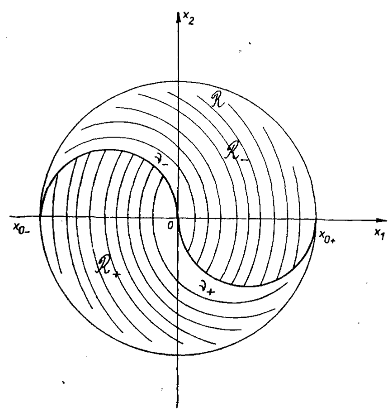

Fig. 1. Switching locus and synthesis of minimal time optimal control to origin for $\dot{x}_1 = x_2$, $\dot{x}_2 = -x_2 + u$, $|u| = 1$ (analogue simulation).

It is not difficult to see that the set $\mathscr{R}$ of reachability is the open region bounded by the circle of radius 2 with center at the origin (see Fig. 1). Thus,

$$\mathscr{R} = \mathscr{R}_+ \cup \mathscr{R}_- = \{(x_1, x_2) : x_1^2 + x_2^2 \leqq 2\}$$

where $\mathscr{R}_-$ are the set of states which can be forced to the v_+ curve in no more than π seconds by the control $u = -1$, and $\mathscr{R}_+$ are the set of states which can be forced to the v_- curve in no more than π seconds by control $u = +1$.

Every solution of $v_\pm$ leaves the point $x_{0\pm}$ and crosses the x_1-axis once at most, and so each response initiating in $\mathscr{R}$ and leading to the origin will have at most one switch. Obviously, for every state in $\mathscr{R}_-$ the time-optimal control sequence is $\{-1, +1\}$, and for every state in $\mathscr{R}_+$ the control sequence is $\{+1, -1\}$. Hence the unique time-optimal control law as a function of state (x_1, x_2) is given by

$$u^* = u^*(x_1, x_2) = +1 \quad \text{for all} \quad (x_1, x_2) \in v_+ \cup \mathscr{R}_- ,$$

$$u^* = u^*(x_1, x_2) = -1 \quad \text{for all} \quad (x_1, x_2) \in v_- \cup \mathscr{R}_+ .$$

More precisely,

$$u^*(x_1, x_2) = -\operatorname{sgn}\left[x_2 + \operatorname{sgn} x_1 \sqrt{(1 - (x_1 - \operatorname{sgn} x_1)^2)}\right].$$

Thus, the synthesis of feedback time-optimal control $u^*(x)$ reduces to a determination of the reachability set and to the construction of the v_- switch curve.

Remark 3. Note that the shape of the reachability set does not change under change of coordinates since $x_0 = y_0 \in \partial\mathscr{R}$ (boundary relative to $\mathscr{E}$).

It will now be show that the time-optimal control ensures asymptotic stability in the controllability space by demonstrating that the minimal-time function is a Lyapunoff function. Assume that the point x_0 with coordinate $(2, 0)$ can be steered to the origin. The ***minimal time*** of the control action is precisely

$$\xi = T(x_0) = \pi \text{ seconds} .$$

Again consider the point (x_1, x_2) which can be forced to the origin in the time

$$T(x_1, x_2) = \int_0^{x_1} \frac{1}{x_2}\, dx_1 = \int_0^{x_1} \frac{1}{\sqrt{[1 - (x_1 - 1)^2]}}\, dx_1 . \tag{10}$$

Substitute $z = x_1 - 1$ in the integrand (10). The changes in the limit of integration are

$$x_1 = 0 \qquad \text{then} \quad z = -1 ,$$

$$x_1 = \sqrt{(1 - x_2^2)} + 1 \quad \text{then} \quad z = \sqrt{(1 - x_2^2)}$$

and

$$T(x_1, x_2) = \int_{-1}^{\sqrt{(1-x_2^2)}} \frac{1}{1 - z^2} \, \mathrm{d}z = [\arc \sin z]_{-1}^{\sqrt{(1-x_2^2)}} =$$

$$= \arc \sin \sqrt{(1 - x_2^2)} + \tfrac{1}{2}\pi = \arc \cos x_2 + \tfrac{1}{2}\pi =$$

$$= \tfrac{1}{2}\pi + (\tfrac{1}{2}\pi - \arc \sin x_2) = \pi - \arc \sin x_2 .$$

From (9) we see that $x_2 = \sin t$ and so

$$T(x(t)) = \pi - \arc \sin (\sin t) =$$

$$= \pi - t ,$$

obviously,

$$\frac{\mathrm{d}T(x(t))}{\mathrm{d}t} = -1 .$$

Hence the optimal control law is a stable control law in the controllability space $\mathscr{E}$.

Suppose we have established the minimal-time function in the y coordinates, then for the purpose of synthesis of the actual system we must express the gradient of the minimal time function in terms of the x coordinates. More precisely, assume that

$$T(y_1, y_2) = \tilde{T}(x_1, x_2) .$$

From relation (8) we obtain

$$y_j = \sum_{k=1}^{n} Q_{jk}(t) \, x_k(t) , \quad j = 1, \ldots, n .$$

Thus the gradient of $\tilde{T}(x)$ is

$$\frac{\mathrm{d}\tilde{T}(x_1, x_2)}{\mathrm{d}x_k} = \sum_j \frac{\partial T(x_1, x_2)}{\partial y_j} \frac{\mathrm{d}y_j}{\mathrm{d}x_k} = \sum_j \frac{\partial T(x_1, x_2)}{\partial y_j} Q_{jk}$$

or

$$\begin{pmatrix} \dfrac{\mathrm{d}T}{\mathrm{d}x_1} \\ \vdots \\ \dfrac{\mathrm{d}T}{\mathrm{d}x_n} \end{pmatrix} = \mathrm{e}^{-At} \begin{pmatrix} \dfrac{\partial T}{\partial y_1} \\ \vdots \\ \dfrac{\partial T}{\partial y_n} \end{pmatrix} .$$

5. EXAMPLE 2: CONTROL OF D. C. SERVOMOTOR

Consider the block diagram of Fig. 2. The system to be controlled is characterised by the transfer function

$$KG(p) = \frac{K}{p(\tau p + a)} = \frac{C(p)}{U(p)}$$

where $a = 1$, $\tau = 10$ and $K = 1$. We seek an optimal control action which reduces the error $e(t)$ and its derivative $\dot{e}(t)$ to zero in the shortest possible time. The reference input signal $v(t)$ is assumed constant. The control action $u(t)$ is constrained by $|u| \leqq$ $\leqq 10$.

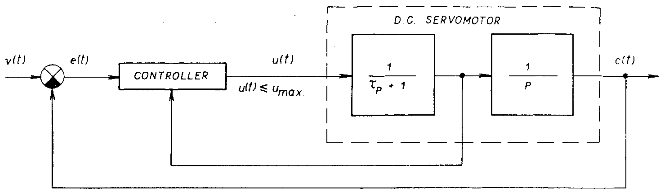

Fig. 2. Control of D. C. servomotor.

The differential equation for the system is

$$\ddot{c}(t) + \frac{\dot{c}(t)}{10} = \frac{u(t)}{10}. \tag{11}$$

The differential equation (11) may be written in terms of the error as

$$-\ddot{e}(t) - \frac{\dot{e}(t)}{10} = -\ddot{v}(t) - \frac{\dot{v}(t)}{10} + \frac{u(t)}{10}.$$

Obviously,

$$-\ddot{e}(t) - \frac{\dot{e}(t)}{10} = \frac{u(t)}{10}.$$

To form the state equations, let

$$x_1 = -e(t) \quad \text{and} \quad x_2(t) = \dot{e}(t).$$

An equivalent system of first-order equations is

$$\dot{x}_1(t) = e(t) = x_2(t),$$

$$\dot{x}_2(t) = \dot{e}(t) = -\frac{x_2(t)}{10} - \frac{u(t)}{10},$$

and in matrix notation

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & -\frac{1}{10} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} \frac{u}{10}$$

 or in vector form as

$$\dot{x} = Ax + B\bar{u}$$

where

$$A = \begin{pmatrix} 0 & 1 \\ 0 & -\frac{1}{10} \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \quad \text{and} \quad \bar{u} = u/10$$

The fundamental matrix solution

$$X(t) = e^{At} = \begin{pmatrix} 1 & 10(1 - e^{-t/10}) \\ 0 & e^{-t/10} \end{pmatrix}$$

and

$$X^{-1}(t) = e^{-At} = \begin{pmatrix} 1 & 10(1 - e^{t/10}) \\ 0 & e^{t/10} \end{pmatrix}.$$

In this case,

$$Y(t) = e^{-At}B = \begin{pmatrix} -10(1 - e^{t/10}) \\ -e^{t/10} \end{pmatrix} \quad \text{for} \quad u = +10.$$

Clearly, the equivalent system is (see (2))

(12) $$\dot{y}_1(t) = -10(1 - e^{t/10}),$$

(13) $$\dot{y}_2(t) = -e^{t/10}.$$

Let $\mathcal{R}_-$ denote the set of states to the right of the v switch curve, and let $\mathcal{R}_+$ denote the set of states to the left of the v switch curve. Clearly,

$$\mathcal{R}_- = \{(x_1, x_2) : x_1 + 10[10 \ln(1 - x_2/10) + x_2] < 0\},$$

$$\mathcal{R}_+ = \{(x_1, x_2) : x_1 + 10[10 \ln(1 - x_2/10) + x_2] > 0\}$$

(see Fig. 3).

The time-optimal control as a function of the state (x_1, x_2) is given by

$$u^* = u^*(x_1, x_2) = +10 \quad \text{for all} \quad (x_1, x_2) \in v_+ \cup \mathcal{R}_-,$$

$$u^* = u^*(x_1, x_2) = -10 \quad \text{for all} \quad (x_1, x_2) \in v_- \cup \mathcal{R}_+.$$

More precisely,

$$u(x_1, x_2) = 10 \operatorname{sgn} \{x_1 - 10[10 \operatorname{sgn} x_2 \ln(1 + |x_2|/10) - x_2]\}.$$

It is bang-bang ($|u| = 10$ almost everywhere), and changes sign once at most. The optimal control action $u^*(t)$ can be understood as a maximal accelerating force followed by a maximal braking deceleration until the motor stops just at the required position.

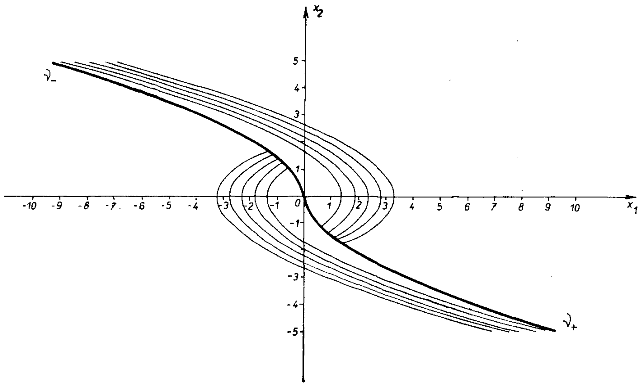

Fig. 3. Minimal time optimal responses to origin of the D.C. servomotor. Switching diagram and synthesis (analogue simulation).

Integrating (12) and (13) from 0 to t we obtain

$$y_1 = -10(t - 10\,\mathrm{e}^{t/10} + 10)\,,$$
$$y_2 = -10(\mathrm{e}^{t/10} - 1)\,.$$

Thus under change of coordinates (see (7)), when $u = +10$ we obtain

$$x_1 = -10[t - 10(1 - \mathrm{e}^{-t/10})]\,,$$
$$x_2 = -10(1 - \mathrm{e}^{-t/10})$$

and when $u = -10$ we obtain

$$x_1 = 10[t - 10(1 - \mathrm{e}^{-t/10})]\,,$$
$$x_2 = 10(1 - \mathrm{e}^{-t/10})\,.$$

Hence, the set of all states which can be forced to the origin by the control $u = +10$ in positive time is defined as

$$\nu_+ = \{(x_1, x_2) : x_1 + 10[10 \ln(1 - x_2/10) + x_2] = 0,\ x_1 > 0,\ x_2 < 0\}\,,$$

and the set of all states which can be forced to the origin by the control $u = -10$ in positive time is defined as

$$v_- = \{(x_1, x_2) : x_1 - 10[10 \ln (1 + x_2/10) - x_2] = 0,\ x_1 < 0,\ x_2 > 0\} .$$

The v switch curve is given by

$$v = v_+ \cup v_- = \{(x_1, x_2) : x_1 = 10[10 \operatorname{sgn} x_2 \ln (1 + |x_2|/10) - x_2] .$$

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