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An Application of Conjugate Duality for Numerical Solution of Continuous Convex Optimal Control Problems

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In the paper two numerical approaches for solving a class of convex continuous optimal control problems with state-space constraints are presented. Both approaches are based on the conjugate duality so that the solution of the original problem is substituted by the solution of another problem or a sequence of another problems that can be easier and hence more easily solvable.

INTRODUCTION

Duality theory became an important part of the theory of extremal problems including the mathematical programming, the optimal control theory, the combinatorical problems and other areas. For the finite-dimensional problems the usage of dual methods has reached a satisfactory stage of development. For infinite-dimensional problems, however, the conjugate duality was exploited up to now preferably for theoretical purposes such as the existence of solutions, constraint qualifications, optimality conditions etc. (cf. [5], [13]). Concerning numerical methods for infinite-dimensional optimal control problems, Rupp ([17], [18]) has applied the original method of Hestenes and Powell to a class of terminally constrained problems in such a way that the dynamic system equation is eliminated by means of a generalized penalty functional. More complex problems including mixed state-control inequality constraints have been studied by Glad [4], who treates all problem-constraints via generalized penalty.

According to our opinion the source of principal difficulties in solving optimal control problems numerically is the presence of state-space constraints. Therefore, in the first (more applicable) approach of this paper it is proposed to eliminate by means of a quadratic generalized penalty functional merely the state-space constraints. Provided the control constraints are of a simple form (e.g. linear), we obtain a sequence of much simpler extremal problems still possessing the original optimal control structure.

The Fenchel dualisation scheme was many times used in the literature in situations where infinite-dimensional problems could be converted into finite-dimensional ones (cf. [8]). We propose, with respect to the works of Lemarechal [6], [7], to apply this scheme even for problems, where the space of dual variables is Hilbert. Clearly, the class of problems to which this approach is applicable is rather restricted, but the numerical results are encouraging.

We shall extensively use the theory developed in [2], [14], [15], [16] and employ the following notation throughout the sequel: R^n is the Eucleidean n-space, $\bar{R}$ is the extended real line, $L_2[0, T, R^n]$ is the space of (equivalence classes of) square integrable vector-valued functions on $[0, T]$ with values in R^n, $W_2^1[0, T, R^n]$ is the space of absolutely continuous vector-valued functions on $[0, T]$ with values in R^n and derivatives in $L_2[0, T, R^n]$, $C[0, T, R^n]$ is the Tschebyshev space, $\mathscr{L}[X, Y]$ is the space of all continuous linear operators mapping X into Y, x^j is the j-th coordinate of a vector x, Π^* is the conjugate operator to a linear operator Π, A^T is the transpose of a matrix A, $\text{prox}_f\, z$ denotes the proximal mapping of z with respect to a function f (cf. [9]), $C(x^* \mid \Omega)$ denotes the contact set of $\Omega \subset X$ with respect to the direction $x^* \in X^*$, i.e.

$$C(x^* \mid \Omega) = \{x \in \Omega \mid \langle x, x^* \rangle_X = \sup_{y \in \Omega} \langle y, x^* \rangle_X\},$$

X^* is the space of all continuous linear functionals over a space X, $\delta(\cdot \mid \Omega)$ is the indicatory functional of a set Ω, $(x)^D$ is the projection of an element x onto a set D, $\partial f(x)$ is the subdifferential of a functional f at x, and Θ is the zero vector.

1. PROBLEM FORMULATION

In the whole paper the following convex optimal control problem will be investigated.

Basic problem (BP):

$$\varphi(x(T)) + \int_0^T \psi(x,(t)), \ u(t))\, dt \to \inf$$

subj. to

$$\dot{x} = A\, x(t) + B\, u(t) \quad \text{a.e. on} \quad [0, T],$$

$$x(0) = x_0,$$

$$u(t) \in \omega \subset L_2[0, T, R^l],$$

$$\bar{q}(x)\,(t) \leqq \Theta \quad \text{on} \quad [0, T],$$

$$a(x(T)) \leqq \Theta$$

where A, B are constant $[n \times n]$, $[n \times l]$ matrices, respectively, $u \in L_2[0, T, R^l]$, $x \in W_2^1[0, T, R^n]$, $\varphi[R^n \to \bar{R}]$, $\psi[R^n \times R^l \to \bar{R}]$, $\bar{q}[R^n \to R^m]$, $a[R^n \to R^k]$, and the inequality sign is valid for all coordinates. We assume that

(i) φ is a proper convex, lower semi-continuous (l.s.c.) function,
(ii) ψ is a proper convex, l.s.c. function,
(iii) ω is convex and closed,
(iv) functions $\bar{q}^i$, $i = 1, 2, \ldots, m$ are convex and continuous,
(v) functions a^i, $i = 1, 2, \ldots, k$ are convex and continuous,
(vi) there exists a control $u_0 \in \omega$ such that the absolutely continuous trajectory $x(u_0)$ corresponding to u_0 satisfies the inequalities

$$\bar{q}(x(u_0))(t) \leqq \Theta \quad \text{for} \quad t \in [0, T], \quad a(x(u_0)(T)) \leqq \Theta,$$

(the consistency of the constraints or the controllability of the given system within the given constraints),
(vii) either ω is bounded or the functional $S[L_2[0, T, R^l] \to \bar{R}]$

$$S(u) = \varphi(x(u)(T)) + \int_0^T \psi(x(u)(t), u(t))\, dt$$

is on ω coercive.

With respect to both approaches being discussed in this sequel it is convenient to transcribe our BP into the form of so called Abstract Convex Optimal Control Problem (ACOCP):

$$J(u, y) \to \inf$$

subj. to

$$y = \Pi u + y_0 \in Y, \tag{1}$$

$$u \in \omega \subset U,$$

$$-q(y) \in D \subset H,$$

where U, Y and H denote the control space, the generalized state-space and the constraint space, respectively, J is the optimality criterion, Π and y_0 describe the system dynamics, cone D defines the partial ordering in H in the usual way, and $-q(y) \in D$ represents the state-space constraints.

Remark. We call Y the generalized state-space or also response space (cf. [11]) because it need not be necessarily identified with the space of trajectories $x(t)$ as it will be shown in Sec. 3.

The most natural choice is to set $U = L_2[0, T, R^l]$, $Y = W_2^1[0, T, R^n]$ and $H = C[0, T, R^m] \times R^k$. The positive cone in H is

$$D = \{(w, z) \in C[0, T, R^m] \times R^k \mid w(t) \geqq \Theta \quad \text{on} \quad [0, T],\ z \geqq \Theta\} \tag{2}$$

and BP attains the form (1) if we set $y = x(t)$, $y_0 = e^{At}x_0$,

$$\Pi u = \int_0^t e^{A(t-\tau)} B\, u(\tau)\, \mathrm{d}\tau\,, \tag{3}$$

$$q(y) = (\bar{q}(x)(t),\ a(x(T)))\,, \tag{4}$$

and denote its objective J. The assumptions stated above imply that the ACOCP induced by BP by means of (2), (3), (4) satisfies the following set of requirements:

(5) (i) $J[U \times Y \to \bar{R}]$ is a proper convex, l.s.c. functional (cf. [12]),
(ii) $\Pi \in \mathscr{L}[U, Y]$, $y_0 \in Y$,
(iii) D is a closed convex cone in H with a vertex at the origin,
(iv) ω is closed and convex,
(v) q is a D-convex continuous map from Y into H (cf. [20]),
(vi) there exists at least one solution of (1) and the corresponding value of the objective is finite (cf. [2]).

In what follows we shall systematically investigate two dual approaches for solving BP in terms of (1). Each approach requires a slightly different transcription to the general form (1) than the natural one defined above but the general assumptions (5) must be satisfied. Moreover, each of them requires some special additional assumptions which will be given at the appropriate places in terms of BP.

2. PERTURBATIONS OF THE STATE-SPACE CONSTRAINTS

This approach is based on the perturbations of the constraint $-q(y) \in D$ and requires the range space of q to be Hilbert. To satisfy this requirement, we set now

$$H = L_2[0, T, R^m] \times R^k\,,$$

and, correspondingly

$$D = \{(w, z) \in H \mid w(t) \geqq \Theta \text{ a.e. on } [0, T],\ z \geqq \Theta\}\,. \tag{6}$$

It can easily be seen that the map q remains continuous from $W_2^1[0, T, R^n]$ into H so that

(i) the corresponding ACOCP (1) is equivalent with BP (in the sense that they possess the same sets of solutions $(u(t), x(t))$ and optimal values),

(ii) the assumptions (5) are satisfied.

With respect to the numerical method (18, 19) described in this section it is reasonable to assume additionally that

$$\operatorname{dom} S(u) \supset \omega\,.$$

Let us construct the family of dual extremal problems $(\mathscr{D}_r^1)$ to BP exploiting the perturbed essential objective

$$(7)\qquad F_r(u, p) = J(u, \Pi u + y_0) + \delta(u \mid \omega) + \delta(\Pi u + y_0 | \{y | \, p - q(y) \in D\}) + \\ + r\|p\|_H^2$$

where $r \geqq 0$ is a scalar parameter. The assumptions being imposed guarantee that F_r, $r \geqq 0$ is a proper convex, l.s.c. functional on $L_2[0, T, R^l] \times L_2[0, T, R^m] \times R^k$. $(\mathscr{D}_r^1)$ attains the form

$$(8)\qquad - F_r^*(\Theta, p^*) \to \sup \quad \text{subj. to} \quad p^* \in H\,,$$

where F_r^* denotes the convex conjugate functional to F_r, i.e.

$$(9)\qquad F_r^*(\Theta, p^*) = \sup_{\substack{u \in \omega \\ p - q(\Pi u + y_0) \in D}} [\langle p, p^* \rangle_H - J(u, \Pi u + y_0) - r\|p\|_H^2]\,.$$

It can be easily proved, exploiting Th. 2.4 of [20], that

$$(10)\qquad -F_r^*(\Theta, p^*) = \inf_{u \in \omega} L_r(u, p^*)\,,$$

where

$$(11)\quad L_r(u, p^*) = \begin{cases} J(u, \Pi u + y_0) - \left\langle p^*, \left(\dfrac{p^*}{2r} - q\right)^D + q \right\rangle_H + r \left\| \left(\dfrac{p^*}{2r} - q\right)^D + q \right\|_H^2 & \text{if} \quad u \in \omega \\ \infty \quad \text{if} \quad u \notin \omega\,. \end{cases}$$

(Arguments at q were dropped due to the notational simplicity). The functional L_r will be termed *augmented Lagrangian* and corresponds exactly to the functional Λ in [20] where it was defined for duality purposes as a certain modification of the penalty functional Ψ.

Let us denote

$$(12)\qquad g_0(p^*) = \inf_{\substack{u \in \omega \\ p - q(\Pi u + y_0) \in D}} [J(u, \Pi u + y_0) - \langle p, p^* \rangle_H]\,,$$

and

$$(13)\qquad g_r(p^*) = \inf_{u \in \omega} L_r(u, p^*)$$

the dual objectives in the case $r = 0$, $r > 0$, respectively. Both functionals g_r and L_r possess various useful properties, listed for the finite-dimensional problems e.g. in [14], [15]. One of the probably most important properties of the dual objective g_r is expressed in the following proposition the proof of which can be found in [10].

Theorem 2.1. For all $r > 0$

$$g_r(p^*) = \max_{a^* \in H} \left[g_0(a^*) - \frac{1}{4r} \|p^* - a^*\|_H^2 \right]. \tag{14}$$

Thus, the dual problems $(\mathscr{D}_r^1)$, $r > 0$ all have the same optimal solutions and supremum as the ordinary dual $(\mathscr{D}_0^1)$. Moreover (assuming $g_0 \not\equiv -\infty$), g_r is everywhere finite and continuously Fréchet differentiable on H. Especially, if for a given p^* the infimum defining $g_r(p^*)$ happens to be attained at a point $\bar{u} \in \omega$, then the Fréchet gradient

$$\nabla g_r(p^*) = \nabla_{p^*} L_r(\bar{u}, p^*) = -\left[q(\Pi\bar{u} + y_0) + \left(\frac{p^*}{2r} - q(\Pi\bar{u} + y_0) \right)^D \right]. \tag{15}$$

Remark. In our case clearly $g_0 \not\equiv -\infty$.

The perturbational functional $h_r(p)$ corresponding to (7) attains the form

$$h_r(p) = h_0(p) + r\|p\|_H^2, \tag{16}$$

where

$$h_0(p) = \inf_{\substack{u \in \omega \\ p - q(\Pi u + y_0) \in D}} J(u, \Pi u + y_0) \tag{17}$$

is the perturbational functional corresponding to (7) for $r = 0$. We are not able to prove its subdifferentiability at Θ in H (D possesses no interior), but only its lower-semicontinuity at Θ, if we take into account the assumptions of the previous section.

Theorem 2.2. BP is with respect to perturbations (7) normal, i.e. its perturbational functional $h_r(p)$, $r \geqq 0$ is at Θ finite and l.s.c.

Proof. The finiteness of $h_r(p)$ at Θ is evident. Let us now assume that $h_r(p)$ is at Θ not l.s.c., i.e. that in any open ball with the centre in Θ there are points $\tilde{p}$ and an $\varepsilon > 0$ such that

$$\inf_u [F_0(u, \tilde{p}) + r\|\tilde{p}\|_H^2] < \inf_u F_0(u, \Theta) - \varepsilon = \mu - \varepsilon,$$

where F_0 is the perturbed essential objective for $r = 0$ and μ is the optimal cost value for BP which is finite. Let us take a sequence $\{\tilde{p}_n\}$ converging to Θ. It is possible to find points $\tilde{u}_n$ for which

$$F_0(\tilde{u}_n, \tilde{p}_n) + r\|\tilde{p}_n\|_H^2 < \mu - (\varepsilon/2).$$

The sequence $\{\tilde{u}_n\}$ is bounded due to either the boundedness of ω or the coercivity of S on ω. Therefore, it is possible to select a subsequence $\{\tilde{u}_{n'}\}$ from $\{\tilde{u}_n\}$ weakly converging to some point $u_0 \in \omega$. In the weak topology functional $F_0(u, p) + r\|p\|_H^2$ remains l.s.c. (cf. [2]). Thus,

$$F_0(u_0, \Theta) \leqq \underline{\lim}_{\substack{\tilde{u}_{n'} \xrightarrow{w} u_0 \\ \tilde{p}_{n'} \to \Theta}} [F_0(\tilde{u}_{n'}, \tilde{p}_{n'}) + r\|\tilde{p}_{n'}\|_H^2] \leqq \mu - \varepsilon/2 \leqq F_0(u_0, \Theta) - \varepsilon/2$$

what is the desired contradiction. □

The dualisation (7) with $r > 0$, has two basic advantages in comparison with the classical case $r = 0$. The first is expressed in the Theorem 2.1 (dom $g_r = H$), the second in the following important proposition:

Theorem 2.3. Let $\hat{p}^*$ be a solution of $(\mathscr{D}_r^1)$ for $r > 0$. Then any $\hat{u}$ minimizing $L_r(u, \hat{p}^*)$ is a solution of BP.

The proof differs in no respect from the corresponding finite-dimensional proposition in [14].

For the solution of $(\mathscr{D}_r^1)$ e.g. the well-known primal-dual numerical scheme of [15] can be applied.

Given $r < 0$, $p^* \in H$ and a sequence $\{\alpha_k\}$ with $0 \leqq \alpha_k \to 0$, such that $\sum_{k=1}^{\infty} \sqrt{\alpha_k} < \infty$.

k-th step:

1) Given $p_k^* \in H$, determine $u_k \in \omega$ such that

$$L_r(u_k, p_k^*) \leqq \inf_{u \in \omega} L_r(u, p_k^*) + \alpha_k . \tag{18}$$

2) Set

$$p_{k+1}^* = p_k^* - 2r\left[q(\Pi u_k + y_0) + \left(\frac{p_k^*}{2r} - q(\Pi u_k + y_0)\right)^D\right]. \tag{19}$$

Theorem 2.4 below deals with the convergence properties of this scheme.

Theorem 2.4. Let the sequence of multipliers $\{p_k^*\}$ generated by (18, 19) be bounded. Then the corresponding sequence of controls $\{u_k\}$ is asymptotically minimizing for BP and its every weak cluster point is a solution of BP.

In the proof we involve following lemmas:

Lemma 2.1. For a bounded sequence $\{p_k^*\}$ of elements of H generated by (18, 19)

$$\lim_{k \to \infty} [g_r(p_{k+1}^*) - g_r(\operatorname{prox}_{-2rg_0} p_k^*)] = 0 .$$

Proof. g_r is, as a concave functional continuous over H, locally Lipschitz so that there exists a constant L such that

$$\lim_{k\to\infty} [g_r(p^*_{k+1}) - g_r(\operatorname*{prox}_{-2rg_0} p^*_k)] \leqq \lim_{k\to\infty} |g_r(p^*_{k+1}) - g_r(\operatorname*{prox}_{-2rg_0} p^*_k)| \leqq$$

$$\leqq \lim_{k\to\infty} L\|p^*_{k+1} - \operatorname*{prox}_{-2rg_0} p^*_k\|_H .$$

As shown in [14], condition (18) yields the estimate

$$r\|\nabla_{p^*}L_r(u_k, p^*_k) - \nabla g_r(p^*_k)\|^2_H \leqq \alpha_k . \tag{20}$$

Recalling the basic properties of proximal mapping from [9], Eq. (15) can be written in the form

$$\nabla g_r(p^*) = -\frac{1}{2r}(p^* - \operatorname*{prox}_{-2rg_0} p^*)$$

so that

$$\operatorname{prox} p^* = p^* + 2r\,\nabla g_r(p^*) .$$

Hence, Ineq. (20) and the "up-date" rule (19) imply that

$$\|p^*_{k+1} - \operatorname*{prox}_{-2rg_0} p^*_k\|^2_H \leqq 4r\alpha_k$$

so that

$$\lim_{k\to\infty} \|p^*_{k+1} - \operatorname*{prox}_{-2rg_0} p^*_k\|^2_H = 0$$

and we are done. □

Lemma 2.2. Let $\{p^*_k\}$ be a bounded maximizing sequence for $(\mathscr{D}^1_r)$, $r > 0$ and let for all k the control u_k satisfy Ineq. (18). Then $\{u_k\}$ is an asymptotically minimizing sequence for BP, i.e. in our case

$$\begin{gathered} u_k \in \omega \\ \varrho(-q(\Pi u_k + y_0), D) \to 0 . \\ J(u_k, \Pi u_k + y_0) \to \mu . \end{gathered} \tag{21}$$

where $\varrho(p, D)$ denotes the distance between an element p and the cone D.

The proof is omitted since in the finite-dimensional case it is essentially given in [14] and our case requires only slight modifications.

Lemma 2.3. Let the sequence $\{u_k\}$ of controls be such that relations (21) are satisfied. Then there exists at least one weak cluster point $\hat{u}$ of this sequence and this control is an actual solution of BP.

Proof. The distance $\varrho(-q(\Pi(.) + y_0), D)$ is a convex functional finite over the whole space $L_2[0, T, R^l]$. It is bounded above on any set

$$\mathcal{O}_\varepsilon = \{u \in L_2[0, T, R^l] \mid -q(\Pi u + y_0) \in \mathcal{B}_\varepsilon\}$$

where

$$\mathcal{B}_\varepsilon = \{p \in H \mid p \in \bigcup_{z \in D} N(z, \varepsilon)\},\ \varepsilon > 0,\ N(z, \varepsilon) = \{w \mid \|w - z\|_H < \varepsilon\}$$

by ε. $\mathcal{O}_\varepsilon$ is open as an original of an open set B_ε in a continuous map $-q(\Pi(\cdot) + y_0)$. It is nonempty because it includes the set of feasible solutions of BP. Hence, ϱ is continuous over H.

Any sequence $\{u_k\}$ with properties (21) is bounded due to the coercivity of S on ω. Hence, it possesses weak cluster points according to the Eberlein-Shmulyan theorem. The continuity and convexity of $\varrho(-q(\Pi(\cdot) + y_0, D)$ imply its weak lower semi--continuity so that if $\{u_{k'}\}$ is a subsequence of $\{u_k\}$ weakly converging to a point u_0, then

$$\varrho(-q(\Pi u_0 + y_0), D) \leqq \lim_{k' \to \infty} \varrho(-q(\Pi u_{k'} + y_0), D) = 0 .$$

Hence, u_0 is with respect to state-space constraints feasible. Similarly the convexity and lower semi-continuity of S imply the weak lower semi-continuity so that

$$S(u_0) = J(u_0, \Pi u_0 + y_0) \leqq \lim_{k' \to \infty} J(u_{k'}, \Pi u_{k'} + y_0) = \mu .$$

which was to be proved. □

Proof of Theorem 2.4. If we take into account the assertions of Lemmas 2.2, 2.3, the only thing which remains to be proved is that the sequence $\{p_k^*\}$ generated by (18, 19) is maximizing for $(\mathcal{D}_r^1)$, $r > 0$.

We observe first from (14) that

$$(22)\quad g_r(p_k^*) = g_0(\operatorname{prox}_{-2rg_0} p_k^*) - \frac{1}{4r}\|\operatorname{prox}_{-2rg_0} p_k^* - p_k^*\|_H^2 = g_0(\operatorname{prox}_{-2rg_0} p_k^*) - r\|\nabla g_r(p_k^*)\|_H^2 .$$

But (14) also yields

$$(23)\qquad g_r(\operatorname{prox}_{-2rg_0} p_k^*) \geqq g_0(\operatorname{prox}_{-2rg_0} p_k^*) .$$

Combining (22) with (23), we have

$$(24)\qquad g_r(\operatorname{prox}_{-2rg_0} p_k^*) \geqq g_r(p_k^*) + r\|\nabla g_r(p_k^*)\|_H^2 \quad \text{for all} \quad k .$$

From (24), the assertion of Lemma 2.1, and the fact that g_r is bounded above (since $\sup g_r = \mu$), we are able to conclude that

$$\lim_{k \to \infty} \|\nabla g_r(p_k^*)\|_H = 0 .$$

Since g_r is concave, the last equation implies that

$$\lim_{k\to\infty} g_r(p_k^*) = \sup g_r$$

and the proof of the theorem is complete. □

Let us now turn briefly our attention to the solution of extremal problems (18) (being also termed intermediate problems) in the case of our BP. If we denote $p^* = (\lambda, \mu) \in L_2[0, T, R^n] \times R^k$, the augmented Lagrangian attains the form

(25)
$$L_r(x(t), u(t), \lambda(t), \mu) = \varphi(x(T)) + \int_0^T \psi(x(t), u(t)\, \mathrm{d}t -$$
$$- \int_0^T \sum_{i=1}^m \lambda^i(t) \max\left\{\bar{q}^i(x)(t), \frac{\lambda^i(t)}{2r}\right\} \mathrm{d}t - \sum_{i=1}^k \mu^i \max\left\{a^i(x(T)), \frac{\mu^i}{2r}\right\} +$$
$$+ r\int_0^T \sum_{i=1}^n \max^2\left\{\bar{q}^i(x)(t), \frac{\lambda^i(t)}{2r}\right\} \mathrm{d}t + r\sum_{i=1}^k \max^2\left\{a^i(x(T)), \frac{\mu^i}{2r}\right\} \quad \text{if} \quad u \in \omega\,,$$
$$L_r(x(t), u(t), \lambda(t), \mu) = +\infty \quad \text{if} \quad u \notin \omega\,,$$

with the trajectory $x(t)$ corresponding to the control $u(t)$. Hence, these intermediate problems preserve the original optimal control structure. On the other hand, as they do not possess any state-space and terminal state constraints, they are much easier to be solved than the original BP.

The multiplier iterations (19) are in the case of BP of the form

$$\lambda_{k+1}^i(t) = \lambda_k^i(t) - 2r \max\left\{\bar{q}^i(x)(t), \frac{\lambda_k^i(t)}{2r}\right\}, \quad i = 1, 2, \ldots, m\,,$$
$$\mu_{k+1}^i = \mu_k^i - 2r \max\left\{a^i(x(T)), \frac{\mu_k^i}{2r}\right\}, \quad i = 1, 2, \ldots, k.$$

For a more detailed study about intermediate problems (18) and multiplier iterations (19) see [10]. Here we illustrate only this whole approach by a simple numerical example, where, for the intermediate problems a variant of steepest descent method was used and these minimization were performed "exactly" in order to get the information about the increase of the dual objective at every dual iteration.

Numerical example. Suppose, we have to solve BP with

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad x_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad J = \tfrac{1}{2}\|u\|_{L_2}^2\,,$$
$$\omega = \{u \mid -1 \leqq u(t) \leqq 1 \quad \text{a.e. on} \quad [0, T]\}\,, \quad T = 4\,,$$
$$x^2(t) \geqq -0{\cdot}7 \quad \text{on} \quad [0, T], \quad x(T) \leqq \Theta\,.$$

We have chosen $r = 5$, and initial values for the multipliers $\mu_0 = \Theta$, $\lambda_0(t) = 0$ on $[0, 4]$. All integrations were performed using the 3rd order variable step Runge--Kutta method with the overall permitted error $e_{max} = 10^{-4}$. Definite integrals were evaluated using the Simpson's rule. The iterational process was stopped after 5th dual iteration where $\|\lambda_6 - \lambda_5\| = 0{\cdot}000148$, $\|\mu_6 - \mu_5\|_{R^2} = 0{\cdot}000137$. In such a way the maximal violance of state-space constraints is specified.

Table 1. Values of μ and g_r.

	μ^1	μ^2	g_r
1. iter.	0·	0.	0·59801
2. iter.	−0·321	0.	0·60663
3. iter.	−0·348	0.	0·60677
4. iter.	−0·354	0.	0·60679
5. iter.	−0·355	0.	0·60684

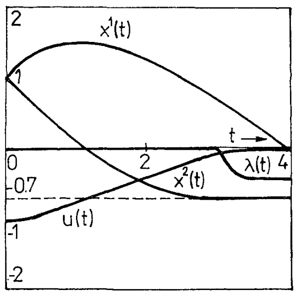

Fig. 1. Resulting control u, trajectories x^1, x^2 and multiplier λ.

3. PERTURBATIONS OF THE SYSTEM DYNAMICS

This approach is based on the perturbations of the equality constraint $y = \Pi u + y_0$. This time it is advantageous to set $y = (x(t), x(T))$, and to identify the space of trajectories with $L_2[0, T, R^n]$ which is clearly possible. Hence,

$$Y = L_2[0, T, R^n] \times R^n .$$

The control space U, constraint space H and ordering cone D remain as in the previous section.

Let us now denote $\tilde{q}[L_2[0, T, R^n] \times R^n \to L_2[0, T, R^m] \times R^k]$ a map defined by

$$\tilde{q}(y) = (\bar{q}(x)(t), a(x(T))) .$$

This map is clearly D-convex. Moreover, the operators $\bar{q}^i(x)(t)$, $i = 1, 2, \ldots, m$ are, as Nemycki operators, continuous from $L_2[0, T, R^n]$ into $L_2[0, T, R^1]$ iff there exist functions $\alpha^i \in L_2[0, T, R]$ and scalars $\beta^i \geqq 0$ such that

$$(26) \qquad |\bar{q}^i(x)| \leqq \alpha^i(t) + \beta^i(\sum_{j=1}^{n} (x^i)^2)^{1/2} \quad \text{for every} \quad x \in R^n , \quad i = 1, 2, \ldots, m .$$

As we wish $\tilde{q}$ to be continuous, we shall henceforth assume that the condition (26) is satisfied.

Let now $\tilde{J}(u, y)[L_2[0, T, R^l] \times L_2[0, T, R^n] \times R^n \to \bar{R}]$ denote such a proper convex, l.s.c. functional that

$$(27) \qquad \tilde{J}(u, y) = J(u, x(t)) \quad \text{for} \quad x(t) \in W_2^1[0, T, R^n] ,$$

and $\tilde{\Pi}[L_2[0, T, R^l] \to L_2[0, T, R^n] \times R^n]$ be a linear continuous operator given by system dynamic equation so that

$$(28) \qquad y = \tilde{\Pi}u + \tilde{y}_0 , \quad \tilde{y}_0 = (e^{At}x_0, e^{AT}x_0) .$$

The extremal problem

$$(29) \qquad \tilde{J}(u, y) \to \inf$$

$$\text{subj. to}$$

$$y = \tilde{\Pi}u + \tilde{y}_0 ,$$

$$u \in \omega ,$$

$$-\tilde{q}(y) \in D ,$$

has the general ACOCP form, is equivalent with BP in the specified sense and satisfies the assumptions (5).

Let us construct the dual extremal problem $(\mathscr{D}^2)$ to (29) exploiting the perturbed essential objective $G[L_2[0, T, R^l] \times L_2[0, T, R^n] \times R^n \to \bar{R}]$

$$(30) \qquad G(u, p) = \tilde{J}(u, \tilde{\Pi}u + \tilde{y}_0 - p) + \delta(u \mid \omega) + \delta(\tilde{\Pi}u + \tilde{y}_0 - p|\{y| - \tilde{q}(y) \in D\}) .$$

Assumptions being imposed guarantee that G is a proper convex, l.s.c. functional on $U \times Y$, i.e. on $L_2[0, T, R^l] \times L_2[0, T, R^n] \times R^n$.

We denote now

$$(31) \qquad A(u, y) = \tilde{J}(u, y) + \delta(u \mid \omega) + \delta(y|\{y| - \tilde{q}(y) \in D\}) ,$$

and assume that the nature of the given BP (more precisely its optimality criterion, control and state-space constraints) enables us to express explicitly the Fenchel--conjugate of Λ, i.e. the functional

$$(32) \qquad \Lambda^*(u^*, y^*) = \sup_{\substack{u \in \omega \\ -\tilde{q}(y) \in D}} [\langle u, u^* \rangle_U + \langle y, y^* \rangle_Y - \tilde{J}(u, y)]$$

defined, of course, on $U^* \times Y^* = (L_2[0, T, R^l] \times L_2[0, T, R^l] \times R^n)$.

The corresponding dual problem $(\mathscr{D}^2)$ attains now the form (cf. [2])

$$(33) \qquad -\langle p^*, \tilde{y}_0 \rangle_Y - \Lambda^*(\tilde{\Pi}^* p^*, -p^*) \to \sup$$

subj. to

$$p^* \in Y^* .$$

Let us denote

$$(34) \qquad \gamma(p) = \inf_{\substack{u \in \omega \\ -\tilde{q}(\tilde{\Pi} u + \tilde{y}_0 - p) \in D}} \tilde{J}(u, \tilde{\Pi} u + \tilde{y}_0 - p)$$

the perturbational functional corresponding to (30). We are again not able to prove its subdifferentiability at Θ in Y (D possesses no interior), but, under slightly more restrictive assumptions, its lower semi-continuity not only for $p = \Theta$ but over the whole space Y.

Theorem 3.1. Let in the case of unbounded ω the functional $\tilde{J}(u, \tilde{\Pi} u + \tilde{y}_0)$ be coercive on ω for all $\tilde{y} \in Y$ not only for $\tilde{y}_0 = (e^{At} x_0, e^{AT} x_0)$. Then the perturbational functional $\gamma(p)$ is l.s.c. on Y, which implies in particular that problem (29) is with respect to perturbations (30) normal.

Proof. The finiteness of $\gamma(p)$ at Θ is evident. Let us now assume that $\gamma(p)$ is at some p_0 not l.s.c., i.e. that in any open ball with the centre in p_0 there are points $\tilde{p}$ and an $\varepsilon > 0$ such that

$$\inf_u G(u, \tilde{p}) < \inf_u G(u, p_0) - \varepsilon = \gamma(p_0) - \varepsilon .$$

Let us take a sequence $\{\tilde{p}_n\}$ converging to p_0. It is possible to find points $\tilde{u}_n$ for which

$$G(\tilde{u}_n, \tilde{p}_n) < \gamma(p_0) - \varepsilon/2 .$$

The sequence $\{\tilde{u}_n\}$ is bounded; due to either the boundedness of ω or the above specified coercivity of $\tilde{J}$. Therefore, it is possible to select a subsequence $\{\tilde{u}_{n'}\}$ from $\{\tilde{u}_n\}$ weakly converging to some point $u_0 \in \omega$. In the weak topology functional $G(u, p)$ remains l.s.c. Thus,

$$G(u_0, p_0) \leqq \lim_{\substack{\tilde{u}_{n'} \xrightarrow{w} u_0 \\ \tilde{p}_{n'} \to p_0}} G(\tilde{u}_{n'}, \tilde{p}_{n'}) \leqq \gamma(p_0) - \varepsilon/2 \leqq G(u_0, p_0) - \varepsilon/2$$

what is the desired contradiction. □

The dual problem (33) may be implicitly constrained due to the incidental unboundedness of sets ω, $\Omega = \{y \mid -q(y) \in D\}$. However, as its solution exists, these sets may be substituted by

$$\omega' = \{u \in \omega \mid \|u\|_U \leqq M_1\}, \quad \Omega' = \{y \in \Omega \mid \|y\|_Y \leqq M_2\}, \tag{35}$$

where M_1, M_2 are sufficiently large real numbers. Indeed, in this case the functional Λ is cofinite and hence

$$\operatorname{dom} \Lambda^* = U^* \times Y^* .$$

In such a way implicit dual constraints will be eliminated.

At present there is a lot of numerical methods suitable for solving (33) and various subgradient techniques can very well handle its nondifferentiable cost. For this purpose we recommend [6], [7], [21] etc.

The necessary and sufficient optimality conditions can be stated in the form of the following assertion:

Theorem 3.2. An element $\hat{p}^* \in H$ is a solution of (33) iff there exists a control $\hat{u}$ such that

$$-\langle \hat{p}^*, \tilde{y}_0 \rangle - \Lambda^*(\tilde{\Pi}^* \hat{p}^*, -\hat{p}^*) = \Lambda(\hat{u}, \tilde{\Pi}\hat{u} + y_0) \tag{36}$$

which is equivalent to

$$(\tilde{\Pi}^* \hat{p}^*, -\hat{p}^*) \in \partial \Lambda(\hat{u}, \tilde{\Pi}\hat{u} + \tilde{y}_0), \tag{37}$$

or

$$(\hat{u}, \tilde{\Pi}\hat{u} + \tilde{y}_0) \in \partial \Lambda^*(\tilde{\Pi}^* \hat{p}^*, -\hat{p}^*). \tag{38}$$

For the proof it is necessary just to combine some basic propositions of convex analysis with the assumptions being imposed.

Remark. In another words the previous theorem states that, under specified conditions, $\hat{p}^*$ belongs to the set of solution of (33) iff among the couple

$$(\bar{u}, \bar{y}) = \arg \max_{\substack{u \in \omega \\ -\tilde{q}(y) \in D}} [\langle u, \tilde{\Pi}^* \hat{p}^* \rangle_U + \langle y, -\hat{p}^* \rangle_Y - \tilde{J}(u, y)] \tag{39}$$

there exists a couple $(\hat{u}, \hat{y})$ satisfying the relation $\hat{y} = \tilde{\Pi}\hat{u} + \tilde{y}_0$; $\hat{u}$ is then an optimal control for (29) and hence also for BP.

If the functional Λ can be decomposed into control- and state-dependent part Λ_1 and Λ_2, i.e.

$$\Lambda(u, y) = \tilde{J}_1(u) + \delta(u \mid \omega) + \tilde{J}_2(y) + \delta(y \mid \{y \mid -\tilde{q}(y) \in D\}) = \Lambda_1(u) + \Lambda_2(y), \tag{40}$$

an evident consequence of Theorem 3.2. can be formulated as follows:

Corrollary 3.2.1. In the case of Λ given by (40) an element $\hat{p}^* \in H$ is a solution of (33) if there exists a control $\hat{u}$ such that

$$(41) \qquad \hat{u} \in \partial \Lambda_1^*(\tilde{\Pi}^* \hat{p}^*) , \quad \tilde{\Pi}\hat{u} + \tilde{y}_0 \in \partial \Lambda_2^*(-\hat{p}^*) .$$

Theorem 3.2 and its Corrollary provide us in many cases with a satisfactory tool for computing a solution of (29) from a known solution $\hat{p}^*$ of (33). This inverse transformation is especially simple if all couples satisfying (39) have the same first element (control part) because then we immediately have for optimal $\hat{p}^*$ the optimal control $\hat{u}$. Such case appears if e.g. the optimality criterion $\tilde{J}$ is strictly convex with respect to u or does not depend on u at all ($\tilde{J}_1 \equiv 0$), ω is a strictly convex set and $\hat{p}^* \notin \ker \tilde{\Pi}^*$. Note that if $\hat{p}^* = \Theta$ then

$$\inf_{\substack{u \in \omega \\ -\tilde{q}(y) \in D}} J(u, y) = J(\hat{u}, \tilde{\Pi}\hat{u} + \tilde{y}_0) ,$$

so that the system dynamics equality constraint can be principially "taken off".

As it was mentioned above, it is generally necessary to apply for the solution of (33) some numerical procedure capable of handling nondifferentiable costs. But in some cases the dual cost in (33) is even Fréchet differentiable over Y^* or over some open subset of Y^*. These situations we have studied in [11] with the help of the concept of rotundity defined in [1]. To this sequel we include only three basic differentiability conditions obtained from more general statements of [11] for the case of problems (29), 33).

Theorem 3.3. Let $\tilde{J}(\cdot, \cdot)$ be strictly convex over its effective domain and elements $\bar{u} \in U$, $\bar{y} \in Y$ be (uniquely) determined by

$$(42) \qquad (\bar{u}, \bar{y}) = \arg\max_{\substack{u \in \omega \\ -\tilde{q}(y) \in D}} [\langle u, \tilde{\Pi}^* p^* \rangle_U + \langle y, - p^* \rangle_Y - \tilde{J}(u, y)] .$$

(Provided ω, Ω are, if necessary replaced by ω', Ω'). Then the objective in (33) is Fréchet differentiable over Y^* with

$$(43) \qquad \bar{v} = -\hat{y}_0 - \tilde{\Pi}\bar{u} + \bar{y}$$

being its gradient.

Theorem 3.4. Let the optimality criterion $\tilde{J}$ be only control-dependent, i.e. Λ can be written in the form (40) with $\tilde{J}_2(\cdot) \equiv 0$. Let $\tilde{J}_1(\cdot)$ be strictly convex over its effective domain, Ω be a strictly convex set, and elements $\bar{u}$, $\bar{y}$ be (uniquely) determined by

$$(44) \qquad \bar{u} = \arg\max_{u \in \omega} [\langle u, \tilde{\Pi}^* p^* \rangle_U - \tilde{J}_1(u)] , \quad \bar{y} \in C(-p^* \mid \Omega) .$$

(Provided ω, Ω are if necessary, replaced by ω', Ω'). Then the objective in (33) is Fréchet differentiable at all points except $p^* = \Theta$ with the gradient given by (43).

Theorem 3.5. Let the optimality criterion $\tilde{J}$ be only state-dependent, i.e. Λ can be written in the form (40) with $\tilde{J}_1(\cdot) \equiv 0$. Let $\tilde{J}_2$ be strictly convex over its effective domain, ω be a strictly convex set, and elements $\bar{u}$, $\bar{y}$ be (uniquely) determined by

$$\bar{u} \in C(\tilde{\Pi}^* p^* \mid \omega), \quad \bar{y} = \arg\max_{-\tilde{q}(y)\in D} [\langle y, -p^*\rangle - \tilde{J}_2(y)]. \tag{45}$$

(Provided ω, Ω are, if necessary, replaced by ω', Ω'). Then the objective in (33) is Fréchet differentiable at all points except $p^* \in \ker \tilde{\Pi}^*$ with the gradient given by (43).

We shall now illustrate this dualisation on one example for which the appropriate dual problem (33) in terms of our BP will be found.

Let the objective in BP be given by

$$\langle c, x(T)\rangle + \tfrac{1}{2}\langle u, u\rangle_{L_2} + \tfrac{1}{2}\langle x, x\rangle_{L_2}, \tag{46}$$

the set of admissible controls

$$\omega = \{u \in L_2[0, T, R^l] \mid |u^i(t)| \leqq 1, \quad \text{a.e. on} \quad [0, T], \quad i = 1, 2, \ldots, l\}, \tag{47}$$

and the state-space constraints be given by

$$x^i(t) \leqq b^i \quad \text{a.e. on } [0, T], \quad i = 1, 2, \ldots, n, \qquad x^i(t) \geqq d^i \quad \text{a.e. on } [0, T], \quad i = 1, 2, \ldots, n, \tag{48}$$

where $b^i > d^i$, $i = 1, 2, \ldots, n$ are some given scalars, and

$$\|x(T)\|_{R^n} \leqq \varrho, \quad 0 < \varrho \in R^1. \tag{49}$$

As the assumptions (i)–(v) and (vii) of Sec. 1 and the additional assumption of Sec. 3 are clearly satisfied, it remains merely to assume that also the controllability condition (vi) is fulfilled.

It can easily be derived that if $p^* = (p_1^*, p_2^*)$, $p_1^* \in L_2[0, T, R^n]$, $p_2^* \in R^n$ then

$$\tilde{\Pi}^* p^* = \int_t^T B^T(e^{A(\tau - t)})^T p_1^* \,d\tau + B^T(e^{A(T-t)})^T p_2^*. \tag{50}$$

By definition

$$\begin{aligned} \Lambda^*(\tilde{\Pi}^* p^*, -p^*) &= \Lambda_1^*(\tilde{\Pi}^* p^*) + \Lambda_2^*(-p^*) = \\ &= \sup_{u\in\omega} \int_0^T [\langle u, \tilde{\Pi}^* p^*\rangle_{R^l} - \tfrac{1}{2}\langle u, u\rangle_{R^l}] \,dt + \\ &+ \sup_{\substack{d^i \leqq x^i(t) \leqq b^i \\ i=1,2,\ldots,n}} \int_0^T [\langle x, -p_1^*\rangle_{R^n} - \tfrac{1}{2}\langle x, x\rangle_{R^n}] \,dt + \\ &+ \sup_{\|x(T)\| \leqq \varrho} [\langle x(T), -p_2^*\rangle_{R^n} - \langle x(T), c\rangle_{R^n}]. \end{aligned} \tag{51}$$

Let us denote for simplicity $v^* = \tilde{\Pi}^* p^*$ and introduce the sets

$$(52)\quad \begin{aligned} \varkappa_1^i &= \{t \in [0, T] \mid v^{i*}(t) < -1\},\\ \varkappa_2^i &= \{t \in [0, T] \mid -1 \leqq v^{i*}(t) \leqq 1\},\\ \varkappa_3^i &= \{t \in [0, T] \mid v^{i*}(t) > 1\}, \quad i = 1, 2, \ldots, l,\\ \xi_1^j &= \{t \in [0, T] \mid p_1^{j*}(t) < -d_j\},\\ \xi_2^j &= \{t \in [0, T] \mid -b_j \leqq p_1^{j*}(t) \leqq -\mathrm{d}^j\},\\ \xi_3^j &= \{t \in [0, T] \mid p_1^{j*}(t) < -b_j\}, \quad j = 1, 2, \ldots, n. \end{aligned}$$

The first two extremal subproblems in (51) are clearly solved by

$$(53)\quad \bar{u}^i = \begin{cases} -1 & \text{on } \varkappa_1^i \\ v^{i*} & \text{on } \varkappa_2^i \\ 1 & \text{on } \varkappa_3^i, \quad i = 1, 2, \ldots, l, \end{cases}$$

and

$$(54)\quad \bar{x}^j = \begin{cases} d^j & \text{on } \xi_1^j \\ -p_1^{j*} & \text{on } \xi_2^j \\ b^j & \text{on } \xi_3^j, \quad j = 1, 2, \ldots, n. \end{cases}$$

The third one attains its maximum at

$$(55)\quad \bar{x}(T) = -\varrho \frac{c + p_2^*}{\|c + p_2^*\|_{R^n}} \quad \text{if} \quad p_2^* \neq -c,$$

and at any feasible $\bar{x}(T)$ if $p_2^* = -c$. Thus,

$$\sup_{u \in \omega} \int_0^T [\langle u, v^* \rangle_{R^l} - \tfrac{1}{2}\langle u, u \rangle_{R^l}] \, \mathrm{d}t = \sum_{i=1}^{l} r_i^*(v^{i*}),$$

where

$$r_i^*(v^{i*}) = \int_{\varkappa_1^i} (-v^{i*} - \tfrac{1}{2}) \, \mathrm{d}t + \tfrac{1}{2} \int_{\varkappa_2^i} (v^{i*})^2 \, \mathrm{d}t + \int_{\varkappa_3^i} (v^{i*} - \tfrac{1}{2}) \, \mathrm{d}t, \quad i = 1, 2, \ldots, l,$$

$$\sup_{\substack{d^j \leqq x^j(t) \leqq b^j \\ j = 1,2\ldots n}} \int_0^T [\langle x, -p_1^* \rangle_{R^n} - \tfrac{1}{2}\langle x, x \rangle_{R^n}] \, \mathrm{d}t = \sum_{j=1}^{n} s_j^*(p_1^{j*}),$$

where

$$s_j^*(p_1^{j*}) = \int_{\xi_1^j} (d^j p_1^{j*} - \tfrac{1}{2}(d^j)^2) \, \mathrm{d}t + \tfrac{1}{2} \int_{\xi_2^j} (p_1^{j*})^2 \, \mathrm{d}t + \int_{\xi_3^j} (-b^j p_1^{j*} - \tfrac{1}{2}(b^j)^2) \, \mathrm{d}t,$$

$$j = 1, 2, \ldots, n,$$

and

$$\sup_{\|x(t)\| \leqq \varrho} [\langle x(T), -p_2^* \rangle_{R^n} - \langle x(T), c \rangle_{R^n}] = \varrho \|c + p_2^*\|_{R^n}.$$

 Dual problem (33) attains now the form

$$(56)\qquad -\int_0^T \langle e^{At}x_0, p_1^*\rangle_{R^n}\,dt - \langle e^{AT}x_0, p_2^*\rangle_{R^n} - \sum_{i=1}^{l} r_i^*(v^{i*}) - \sum_{j=1}^{n} s_j^*(p_1^{j*}) -$$

$$- \varrho\|c + p_2^*\|_{R^n} \to \sup$$

subj. to

$$p_1^* \in L_2[0, T, R^n]\,, \quad p_2^* \in R^n\,.$$

None from three differentiability conditions stated above is directly applicable to (56). However, the following assertion is true:

Theorem 3.6. The dual cost in (56) is Fréchet differentiable at all couples $(p_1^*, p_2^*) \in H$ except of those where $p_2^* = -c$. The gradient is given by

$$\bar{v} = -\tilde{y}_0 - \tilde{\Pi}\bar{u} + \bar{y}$$

where $\bar{u}$ is given by (53) and $\bar{y}$ by (54), (55).

Proof. With respect to the theory developed in [11] it remains to prove that the infimum of the functional $\gamma(\cdot) - \langle\cdot, p^*\rangle$ is attained strongly at any p^* with $p_2^* \neq c$ (γ is the corresponding perturbational functional) with respect to the norm topology of Y. Clearly,

$$\inf_{p\in Y}[\gamma(p) - \langle p, p^*\rangle_Y] = \inf_{p\in Y}\Big[\inf_{\substack{u\in\omega\\ -\tilde{q}(\tilde{\Pi}u+\tilde{y}_0-p)\in D}} \tilde{J}(u, \tilde{\Pi}u + \tilde{y}_0 - p) - \langle p, p^*\rangle_Y\Big] =$$

$$= \inf_{y\in Y}\Big[\inf_{\substack{u\in\omega\\ -\tilde{q}(y)\in D}} \tilde{J}(u, y) - \langle\tilde{\Pi}u + \tilde{y}_0 - y, p^*\rangle_Y\Big] =$$

$$= -\sup_{\substack{u\in\omega\\ -\tilde{q}(y)\in D}}\big[\langle u, \tilde{\Pi}^*p^*\rangle_U - \langle y, p^*\rangle_Y - \tilde{J}(u, y)\big] - \langle\tilde{y}_0, p^*\rangle_Y =$$

$$= -\sup_{u\in\omega}\int_0^T\big[\langle u, \tilde{\Pi}^*p^*\rangle_{R^l} - \tfrac{1}{2}\langle u, u\rangle_{R^l}\big]\,dt -$$

$$-\sup_{\substack{d^i\leq x^i(t)\leq b^i\\ i=1,2,\ldots,n}}\int_0^T\big[\langle x, p_1^*\rangle_{R^n} - \tfrac{1}{2}\langle x, x\rangle_{R^n}\big]\,dt - \sup_{\|x(T)\|\leq\varrho}\langle x(T), -p_2^* - c\rangle_{R^n} -$$

$$-\int_0^T\langle e^{At}x_0, p_1^*\rangle_{R^n}\,dt - \langle e^{AT}x_0, p_2^*\rangle_{Rn}\,.$$

The suprema in the first and second term are attained strongly with respect to norm topologies in $L_2[0, T, R^l]$ and $L_2[0, T, R^n]$ due to the strict convexity of the corresponding quadratic forms at $\bar{u}$ and $\bar{x}$, respectively. The supremum of the third term is attained strongly at $\bar{x}(T)$ due to the nature of the support function of the ball $\|x(T)\| \leq \varrho$ whenever $p_2^* \neq -c$. □

Still one aspect of this approach seems to be worth mentioning. At evaluation of Λ^* and $\tilde{\Pi}^* p^*$ some methods of numerical quadrature are to be applied. In the case of the rectangular rule, we obtain finally a piecewise constant approximation of an actual optimal control. This corresponds to the case, if, by using some primal method, the Euler integration rule would be applied for the solution of the system equation and the rectangular rule for the evaluation of the objective. For more advanced quadrature formulae better approximations of an actual optimal control can be obtained and their appropriate choice should be further investigated.

The differentiable case will now be illustrated by the following numerical example where, for the solution of $(\mathscr{D}^2)$, the well-known Polak - Ribière - Powell conjugate gradient algorithm was applied.

Numerical example. Suppose we have to solve BP with

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad x_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

$$J = \tfrac{1}{2}\|x\|^2_{L_2} + \tfrac{1}{2}\|u\|^2_{L_2},$$

$$\omega = \{u \mid -1 \leqq u(t) \leqq 1 \quad \text{a.e. on } [0, T]\}, \quad T = 4,$$

$$x^2(t) \geqq -0{\cdot}3 \quad \text{on} \quad [0, T].$$

The interval $[0, 4]$ was discretized into 80 equidistant subintervals, the definite integrals were evaluated using the rectangular rule, the maximization procedure was

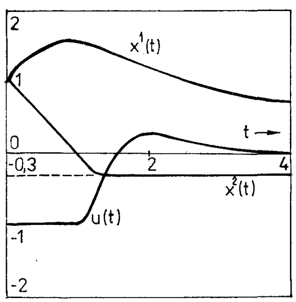

Fig. 2. Resulting control u and trajectories x^1, x^2.

started from $p^* = \Theta$ and stopped at the moment when no increase of the objective in the gradient direction could be found due to the errors. At this moment the value of the objective was 5·3053.

4. CONCLUSION

It is far not yet clear, whether, for a given problem, a primal or dual technique will be better and which particular method suits its structure best. This analysis offers an important and attractive research area for the future and is especially important if we wish to control some nonlinear plant by means of the convex feedback method as described in [3].

The first approach described in this sequel is more universal; on the other hand usually more intermediate problems are to be solved even if the dual iterational scheme converges relatively fast. At any dual iteration we have an admissible control but the corresponding trajectory may not neccessarily satisfy the state-space constraints. The Fréchet gradient of the dual objective at some multiplier p^* is given by (15) and equals such a perturbation vector $-\bar{p}$ that $\bar{p}$ solves the extremal problem (9). In other words, this gradient measures in a certain sense the violance of state-space constraints. If the structure of ω is sufficiently complicated, it is possible to treat the control constraints also by means of the generalized penalty term but we do not recommend it for the case of upper and lower bounds or affine inequality constraints (cf. [19]).

The second approach requires much more special assumptions so that its applicability is substantially narrower. Its advantage over the first one is that we need to perform merely one unconstrained minimization over Y. For any multiplier p^* the current control and trajectory are admissible with respect to the appropriate constraints but they do not satisfy the system dynamic equation. The appropriate difference in Y provide us in the differentiable case with the Fréchet gradient of the dual objective.

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