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CONTROLLABILITY INDICES OF LINEAR SYSTEMS WITH DELAYS[1]

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The purpose of this paper is to exhibit new lists of controllability indices relative to $\mathbb{R}^n$-controllability and controllability over the field $\mathbb{R}(\nabla)$, for linear systems with delays. In fact, as for linear systems without delays, we define two lists of indices for each notion of controllability. Moreover we define the bijection which links both lists for each type of controllability. Finally, using controllability indices over $\mathbb{R}(\nabla)$, we give some further informations about coefficient assignment by state feedback, particulary about the polynomial forms the coefficients can take.

1. INTRODUCTION

Controllability and controllability indices [11] are completely characterized for linear continuous systems without delay. It is natural to say that such system is (state) controllable if, by suitable choice of its inputs, the state can be made to behave in some desirable way, and this in a time as short as possible. Moreover two dual lists of controllability indices have been defined ([11]). In the case of linear systems with delays, controllability takes lots of different forms according to the system representation and the practical properties we search ([6,7,10,12]). This paper is focused on two classical kinds of controllability according to the modelization we adopt for linear continuous systems with delays: $\mathbb{R}^n$-controllability and controllability over the field $\mathbb{R}(\nabla)$ of rational functions with real coefficients.

For each case we define two new lists of controllability indices, taking account the characterization of a system with delays, say that the time plays a key role when controllability is concerned.

With this in view, we first define, for each type of controllability respectively, two new specific notions for linear continuous systems with delays: the classes and the orders of these systems, which are characteristic of the way the delays contribute to the definition of controllability. Next the original point of the determination of the new controllability indices is that they are exhibited, according to these classes or orders.

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These new controllability indices can be used in order to analyze control problems. As illustration, we show that the possible transformations by state feedback of the coefficients of the characteristic polynomial depend on the controllability indices over $I\!R(\nabla)$.

2. PROBLEM STATEMENT AND STATE OF THE ART

State controllability of linear systems with delays is characterized through a lot of definitions, according to the system representation, i.e. the way the delay is modelled.
(For instance the system may have commensurate, non commensurate, distributed delays ,...). Furthermore, each kind of controllability appearing in the litterature corresponds to different practical properties of states trajectories.

Consider a square, linear, time-invariant system having commensurate delays in state, inputs and outputs (all delays are multiple of a unit one h):

$$\sum \begin{cases} \dot{x}(t) = A_0\, x(t) + A_1\, x(t-h) + \cdots + A_a\, x(t-ah) \\ \qquad\qquad +B_0\, u(t) + B_1\, u(t-h) + \cdots + B_b\, u(t-bh) \\ y(t) = C_0\, x(t) + C_1\, x(t-h) + \cdots + C_c\, x(t-ch) \end{cases} \tag{2.1}$$

with $x(t) \in I\!R^n$, $u(t) \in I\!R^m$, $y(t) \in I\!R^p$, a, b, and $c \in I\!N$, $h \in I\!R$.

Usually this system is rewritten using the operator ∇ defined by $\nabla\, x(t) = x(t-h)$ ([6]). We then consider the following representation for $\sum$:

$$\sum \begin{cases} \dot{x}(t) = A(\nabla)\, x(t) + B(\nabla)\, u(t) \\ y(t) = C(\nabla)\, x(t), \end{cases} \tag{2.2}$$

where $A(\nabla) = A_0 + \nabla A_1 + \cdots + \nabla^a A_a$, $B(\nabla) = B_0 + B_1 + \cdots + \nabla^b B_b$ and $C(\nabla) = C_0 + \nabla C_1 + \cdots + \nabla^c C_c$, are matrices over the ring $I\!R[\nabla]$ of polynomials in ∇ with real coefficients.

The controllabilty submodule $\langle A(\nabla)/\mathrm{Im}\, B(\nabla)\rangle$ of the pair $(A(\nabla), B(\nabla))$ of (2.2) is given by [6]:

$$\langle A(\nabla)/\mathrm{Im}\, B(\nabla)\rangle = \mathrm{Im}\, B(\nabla) + A(\nabla)\,\mathrm{Im}\, B(\nabla) + \cdots + A^{n-1}(\nabla)\,\mathrm{Im}\, B(\nabla). \tag{2.3}$$

Its matrix representation is:

$$\langle A(\nabla)/B(\nabla)\rangle = \left[B(\nabla)\,|\,A(\nabla)\,|\cdots|\,A^{n-1}(\nabla)\,A(\nabla)\right]. \tag{2.4}$$

Finally we note $\underline{m}$, the set of integers $\{1, 2, \ldots, m\}$ and $\{n_i^j\}_m$, for all $j \in I\!N$, the list of integers $n_i^j \in \underline{m}$. M^T is the transpose matrix of the matrix M and $C^0([T_1, T_2]; R)$ the set of continuous functions defined on $[T_1, T_2]$ with values in $I\!R$. $\mathcal{E}$ stands for $\mathrm{Im}\, E$, the image of any application E and I_n for the identity matrix of dimension n.

In this part we only present the three main characterizations of the concept of controllability.

2.1. The $I\!R^n$-controllability

The $I\!R^n$-controllability has been studied since the end of the sixties ([10, 12]) and is related to the representation (2.1). The solution of the state equations (2.1) needs the initial conditions $x(0) = x_0$ and $x(t) = \varphi(t)$ for $-ah \le t < 0$.

Definition 2.1.1. ([10], [12]) The system (2.1) is $I\!R^n$-*controllable* at $t_0 = 0$ if, whatever $\varphi \in C^0[-ah, 0]$, $x_0 = x(0)$ and x_1, there exists $t_1 > 0$ and a control $u(t)$, $t \in [0, t_1]$, such that $x(t_1) = x_1$.

Remark 2.1.1.

- if $x_1 = 0$ the controllability is called controllability to the origin.
- this controllability expresses the only x_1 (or zero)-crossing of the state.

In order to characterize the $I\!R^n$-controllability of (2.1), we need the following matrices of $R^{n\times m}$ ([10]),

$$Q_{k+1}(j) = \sum_{i=0}^{R} A_i\, Q_k(j-i) \tag{2.5}$$

with

$$\begin{cases} Q_0(i) = B_i, & i = 0, 1, \ldots, R \\ Q_k(j) = 0 & \text{if } k < 0 \text{ or } j < 0, \end{cases}$$

where $R = \max\{a, b\}$, $A_i = 0$ for $i > a$ and $B_i = 0$ for $i > b$.

These matrices have the following properties ([9, 10]):

i) $Q_k(j) = 0$ for $j > (k+1)\,R$,

ii) $\forall\, j \in I\!N$, $\quad Q_n(j) = \sum_{i=0}^{n-1} \sum_{p=0}^{j} \alpha_{ipj}\, Q_i(p)$, where $\alpha_{ipj} \in I\!R$.

Remark 2.1.2. It is proved ([6, 9]) that the matrices $Q_k(j)$ associated with (2.1) and the moments $A(\nabla)^i\, B(\nabla)$, of the controllability submodule, associated with (2.2), are linked by:

$$\forall\, i \in I\!N, \quad A^i(\nabla)\, B(\nabla) = \sum_{j=0}^{(i+1)\,R} Q_i(j)\, \nabla^j. \tag{2.6}$$

We then have ([10]):

Theorem 2.1.2. The system (2.1) is $I\!R^n$-controllable if and only if:

$$\text{rank}\,[Q_0(0),\, Q_0(1), \ldots, Q_0(R),\, Q_1(0),\, Q_1(1), \ldots, Q_1(2R), \ldots \\ \ldots, Q_{n-1}(0), \ldots, Q_{n-1}(nR)] = n.$$

Remark 2.1.3. This condition is only sufficient for the $I\!R^n$-controllability to the origin ([10]).

2.2. Controllability over $\mathbb{R}[\nabla]$

This controllability, defined over $\mathbb{R}[\nabla]$, the ring of polynomials in ∇ with real coefficients ([6, 8]) is the extension of the usual notion of controllability established for linear systems without delays.

Definition 2.2. ([4,6]) The system (2.2) is controllable over the ring $\mathbb{R}[\nabla]$ if and only if its controllability submodule satisfies the following equivalent characterizations:

(i) span $\langle A(\nabla)/\mathrm{Im}\, B(\nabla)\rangle = \mathbb{R}^n[\nabla]$.

(ii) the Smith form of $\langle A(\nabla)/B(n)\rangle$ is $[I_n|0]$.

(iii) $\langle A(\nabla)/B(\nabla)\rangle$ has a right inverse $Q(\nabla)$ over $\mathbb{R}[\nabla]$.

An interpretation of this controllability is that, like for linear systems without delays, a system (2.2) which is controllable over the ring $\mathbb{R}[\nabla]$ can reach, from $x(0)$, any state x_1 at a given time T, with T *as small* as possible (and "ad hoc" control $u(t)$, $t \in [0,T]$).

This similarity between linear systems with delays and linear systems without delays uniquely concerns controllability properties and is not always true for control problems (decoupling, coefficient assignment,...).

2.3. Controllability over the field $\mathbb{R}(\nabla)$

Let us first note that two submodules of $\mathbb{R}^n[\nabla]$, $\mathcal{S}_1$ and $\mathcal{S}_2$, such that $\mathcal{S}_2 \subset \mathcal{S}_1$ and $\dim \mathcal{S}_1 = \dim \mathcal{S}_2$, may be different.

This type of controllability includes the previous one but its interpretation is completely different. In this case we cannot reach from $x(0)$ any state x_1 at a given time T, with T as small as possible; in fact a minimal time will be neccessary to reach x_1. A trivial example of these systems is a system with pure delays in controls.

Definition 2.3. ([4,6]) The system (2.2) is controllable over the field $\mathbb{R}(\nabla)$ if its controllability submodule satisfies the following equivalent characterizations:

(i) rank $\langle A(\nabla)/B(\nabla)\rangle = n$.

(ii) the Smith form of $\langle A(\nabla)/B(\nabla)\rangle$ has non zero elements on its diagonal.

(iii) $\langle A(\nabla)/B(\nabla)\rangle$ has a right inverse $Q(\nabla)$ over $\mathbb{R}(\nabla)$.

Remark 2.3. This definition is less restrictive than Definition 2.2. Its practical interpretation will be deduced from the controllability indices we define later.

Finally, as a controllable system over the ring $\mathbb{R}[\nabla]$ is a particular case of controllable systems over the field $\mathbb{R}(\nabla)$, we will then only lean on both other types of controllability, say $\mathbb{R}^n$-controllability and controllability over the field $\mathbb{R}(\nabla)$.

3. CONTROLLABILITY INDICES OF SYSTEMS WITH DELAYS

We first define here two new notions, the class and the order of respectively (2.1) and (2.2). These notions are specific to the presence of delays and characterize the fact that the time may arise when controllability is concerned. These classes and orders are next used to define two new lists of controllability indices for each kind of controllability defined in the previous part.

3.1. Controllability indices relative to $\mathbb{R}^n$-controllability

In the following we note, by (2.5):

$$\mathcal{Q}_i(k) = \operatorname{Im} Q_i(k), \quad \text{for } i = 0, 1, \dots, n-1, \text{ for all } k \in \mathbb{N}.$$

We then first define the nxnm real matrix M_k and the subspace $\mathcal{M}_k$, for all $k \in \mathbb{N}$, by:

$$\begin{aligned} M_k &= [Q_0(k),\, Q_1(k), \dots, Q_{n-1}(k)] \\ \mathcal{M}_k &= [\mathcal{Q}_0(k) + \mathcal{Q}_1(k) + \dots + \mathcal{Q}_{n-1}(k)] = \operatorname{Im}(M_k). \end{aligned} \tag{3.1}$$

We then define the notion of *class* as follows:

Definition 3.1.1. We say that the real matrix M_k represents the class k of (2.1), $\forall\, k \in \mathbb{N}$.

By (2.6) (Remark 2.1.2), these previous matrices can be used to develop the matrix $\langle A(\nabla)/B(\nabla)\rangle$, defined by (2.4), as:

$$\langle A(\nabla)/B(\nabla)\rangle = M_0 + M_1\,\nabla + \dots + M_k\,\nabla^k + \dots, \tag{3.2}$$

where M_k is d fined by (3.1).

A natural interpretation of these classes is the following:

- the class 0 represents the states $x(t_1)$ that we can reach at $t_1 = 0 + \varepsilon$, from $x(0) = 0$, ε being as small as possible,
- more generally, the set of classes $\{0, 1, \dots, i\}$ represents all the states $x(t_1)$ that we can reach from $x(0) = 0$ at $t_1 = i \cdot h + \varepsilon$.

Remark 3.1.1. This notion of class may be generalized to any polynomial matrix $H(\nabla) = H_0 + H_1\,\nabla + \dots + H_i\,\nabla^i + \dots$, where H_i, $i \in \mathbb{N}$, is a real matrix.

3.1.1. "First type" controllability indices

In the case of linear system described by $\dot{x}(t) = A\,x(t) + B\,u(t)$, where $A \in \mathbb{R}^{n\times m}$ and $B \in \mathbb{R}^{n\times m}$, some invariants, called "of the first type", are defined by [11]:

$$p_1 = d(\mathcal{B}); \quad p_i = d\left(\frac{\mathcal{B} + A\mathcal{B} + \dots + +A^{i-1}\mathcal{B}}{\mathcal{B} + A\mathcal{B} + \dots A^{i-2}\mathcal{B}}\right), \quad i = 1, 2, \dots, n. \tag{3.3}$$

We will use now the classes of (2.1) in order to define, in a similar way to (3.3), new controllability indices by the following procedure.

First, we define the indices of class 0 as follows: for $i \in \underline{n}$, each index, p_i^0 (of class 0), corresponds to the increasing of the column rank between $\{Q_{i-1}(0), Q_{i-2}(0), Q_{i-3}(0), \ldots, Q_0(0)\}$ and $\{Q_{i-2}(0), Q_{i-3}(0), \ldots, Q_0(0)\}$.

More generally we define the indices of class k: for $i \in \underline{n}$, each index, p_i^k (of class k), corresponds to the increasing of the column rank between $\{Q_{i-1}(k), Q_{i-2}(k), \ldots, Q_0(k), M_{k-1}, M_{k-2}, \ldots, M_0\}$ and $\{Q_{i-2}(k), Q_{i-3}(k), \ldots, Q_0(k), M_{k-1}, M_{k-2}, \ldots, M_0\}$.

Note

$$\sigma = \operatorname{rank}[Q_0(0), Q_0(1), \ldots, Q_0(R), Q_1(0), \ldots \\ \ldots, Q_1(2R), \ldots, Q_{n-1}(0), \ldots, Q_{n-1}(nR)]. \tag{3.4}$$

Definition 3.1.2. The n "first type" controllability indices of class k are defined, for all $k \in I\!N$, by:

$$\begin{aligned} p_1^k &= d\left(\frac{\mathcal{M}_0 + \mathcal{M}_1 + \cdots + \mathcal{M}_{k-1} + \mathcal{Q}_0(k)}{\mathcal{M}_0 + \mathcal{M}_1 + \cdots + \mathcal{M}_{k-1}}\right) \\ p_i^k &= d\left(\frac{\mathcal{M}_1 + \mathcal{M}_1 + \cdots \mathcal{M}_{k-1} + \mathcal{Q}_0(k) + \mathcal{Q}_1(k) + \cdots + \mathcal{Q}_{i-1}(k)}{\mathcal{M}_1 + \mathcal{M}_1 + \cdots + \mathcal{M}_{k-1} + \mathcal{Q}_0(k) + \mathcal{Q}_1(k) + \cdots + \mathcal{Q}_{i-2}(k)}\right), \\ & \qquad i = 2, 3, \ldots, n. \end{aligned}$$

Remark 3.1.2.

- We note $\{p_i^k\}_n$ the list of these indices of class k, for all $k \in I\!N$.
- For systems without delays, the "first type" indices are completely characterized by the class 0.
- For all $k \in I\!N$ these indices are such that

$$\sigma_k := \sum_{c=0}^{k} \sum_{i=1}^{n} p_i^c = d(\mathcal{M}_0 + \mathcal{M}_1 + \cdots + \mathcal{M}_k) = \operatorname{rank}[Q_0(0), Q_1(0), \ldots \\ \ldots Q_{n-1}(0), Q_0(1), \ldots, Q_{n-1}(1), \ldots, Q_0(k), \ldots, Q_{n-1}(k)].$$

In order to determine the complete list of "first type" controllability indices, we consider the dimension σ_k: if $\sigma_k = \sigma$, the list is complete; if *not* we must continue this procedure and study the class $k+1$ of the system and so on. This procedure is convergent, because $\{\sigma_k\}_{k \in I\!N}$ is a non-decreasing series bounded by σ. Hence there always exists an integer $K \leq nR$ ((2.5)) such that:

$$\sum_{c=0}^{K} \sum_{i=1}^{n} p_i^c = \operatorname{rank}[Q_0(0), Q_1(0), \ldots, Q_{n-1}(0), Q_0(1), \ldots, Q_{n-1}(1), \ldots \\ \ldots Q_0(K), \ldots, Q_{n-1}(K)] = \sigma.$$

These integers p_i^c allow us to characterize the $I\!R^n$-controllability of (2.1) as follows:

Proposition 3.1.3. The system (2.1) is $\mathbb{R}^n$-controllable if and only if there exists an integer $K \leq nR$ such that:

$$\sum_{c=0}^{K}\sum_{i=1}^{n} p_i^c = n.$$

The proof is straightforward by the Remark 3.1.2 and by Theorem 2.1.2. □

3.1.2. "Second type" controllability indices

In the case of linear system described by $\dot{x}(t) = A\,x(t) + B\,u(t)$, some invariants "of the second type", are defined by [11]:

$$n_i = \min\{p \in \mathbb{N} \text{ such that } A^p\, b_i \text{ belongs to the subspace generated by the columns } A^p\, b_{i-1}, \ldots, A^p\, b_1,\ A^{p-1}\, b_m, \ldots, A^{p-1}\, b_1, \ldots, b_m, \ldots, b_1\}.$$

In a same spirit, we can define a "second list" of controllability indices for systems with delays.

In this list, each index n_i^k, for $i \in \underline{m}$ and $j \in \mathbb{N}$, is associated with the ith column of $Q_0(k)$ and corresponds (for a class k) to the number of independent columns generated by this ith input in the subspace $\mathcal{M}_k$, $\mathcal{M}_{k-1}$, $\mathcal{M}_{k-2}, \ldots, \mathcal{M}_0$.

$Q_j^i(k)$ representing the ith column of $Q_j(k)$, we denote $\mathcal{X}_{j=1}^{i-1}(k)$ the subspace generated by:

$$[Q_{j-1}^{i-1}(k), \ldots, Q_{j-1}^{1}(k),\, Q_{j-2}^{m}(k), \ldots Q_{j-2}^{1}(k), \ldots, Q_0^m(k), \ldots$$
$$\ldots, Q_0^1(k),\, Q_{n-1}^m(k-1), \ldots, Q_0^1(k-1), \ldots, Q_{n-1}^m(0), \ldots, Q_0^1(0)]$$

Then, we can define the "second type" controllability indices of class k, $k \in \mathbb{N}$. (Some developments relative to this definition, which are only technical, can be found in [9] with illustrative examples):

Definition 3.1.4. $\{n_i^k\}_m$ is the list of "second type" controllability indices of class k, for all $k \in \mathbb{N}$, with

$$n_i^k \triangleq \overline{q}_i^k - g_i^k, \quad i \in \underline{m},$$

where $\overline{q}_i^k = \max\{j \in \mathbb{N}$ such that $Q_{j-1}^i(k) \notin \mathcal{X}_{j-1}^{i-1}(k)\}$, and $q_i^k \triangleq \text{card}\{j \in (1, 2, \ldots, \overline{q}_i^k - 1)$ such that $Q_{j-1}^i(k)$ is a linear combination of the columns of the matrices $Q_{n-1}(k-1), \ldots, Q_0(k-1), \ldots, Q_{n-1}(0), \ldots, Q_0(0)\}$.

Remark 3.1.3. Some of the indices n_i^k, $i \in \underline{m}$, may be zero.

By construction there exists an integer N such that

$$\sum_{c=0}^{N}\sum_{i=1}^{m} n_i^c = \sigma, \tag{3.5}$$

where σ is defined by (3.4).

We can then characterize the $\mathbb{R}^n$-controllability of (2.1) by:

Proposition 3.1.5. The system (2.1) is $\mathbb{R}^n$-controllable if and only if there exists $N \in \mathbb{N}$ such that:

$$\sum_{c=0}^{N}\sum_{i=1}^{m} n_i^c = n.$$

The proof is straightforward by (3.5). □

We so define two new lists of controllability indices (by classes) relative to the $\mathbb{R}^n$-controllability. Moreover we can exhibit the bijection which links these two lists of integers.

Proposition 3.1.6. The lists $\{p_i^k\}_n$ and $\{n_i^k\}_m$ are linked, for all $k \in \mathbb{N}$, by:

$$\sum_{i=1}^{m} n_i^k = \sum_{i=1}^{n} p_i^k.$$

Proof. This proposition is true for the class 0, because:

$$\sum_{i=1}^{m} n_i^0 = \operatorname{rank}\left[Q_0(0),\, Q_1(0), \dots, Q_{n-1}(0)\right] = d(\mathcal{M}_0) = \sum_{i=1}^{n} p_i^0.$$

For $k \neq 0$, we have:

$$\begin{aligned}
\sum_{i=1}^{n} p_i^k &= d\left(\frac{\mathcal{M}_0 + \mathcal{M}_1 + \cdots + \mathcal{M}_k}{\mathcal{M}_0 + \mathcal{M}_1 + \cdots + \mathcal{M}_{k-1}}\right)\\
&= \operatorname{rank}\left[Q_0(0), Q_1(0), \dots, Q_{n-1}(0), Q_0(1), \dots, Q_{n-1}(1), \dots, Q_0(k), \dots, Q_{n-1}(k)\right]\\
&\quad - \operatorname{rank}\left[Q_0(0), Q_1(0), \dots, Q_{n-1}(0), Q_0(1), \dots, Q_{n-1}(1), \dots, Q_0(k-1), \dots, Q_{n-1}(k-1)\right]\\
&= \sum_{i=1}^{m} n_i^k.
\end{aligned}$$

□

Remark 3.1.4. This bijection is always true for linear systems without delays (in this case, we only consider $k = 0$).

The following example illustrates the previous results relative to $\mathbb{R}^n$-controllability.

Example 3.1. Let us consider the system

$$A_0 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},\quad A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},\quad B_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix},\quad B_1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

With the previous notations,

$$M_0 = \left[\begin{array}{cc|cc|cc|cc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots \\ 0 & 1 & 0 & 0 & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right],$$

$$M_1 = \left[\begin{array}{cc|cc|cc|cc} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{array}\right],$$

$$M_0 = [\,Q_0(0)\,|\,Q_1(0)\,|\,Q_2(0)\,|\,Q_3(0)\,], \quad M_1 = [\,Q_0(1)\,|\,Q_1(1)\,|\,Q_2(1)\,|\,Q_3(1)\,].$$

We so have (Theorem 2.1.2):

$$\text{rank}[Q_0(0),\ Q_1(0),\ Q_2(0),\ Q_3(0),\ Q_0(1),\ Q_1(1),\ Q_2(1),\ Q_3(1)] = 4.$$

The system is so $\mathbb{R}^n$-controllable.

First part: We now apply the procedure of the Definition 3.1.2 to determine the "first type" controllability indices.

$$p_1^0 = d(\mathcal{Q}_0(0)) = 2, \quad p_2^0 = d\left(\frac{\mathcal{Q}_0(0) + \mathcal{Q}_1(0)}{\mathcal{Q}_0(0)}\right) = 0, \quad p_3^0 = p_4^0 = 0.$$

The indices of *class* 0 are such that $\sum_{i=1}^{n} p_i^0 = 2 < 4$.

We now determine the indices of class 1.

$$p_1^1 = d\left(\frac{\mathcal{M}_0 + \mathcal{Q}_0(1)}{\mathcal{M}_0}\right) = 1, \quad p_2^1 = d\left(\frac{\mathcal{M}_0 + \mathcal{Q}_0(1) + \mathcal{Q}_1(1)}{\mathcal{M}_0 + \mathcal{Q}_0(1)}\right) = 1, \quad p_3^1 = p_4^1 = 0.$$

The indices of *class* 1 are: $1, 1, 0, 0$. We so have:

$$\sum_{i=1}^{n} p_i^0 + \sum_{i=1}^{n} p_i^1 = 4.$$

The list of "first type" controllability indices is so entirely complete.

Second part: Following the Definition 3.1.4 and by the forms of M_0 and M_1 we have: $Q_0^1(0)$ and $Q_0^2(0)$ are the only independent columns of M_0. Thus,

$$n_1^0 = 1 \quad \text{and} \quad n_2^0 = 1.$$

$Q_0^1(1)$ is the last independent column generated by the first input of M_1. So $\overline{q}_1^1 = 1$ and $q_1^1 = 0$, thus:

$$n_1^1 = 1.$$

Moreover $Q_1^2(1)$ is the last independent column generated by the second input of M_1. Hence $\overline{q}_2^1 = 2$. As $Q_0^2(1) = 0$, then it is a linear combination of any $Q_i^2(0)$, $i = 1, 2, 3, 4$. Hence $q_2^1 = 1$, and so:

$$n_2^1 = 1.$$

So $n_1^0 + n_2^0 + n_1^1 + n_2^1 = 4 = n$.

The list of controllability indices of the second type of the system is: $\{1, 1, 1, 1\}$.

Third part: The bijection of Proposition 3.1.6 is true. In fact the sums of indices by classes are equal to 2 for the class 0 and 2 for the class 1, whatever the list we consider.

3.2. Controllability indices relative to the controllability over the field $\mathbb{R}(\nabla)$

In this part we present two new lists of controllability indices relative now to controllability over (the field) $\mathbb{R}(\nabla)$ for systems described by equations (2.2).

In the previous part we have defined the notion of class of the system (2.1). By (2.6) these classes also correspond to the expansion of $\langle A(\nabla)/B(\nabla)\rangle$ as a polynomial matrix in ∇ . Now, by the Definition 2.3, we have to consider the rank of the matrix $\langle A(\nabla)/B(\nabla)\rangle$. Hence we are not interested by each coefficients of this matrix but by the polynomial matrix of lower degree, deduced from $\langle A(\nabla)/B(\nabla)\rangle$, which has the same rank as the matrix $\langle A(\nabla)/B(\nabla)\rangle$.

Let us first recall that, from (3.2):

$$\langle A(\nabla)/B(\nabla)\rangle = M_1 + M_1\,\nabla + \cdots + M_k\,\nabla^k + \ldots$$

We define now, for all $k \in \mathbb{N}$, the matrix $M^k(\nabla)$, extracted form $\langle A(\nabla)/B(\nabla)\rangle$, and the submodule $\mathcal{M}^k(\nabla)$ by:

$$\begin{aligned} M^k(\nabla) &\triangleq M_0 + M_1\,\nabla + \cdots + M_k\,\nabla^k, \\ \mathcal{M}^k(\nabla) &\triangleq \mathrm{Im}(M^k(\nabla)). \end{aligned}$$

This leads us to define the notion of order of the system (2.2).

Definition 3.2.1. For all $k \in \mathbb{N}$, we say that $M^k(\nabla)$ represents the part of *order* k of the matrix $\langle A(\nabla)/B(\nabla)\rangle$ (defined by (2.4)).

Remark 3.2.1. This definition of order can be applied to any polynomial matrix $H(\nabla) = H_0 + H_1\,\nabla + \cdots + H_i\,\nabla^i + \cdots + H_q\,\nabla^q$, with $q \in \mathbb{N}$ and where H_i, $i \in \mathbb{N}$, is a real matrix.

If we note $R_i^k(\nabla)$, $i = 0, 1, \ldots, n-1$, the part of order k of $(A(\nabla))^i\,B(\nabla)$ and if $\mathcal{R}_i^k(\nabla)$ denote the submodule generated by the columns of $R_i^k(\nabla)$, the matrix $M^k(\nabla)$ and the submodule $\mathcal{M}^k(\nabla)$, $\forall\, k \in \mathbb{N}$, may be rewritten as (see Example 3.2.1):

$$\begin{aligned} M^k(\nabla) &= \left[R_0^k(\nabla),\, R_1^k(\nabla), \ldots, R_{n-1}^k(\nabla)\right], \\ \mathcal{M}^k(\nabla) &= \left[\mathcal{R}_0^k(\nabla),\, \mathcal{R}_1^k(\nabla), \ldots, \mathcal{R}_{n-1}^k(\nabla)\right]. \end{aligned}$$

We now present an example to illustrate the previous definitions.

Example 3.2.1. Let us consider the same example as in part 3.1. With the previous notations:

$$\langle A(\nabla)/B(\nabla)\rangle = \left[\begin{array}{cc|cc|cc|cc} 1 & 0 & \nabla & 0 & 0 & 0 & 0 & 0 \\ \nabla & 0 & 0 & 0 & \cdots & & \cdots & \\ 0 & 1 & 0 & 0 & \cdots & & \cdots & \\ 0 & 0 & 0 & \nabla & 0 & 0 & 0 & 0 \end{array}\right] = M_0 + \nabla\, M_1.$$

The part of order 0 of $\langle A(\nabla)/B(\nabla)\rangle$ is:

$$\begin{aligned} M^0 = M_0 \quad &= \left[\begin{array}{cc|cc|cc|cc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & & \cdots & \\ 0 & 1 & 0 & 0 & \cdots & & \cdots & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \\ &= \left[R_0^0(\nabla)\,|\,R_1^0(\nabla)\,|\,R_2^0(\nabla)\,|\,R_3^0(\nabla)\right]. \end{aligned}$$

$\mathcal{M}^0(\nabla)$ is the submodule generated by the columns of M^0.

The part of order 1 of $\langle A(\nabla)/B(\nabla)\rangle$ is:

$$\begin{aligned} M^1(\nabla) = M_0 + M_1\,\nabla \quad &= \left[\begin{array}{cc|cc|cc|cc} 1 & 0 & \nabla & 0 & 0 & 0 & 0 & 0 \\ \nabla & 0 & 0 & 0 & \cdots & & \cdots & \\ 0 & 1 & 0 & 0 & \cdots & & \cdots & \\ 0 & 0 & 0 & \nabla & 0 & 0 & 0 & 0 \end{array}\right] \\ &= \left[R_0^1(\nabla)\,|\,R_1^1(\nabla)\,|\,R_2^1(\nabla)\,|\,R_3^1(\nabla)\right]. \end{aligned}$$

$\mathcal{M}^1(\nabla)$ is the submodule generated by the columns of $M^1(\nabla)$.

For conver.˙ence and by abuse of language, we will talk about the *order* k of any polynomial matrix $H(\nabla)$ instead of talking about the *part of order* k of $H(\nabla)$.

ડ.2.1. *"First type" controllability indices*

We now define the "first type" controllability indices of order k, for $k \in \mathbb{N}$, by: for $i \in \underline{n}$, each index p_i^k (of order k) corresponds to increasing of the column rank between $\{R_{i-1}^k(\nabla), R_{i-2}^k(\nabla), R_{i=3}^k(\nabla), \ldots, R_0^k(\nabla)\}$ and $\{R_{i-2}^k(\nabla), R_{i-3}^k(\nabla), \ldots, R_0^k(\nabla)\}$.

Note $r = \operatorname{rank}\langle A(\nabla)/B(\nabla)\rangle = d(\langle A(\nabla)/\operatorname{Im} B(\nabla)\rangle)$.

Definition 3.2.2. The n "first type" controllability indices of order k, for all $k \in \mathbb{N}$, are defined by:

$$\begin{aligned} p_1^k &= d(\mathcal{R}_0^k(\nabla)), \\ p_i^k &= d\left(\frac{\mathcal{R}_0^k(\nabla) + \mathcal{R}_1^k(\nabla) + \cdots + \mathcal{R}_{i-1}^k(\nabla)}{\mathcal{R}_0^k(\nabla) + \mathcal{R}_1^k(\nabla) + \cdots + \mathcal{R}_{i-2}^k(\nabla)}\right), \qquad i = 2, 3, \ldots, n. \end{aligned}$$

Remark 3.2.2.

- We note $\{p_i^k\}_n$ the list of indices of order k, for all $k \in \mathbb{N}$.
- These indices are such that:

$$\sum_{i=1}^{n} p_i^k := r_k = d\left(\mathcal{R}_0^k(\nabla) + \mathcal{R}_1^k(\nabla) + \cdots + \mathcal{R}_{n-1}^k(\nabla)\right).$$

In order to determine the complete list of "first type" controllability indices, we consider the dimension r_k: if $r_k = r$, the list is complete; if *not* we must continue this procedure and study the order $k+1$ of the system and so on, knowing that this procedure is convergent, because $\{r_k\}_{k \in \mathbb{N}}$ is an non-decreasing serie bounded by r. Hence there always exists an integer $K \leq a(n-1) + b$ (see (2.2) and (2.4)) such that:

$$\sum_{i=1}^{n} p_i^K = d\left(\mathcal{R}_0^K(\nabla) + \mathcal{R}_1^K(\nabla) + \cdots + \mathcal{R}_{n-1}^K(\nabla)\right) = r.$$

The controllability over $\mathbb{R}(\nabla)$ of the system (2.2) can then be characterized as follows:

Proposition 3.2.3. The system (2.2) is controllable (at the order K) over $\mathbb{R}(\nabla)$ if and only if there exists $K \in \mathbb{N}$ such that

$$\sum_{i=1}^{n} p_i^K = n.$$

The proof is straightforward by Remark 3.2.2 and Definition 2.3. □

3.2.2. "Second type" controllability indices

In a same way as in the previous section, we can define a list of "second type" controllability indices. So let us represent, for $i \in \underline{m}$, $(R_0^k)^i(\nabla)$ the ith column of $R_0^k(\nabla)$ and more generally $(R_p^k)^i(\nabla)$ the ith column of $R_p^k(\nabla)$, for all $k \in \mathbb{N}$. Let us denote $\mathcal{X}_p^{i-1}(k)$ the subspace generated by the columns $(R_p^k)^{i-1}(\nabla), \ldots$
$\ldots, (R_p^k)^1(\nabla), (R_{p-1}^k)^m(\nabla), \ldots, (R_{p-1}^k)^1(\nabla), \ldots, (R_0^k)^m(\nabla), \ldots, (R_0^k)^1(\nabla)$.

Definition 3.2.4. The "second type" controllability indices of order k, for all $k \in \mathbb{N}$, are given by:

$$n_i^k = \min\left\{p \in \mathbb{N} \text{ such that } (R_p^k)^i(\nabla) \text{ belongs } \mathcal{X}_p^{i-1}(k)\right\}.$$

By construction, there exists $N \in \mathbb{N}$ such that $\sum_{i=1}^{m} n_i^N = r$.

These indices allow us to characterize the controllability over $\mathbb{R}(\nabla)$ of (2.2) as follows:

Proposition 3.2.5. The system (2.2) is controllable (at the order N) over $\mathbb{R}(\nabla)$ if and only if there exists $N \in \mathbb{N}$ such that

$$\sum_{i=1}^{m} n_i^N = n.$$

The proof is straightforward by the previous definition. □

These two new lists of controllability indices (by orders) relative to the controllability over $\mathbb{R}(\nabla)$ are linked by the following bijection:

Proposition 3.2.6. Both lists of controllability indices are linked by the following bijection:

$$\text{for all } k \in \mathbb{N} : n_i^K = \operatorname{card}\{p_j^k \geq i,\ j \in \underline{n}\}, \quad i \in \underline{m}.$$

Proof. The proof has to be here order by order. However it is omitted because the Definitions 3.2.2 and 3.2.4 are similar to the usual definitions of first and second type controllability indices ([11]). □

Remark 3.2.3. In the case of controllable systems over the ring $\mathbb{R}[\nabla]$, both previous lists are completely characterized by the order 0 (see Definition 2.2).

We now present an example to illustrate the previous definitions of controllability indices.

Example 3.2.2. Let us consider the Example 3.2.1.

$$\langle A(\nabla)/\Gamma(\nabla)\rangle = \left[\begin{array}{cc|cc|cc|cc} 1 & 0 & \nabla & 0 & 0 & 0 & 0 & 0 \\ \nabla & 0 & 0 & 0 & \cdots & & \cdots & \\ 0 & 1 & 0 & 0 & \cdots & & \cdots & \\ 0 & 0 & 0 & \nabla & 0 & 0 & 0 & 0 \end{array}\right] = M_0 + \nabla\, M_1.$$

First part: $M^0 = M_0 = [\, R_0^0(\nabla) \,|\, R_1^0(\nabla) \,|\, R_2^0(\nabla) \,|\, R_3^0(\nabla)\,]$ and,

$$M^1(\nabla) = M_0 + M_1\,\nabla = [\, R_0^1(\nabla) \,|\, R_1^1(\nabla) \,|\, R_2^1(\nabla) \,|\, R_3^1(\nabla)\,].$$

Following the previous determination of the "first type" indices we find:

$$\text{order 0:} \qquad p_1^0 = d(\mathcal{R}_0^0(\nabla)) = 2, \quad p_i^0 = 0 \quad \text{for} \quad i = 2,\,3,\,4.$$

$$\text{order 1:} \qquad p_1^1 = d(\mathcal{R}_0^1(\nabla)) = 2, \quad p_2^1 = d\left(\frac{\mathcal{R}_0^1(\nabla) + \mathcal{R}_1^1(\nabla)}{\mathcal{R}_0^1(\nabla)}\right) = 2.$$

The system is so controllable at order 1.

Second part: following the previous procedure of calculation of the "second type" indices we obtain:

$$\text{order 0:} \qquad n_1^0 = 1, \quad n_2^0 = 1.$$

$$\text{order 1:} \qquad n_1^1 = 2, \quad n_2^1 = 2.$$

Third part: It is easy to check that:

order 0: $\text{card}\{n_i^0 \geq 1,\ i = 1,\, 2\} = 2 = p_1^0$ and $\text{card}\{n_i^0 \geq 2,\ i = 1,\, 2\} = 0 = p_2^0$.
order 1: $\text{card}\{n_i^1 \geq 1,\ i = 1,\, 2\} = 2 = p_1^1$ and $\text{card}\{n_i^1 \geq 2,\ i = 1,\, 2\} = 0 = p_2^1$.

4. INVARIANCE PROPERTIES

We define here some transformations that keeps the controllability indices invariant, for $\mathbb{R}^n$-controllability and controllability over $\mathbb{R}(\nabla)$.

Let us first consider the class of static state feedbacks such that:

$$u(t) = F_0\, x(t) + F_1\, x(t-h) + \cdots + F_k\, x(t-kh) + \ldots \quad \text{for (2.1)}$$

and

$$F(\nabla) = F_0 + \nabla\, F_1 + \cdots + \nabla^k\, F_k + \ldots \quad \text{for (2.2)},$$

where F_i, $i \in \mathbb{N}$, are real matrices of dimension $m \times n$.

Such feedbacks are said to be realizable (nonpredictive).

We also consider changing of basis $T : \mathbb{R}^n \to \mathbb{R}^n$, where T is real and nonsingular.

Let us recall that the controllability indices relative to $\mathbb{R}^n$-controllability are defined using the notion of *class* of (2.1), and that the controllability indices relative to controllability over $\mathbb{R}(\nabla)$ are defined using the notion of *order* of (2.2).

By (3.2) it is proved that the notion of class is defined through the polynomial expansion in ∇ of $\langle A(\nabla)/B(\nabla)\rangle$.

Hence we can consider for (2.1) and (2.2) the transformations $\mathcal{G}(T, F)$ where,

$$F(\nabla) : \mathbb{R}^n(\nabla) \to \mathbb{R}^m(\nabla) \text{ is realizable,}$$
$$T : \mathbb{R}^n(\nabla) \to \mathbb{R}^n(\nabla) \quad \text{is constant and nonsingular,}$$

which do not change the definitions of classes or orders (linked with the definition of the indices).

Note $g = (T, F)$ any element of $\mathcal{G}(T, F)$ such that, for a system $(A(\nabla),\, B(\nabla))$:

$$g(A, B) = (T^{-1}(A + BF)\,T,\ T^{-1}B).$$

It is then straightforward ([11]) that, provided with the following law:

$$g_2 \circ g_1 = (T_1 T_2,\ F_1 + F_2 T_1^{-1}), \quad \text{where} \quad g_1 = (T_1,\, F_1),\ g_2 = (T_2,\, F_2),$$

$\mathcal{G}(T, F)$ is a group.

Then,

Theorem 4.1. The controllability indices relative to $\mathbb{R}^n$-controllability and the controllability indices relative to controllability over $\mathbb{R}(\nabla)$ are invariant under the group of transformations $\mathcal{G}(T, F)$.

The proof is only technical and is given in Appendix 1. □

5. COEFFICIENT ASSIGNMENT

In this last section we present an application of the controllability indices related to the problem of coefficient assignment by state feedback. The controllability indices previously defined allow us to complete the available results for systems with delays ([1, 2]). In this way, we will be able to precise the only possible polynomial forms we can obtain, by a realizable (non-predictive) state feedback, for the coefficients (of the characteristic polynomial of the closed-loop system).

As in [2], we first consider a single-input ($m = 1$) single-output (SISO) system (2.2). This system is only assumed to be controllable over $\mathbb{R}(\nabla)$ and its list of "second type" controllability indices (Definition 3.2.4), at the order k, is noted $\{n_1^k\}_{k\in\mathbb{N}}$.

The characteristic polynomial $\alpha(\lambda)$ of the open-loop system is:

$$\alpha(\lambda) = \det(\lambda I - A(\nabla)) = \lambda^n - [a_1(\nabla) + a_2(\nabla)\,\lambda + \cdots + a_n(\nabla)\,\lambda^{n-1}],$$

where $\{a_i(\nabla)\}_{i\in\underline{n}}$ are called the coefficients of $\alpha(\lambda)$.

The characteristic polynomial of the closed-loop system is noted:

$$\overline{\alpha}(\lambda) = \det(\lambda I - A(\nabla) - b(\nabla)\,f(\nabla)) = \lambda^n - [\overline{a_1}(\nabla) + \overline{a_2}(\nabla)\,\lambda + \cdots + \overline{a_n}(\nabla)\,\lambda^{n-1}].$$

Finally, note, for $i \in \underline{n}$:

$$\begin{aligned} a_i(\nabla) &= a_{i0} + \nabla\, a_{i1} + \cdots + \nabla^k\, a_{ik} + \dots \\ \overline{a_i}(\nabla) &= \overline{a_{i0}} + \nabla\, \overline{a_{i1}} + \cdots + \nabla^k\, \overline{a_{ik}} + \dots \end{aligned}$$

Then, for such system, we have:

Theorem 5.1. A realizable state feedback $f(\nabla)$ allows us to assign, at most,

$r_0 = n_1^0$ arbitrary coefficients of $\{\overline{a_{i0}}\}_{i\in\underline{n}}$ (the $n - r_0$ others being non-arbitrary),
and then,
$r_1 = n_1^1$ arbitrary coefficients of $\{\overline{a_{i1}}\}_{i\in\underline{n}}$ (the $n - r_1$ others being non-arbitrary),
and more generally,
$r_k = n_1^k$ arbitrary coefficients of $\{\overline{a_{ik}}\}_{i\in\underline{n}}$ (the $n - r_k$ others being non-arbitrary),
for all $k \in \mathbb{N}$.

Moreover we of course can precise which coefficients of $\{\overline{a_i}(\nabla)\}_{i\in\underline{n}}$ we can assign.

The proof is given in Appendix 2. An example which illustrates this result is available in [9]. □

In a detailed way, this means that we can choose, by the following procedure, the coefficients of the closed-loop system and any set $\{j_k \in \mathbb{N}\}_{k\in\mathbb{N}}$, so that:

Step 0: $j_0 \le r_0$ coefficients of $\{\overline{a_i}(\nabla)\}_{i\in\underline{n}}$ have an arbitrary nonzero constant term.

Of course the $n - j_0$ other terms of $\{\overline{a_i}0\}_{i \in \underline{n}}$ are non-arbitrary.

Step 1: $j_1 \le r_1 - j_0$ coefficients of $\overline{a_i}(\nabla)\}_{i\in\underline{n}}$, different from those chosen at the step 0, have an arbitrary term of degree 1 in ∇ (without an arbitrary one of degree 0 in ∇).

Moreover the j_0 coefficients chosen at the step 0 can also have an arbitrary term of degree 1 in ∇.

Of course the $n - j_0 - j_1$ other terms of $\{\overline{a_{i1}}\}_{i\in\underline{n}}$ are non-arbitrary.

Step k: $j_k \le r_k - j_{k-1} - j_{k-2} \cdots - j_0$ coefficients of $\{\overline{a_i}(\nabla)\}_{i\in\underline{n}}$, different from those chosen at the steps $0, 1, \dots, k-1$, have an arbitrary term of degree k in ∇, (without arbitrary ones of degrees $0, 1, \dots, k-1$ in ∇), for $k \in \mathbb{N}$, $k \ge 2$.

Moreover the $j_0 + j_1 + \cdots + j_{k-1}$ coefficients chosen at the steps $0, 1, \dots, k-1$ can also have an arbitrary term of degree k in ∇.

Of course the $n - j_0 - j_1 - \cdots - j_k$ others terms of degree k in ∇, $\{\overline{a_{ik}}\}_{i\in\underline{n}}$, are non-arbitrary.

This result allows us to completely characterize the set of realizable feedbacks for coefficient assignment of SISO systems.

In the *multi-input* case the result cannot be directly extended ([11]).

Let us consider a system whose the pair $(A(\nabla),\, B(\nabla))$ is controllable over $\mathbb{R}(\nabla)$.

If there exist a vector $b(\nabla)$ in $B(\nabla)$ such that the pair $(A(\nabla),\, b(\nabla))$ is controllable over $\mathbb{R}(\nabla)$, then the result of Theorem 5.1 is true.

If not, let us choose any vector $b(\nabla)$ in $B(\nabla)$. As the pair $(A(\nabla),\, b(\nabla))$ is *not* controllable over $\mathbb{R}(\nabla)$, the natural way to proceed is to transform it into a controllable over $\mathbb{R}(\nabla)$ one ([11], [6]).

The first solution is to construct a state feedback to make the closed-loop pair controllable over $\mathbb{R}(\nabla)$. However we are not able to guarantee the realizability of this feedback (except if the system (2.2) is controllable over the ring $\mathbb{R}[\nabla]$, [6]).

Another solution is that there always exists a unimodular matrix $U(\nabla)$ such that [3]:

$$U(\nabla)^{-1} A(\nabla)\, U(\nabla) = \begin{bmatrix} A_1(\nabla) & A_{12}(\nabla) \\ 0 & A_2(\nabla) \end{bmatrix} \; U(\nabla)^{-1} b(\nabla) = \begin{bmatrix} b_1(\nabla) \\ 0 \end{bmatrix},$$

where the pair $(A_1(\nabla),\, b_1(\nabla))$ is controllable over $\mathbb{R}(\nabla)$. By exhibiting both lists of controllability indices over $\mathbb{R}(\nabla)$ of the pair $(A_1(\nabla),\, b_1(\nabla))$ we can, of course, apply Theorem 5.1 to this pair.

Nevertheless these indices do not correspond to the indices of the pair $(A(\nabla),\, B(\nabla))$.

6. CONCLUSION

This paper is focused on two types of controllability: $\mathbb{R}^n$-controllability and controllability over the field $\mathbb{R}(\nabla)$. First of all we define two new notions, the class and the order, suitable for linear systems with delays and compatible with each type of controllability respectively. Next we exhibit, by classes or by orders, two new lists of controllability indices relative to each type of controllability.

Finally we use these new controllability indices over the field $I\!R(\nabla)$ to give the only good forms of the coefficients (of the characteristic polynomial of the closed-loop system) we can assign by a realizable state feedback. The only assumption on the system is to be controllable over the field $I\!R(\nabla)$. Hence this is more realistic than to exhibit a condition, most often restrictive ([1,2]), such that we can arbitrary assign the coefficients.

APPENDIX 1

We present here the proof of Theorem 4.1. Let T be a constant nonsingular automorphism of $I\!R^n(\nabla)$ and $F(\nabla)$ be a realizable state feedback. Note $A'(\nabla)$ for $T^{-1}(A(\nabla)+B(\nabla)\,F(\nabla))\,T$ and $B'(\nabla)$ for $T^{-1}\,B(\nabla)$. In the following, we consider the polynomial expansions:

$$\begin{aligned} \langle A(\nabla)/B(\nabla)\rangle &= M_0+\nabla\,M_1+\cdots+\nabla^k\,M_k+\dots \\ \langle A'(\nabla)/B'(\nabla)\rangle &= M_0'+\nabla\,M_1'+\cdots+\nabla^k\,M_k'+\dots \\ &= \langle T^{-1}(A(\nabla)+B(\nabla)\,F(\nabla))\,T\,/\,T^{-1}\,B(\nabla)\rangle \\ &= T^{-1}\,\langle A(\nabla)/B(\nabla)\rangle. \end{aligned} \tag{A.1}$$

To prove the theorem, we only have to prove the following propositions.

i) $F(\nabla)$ realizable implies:

$$\langle A(\nabla)+B(\nabla)\,F(\nabla)/\mathrm{Im}\,B(\nabla)\rangle = \langle A(\nabla)/\mathrm{Im}\,B(n)\rangle.$$

ii) T constant nonsingular implies:

$$\begin{aligned} &\mathrm{rank}\,T^{-1}\,\langle A(\nabla)/B(\nabla)\rangle^k = \mathrm{rank}\langle A(\nabla)/B(\nabla)\rangle^k, && \text{for all } k\in I\!N \text{ and} \\ &\mathrm{rank}[\,M_i'\cdots M_1'\,M_0'\,] = \mathrm{rank}[\,M_i\cdots M_1\,M_0\,], && \text{for } i=0,1,\dots,k, \end{aligned}$$

where the integer k is the corresponding *order or class*, respectively.

Indeed if (i) and (ii) are true, the orders and classes of $\langle A'(\nabla)/B'(\nabla)\rangle$ will have the same rank than those of $\langle A(\nabla)/B(\nabla)\rangle$.

P r o o f of i): If $F(\nabla)$ is realizable then, whatever $R(\nabla)\in I\!R^{n\times m}[\nabla]$ we have,

$$\mathrm{Im}\,B(\nabla)+(A(\nabla)+B(\nabla)\,F(\nabla))\,\mathrm{Im}\,R(\nabla) = \mathrm{Im}\,B(\nabla)+A(\nabla)\,\mathrm{Im}\,R(\nabla).$$

A similar proof to Wonham's one ([11]) leads to:

$$\langle A(\nabla)+B(\nabla)\,F(\nabla)/\mathrm{Im}\,B(\nabla)\rangle = \langle A(\nabla)/\mathrm{Im}\,B(\nabla)\rangle.$$

P r o o f of ii): If T is nonsingular, then

$$M_i' = T^{-1}\,M_i \quad \text{and so} \quad \mathrm{rank}\,M_i' = \mathrm{rank}\,M_i \quad \text{for all } i\in I\!N.$$

Hence it is immediate that:

$$\begin{aligned} &\mathrm{rank}\,T^{-1}\,\langle A(\nabla)/B(\nabla)\rangle^k = \mathrm{rank}\langle A(\nabla)/B(\nabla)\rangle^k, && \text{for all } k\in I\!N \text{ and} \\ &\mathrm{rank}[\,M_i'\cdots M_1'\,M_0'\,] = \mathrm{rank}[\,M_i\cdots M_1\,M_0\,], && \text{for } i=0,1,\dots,k, \end{aligned}$$

where the integer k is the corresponding order or class, respectively.

APPENDIX 2

We present here the proof of Theorem 5.1. We will follow a similar method to the one given in [11] in the case of linear systems without delays. We first use a changing of basis on $\langle A(\nabla)/\mathrm{Im}\, b(\nabla)\rangle$ in order to transform the pair $(A(\nabla), b(\nabla))$ into a canonical form ([5]). Let us recall that the characteristic polynomial $\alpha(\lambda)$ of the open-loop system is:

$$\alpha(\lambda) = \det(\lambda I - A(\nabla)) = \lambda^n - [a_1(\nabla) + a_2(\nabla)\,\lambda + \cdots + a_n(\nabla)\,\lambda^{n-1}],$$

where $\{a_i(\nabla)\}_{i\in \underline{n}}$ are called the coefficients of $\alpha(\lambda)$.

Let us now consider the following polynomials:

$$\begin{cases} \alpha_0(\lambda) = \alpha(\lambda) \\ \lambda\,\alpha_i(\lambda) = \alpha_{i-1}(\lambda) + a_i(\nabla), \quad \text{for } i \in \underline{n}. \end{cases}$$

Let:

$$e_i(\nabla) = \alpha_i(A(\nabla))\,b(\nabla), \quad \text{for } i \in \underline{n}.$$

Hence $S(\nabla) = (e_1(\nabla), e_2(\nabla), \ldots, e_n(\nabla))$ is a new basis of $\langle A(\nabla)/\mathrm{Im}\, b(\nabla)\rangle$.

As in the case of linear systems without delays we immediatly obtain:

$$S^{-1}(\nabla)\,A(\nabla)\,S(\nabla) = \begin{bmatrix} 0 & 1 & 0 & \cdots & \cdots & 0 \\ \vdots & & \ddots & \ddots & & \\ \vdots & & \ddots & \ddots & \ddots & \\ \vdots & & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & 1 \\ a_1(\nabla) & a_2(\nabla) & \cdots & \cdots & \cdots & a_n(\nabla) \end{bmatrix} \quad S^{-1}(\nabla)\,B(\nabla) = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

Note $\overline{A}(\nabla)$ for $S^{-1}(\nabla)\,A(\nabla)\,S(\nabla)$ and $\overline{b}(\nabla)$ for $S^{-1}(\nabla)\,b(\nabla)$. The pair $(\overline{A}(\nabla), \overline{b}(\nabla))$ does not have the same properties (controllability,...) as the pair $(A(\nabla), b(\nabla))$ but what is important at this step is that:

$$\det(\lambda I - \overline{A}(\nabla)) = \det(\lambda I - A(\nabla)).$$

Then we can choose a polynomial state feedback $\overline{f}(\nabla)$ such that:

$$\overline{f}(\nabla) = \left[\overline{A_1}(\nabla) - a_1(\nabla)\ \overline{a_2}(\nabla) - a_2(\nabla) \cdots \overline{a_n}(\nabla) - a_n(\nabla)\right].$$

This choice is suitable for the coefficients $\overline{a_i}(\nabla)$, $i \in \underline{n}$, to be the coefficients of $\det(\lambda I - \overline{A}(\nabla) - \overline{b}(\nabla)\,\overline{f}(\nabla))$.

Then, in the original base, the state feedback we need to assign the coefficients of $\det(\lambda I - A(\nabla) - b(\nabla)\,f(\nabla))$ is:

$$f(\nabla) = \overline{f}(\nabla)\,S^{-1}(\nabla).$$

As $T^{-1}(\nabla)$ is a matrix over $\mathbb{R}(\nabla)$, the problem is:

How to choose $\overline{f}(\nabla)$ such that $f(\nabla) = \overline{f}(\nabla)\,S^{-1}(\nabla)$ is realizable?

First, the definition of $T(\nabla)$ leads to:

$$S(\nabla) = \langle A(\nabla)/b(\nabla)\rangle\, U(\nabla), \text{ with } U(\nabla) \text{ unimodular}$$

$$U(\nabla) = \begin{bmatrix} -a_2(\nabla) & -a_3(\nabla) & \cdots & -a_n(\nabla) & 1 \\ -a_3(\nabla) & & & \cdot & 0 \\ \vdots & & \cdot & & \vdots \\ -a_n(\nabla) & \cdot & & & \vdots \\ 1 & 0 & \cdots & \cdots & 0 \end{bmatrix}.$$

Then the state feedback can be rewritten as:

$$f(\nabla) = \overline{f}(\nabla)\, U^{-1}(\nabla)\, \langle A(\nabla)/b(\nabla)\rangle^{-1}$$

and $f(\nabla)$ is solution of: (note that $(U^{-1}(\nabla))^T = U^{-1}(\nabla)$)

$$\langle A(\nabla)/b(\nabla)\rangle^T f(\nabla)^T = U^{-1}(\nabla)\,(\overline{f}(\nabla))^T. \tag{A.2}$$

In the following, note:

$$U^{-1}(\nabla)\,\overline{f}(\nabla)^T = \overline{\overline{f}}(\nabla)^T.$$

Its polynomial expansion in ∇ is noted:

$$\overline{\overline{f}}(\nabla) = \overline{\overline{f}}_0 + \nabla\, \overline{\overline{f}}_1 + \cdots + \nabla^k\, \overline{\overline{f}}_k + \ldots$$

As $U^{-1}(\nabla)$ is unimodular, then $\overline{\overline{f}}(\nabla)^T$ also has n degrees of liberty.

In order to impose to the state feedback $f(\nabla)$ to be realizable, we need the following lemma:

Lemma A.1. The linear equation (A.2) is such that, for all $k \in I\!N$:

$$\operatorname{rank} \begin{bmatrix} \overline{\overline{f}}_0^T \\ \overline{\overline{f}}_1^T \\ \vdots \\ \overline{\overline{f}}_k^T \end{bmatrix} = \operatorname{rank} \begin{bmatrix} \overline{f}_0^T \\ \overline{f}_1^T \\ \vdots \\ \overline{f}_k^T \end{bmatrix}.$$

Proof. As $U^{-1}(\nabla)$ is unimodular, then it can be written as:

$$U^{-1}(\nabla) = V_0 + \nabla\, V_1 + \cdots + \nabla^k\, V_k + \ldots,$$

where V_i, $i \in I\!N$, are real matrices, and V_0 is nonsingular.

By equalizing the polynomial expansions in ∇ of $\overline{\overline{f}}(\nabla)$ and $U^{-1}(\nabla)\,\overline{f}(\nabla)$, we obtain:

$$\begin{bmatrix} \overline{\overline{f}}_0^T \\ \overline{\overline{f}}_1^T \\ \vdots \\ \overline{\overline{f}}_k^T \end{bmatrix} = \begin{bmatrix} V_0 & 0 & \cdots & 0 \\ V_1 & V_0 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & 0 \\ V_K & \cdots & V_1 & V_0 \end{bmatrix} = \begin{bmatrix} \overline{f}_0^T \\ \overline{f}_1^T \\ \vdots \\ \overline{f}_k^T \end{bmatrix}.$$

As V_0 is nonsingular then we have:

$$\text{rank}\begin{bmatrix} \overline{\overline{f}}_0^T \\ \overline{\overline{f}}_1^T \\ \vdots \\ \overline{\overline{f}}_k^T \end{bmatrix} = \text{rank}\begin{bmatrix} \overline{f}_0^T \\ \overline{f}_1^T \\ \vdots \\ \overline{f}_k^T \end{bmatrix}.$$

□

Note now, for all $k \in I\!N$, r_k the rank of the order k of $\langle A(\nabla)/b(\nabla)\rangle$.

By Definition 3.2.4 and as $m = 1$, $r_k = n_1^k$, for all $k \in I\!N$.

We can now exhibit a procedure of calculation of the state feedback. Now, to prove that there exists a realizable $f(\nabla)$ for a choice of $\overline{f}(\nabla)$, we develop the proof for each order, step by step.

We present here the sketch of the proof, all steps being part of an algorithm.

In the following, note, for all $k \in I\!N$ and for $i \in \underline{n}$:

$$\begin{aligned} \langle A(\nabla)/B(\nabla)\rangle &= M_0 + \nabla M_1 + \cdots + \nabla^k M_k + \ldots \\ a_i(\nabla) &= a_{i0} + \nabla a_{i1} + \cdots + \nabla^k a_{ik} + \ldots \\ \overline{a_i}(\nabla) &= \overline{a_{i0}} + \nabla \overline{a_{i1}} + \cdots + \nabla^k \overline{a_{ik}} + \ldots \\ \overline{f}(\nabla) &= \overline{f}_0 + \nabla \overline{f}_1 + \cdots + \nabla^k \overline{f}_k + \ldots, \end{aligned}$$

where $\overline{f}_k = [\,\overline{a_{1k}} - a_{1k}\ \overline{a_{2k}} - a_{2k}\ \cdots\ \overline{a_{nk}} - a_{nk}\,]$,

$$\begin{aligned} \overline{\overline{f}}(\nabla) &= \overline{\overline{f}}_0 + \nabla \overline{\overline{f}}_1 + \cdots + \nabla^k \overline{\overline{f}}_k + \ldots \\ f(\nabla) &= f_0 + \nabla f_1 + \cdots + \nabla^k f_k + \ldots, \end{aligned}$$

where $f_k = [\,f_{1k}\ f_{2k}\ \cdots\ f_{nk}\,]$, with $f_{ik} \in I\!R$.

Step 0. At the order 0 of (A.2), there exists f_0 so that the following linear equation:

$$(M_0)^T (f_0)^T = (\overline{\overline{f}}_0)^T, \tag{A.3}$$

has a solution if and only if $\text{Im}(\overline{\overline{f}}_0)^T \subset \text{Im}(M_0)^T$.

As $\text{rank}(M_0)^T = r_0$, hence we can only choose r_0 independent coefficients in $(\overline{\overline{f}}_0)^T$, so that:

$$\text{Im}(\overline{\overline{f}}_0)^T \subset \text{Im}(M_0)^T.$$

Then, by the Lemma A.1, we can only choose r_0 independent coefficients in $\overline{f}_0$ so that there exists a solution for the system (A.3)

We so can assign $j_0 \leq n_1^0$ coefficients of $\{\overline{a_{i0}}\}_{i\in\underline{n}}$ the others $n - j_0$ constant terms of the coeefficients being non-arbitrary.

Of course we can precise these coefficients by the condition of solution of (A.3).

Step k. Let us keep the previous choice of $\overline{f}_0, \overline{f}_1, \ldots, \overline{f}_{k-1}$, and the deduced calculation of $f_0, f_1, \ldots, f_{k-1}$. At the order k of (A.2), there exists $f_0, f_1, \ldots, f_k$, solutions of (A.2), if and only if the following linear equation:

$$\begin{bmatrix} M_0^T & 0 & \cdots\cdots & 0 \\ M_1^T & M_0^T & 0 & \cdots\cdots \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots \\ \cdots\cdots & \cdots\cdots & \cdots & 0 \\ M_k^T & \cdots\cdots & M_1^T & M_0^T \end{bmatrix} = \begin{bmatrix} (f_0)^T \\ (f_1)^T \\ \vdots \\ (f_k)^T \end{bmatrix} = \begin{bmatrix} \overline{\overline{f}}^T \\ \overline{\overline{f}}_1^T \\ \vdots \\ \overline{\overline{f}}_k^T \end{bmatrix} \qquad (A.4)$$

has a solution.

This equation includes the equations of all the previous steps, $0, 1, \ldots, k-1$. Hence we only have to solve :

$$[\ (M_0)^T \quad (M_1)^T \quad \cdots \quad (M_k)^T\ ] \begin{bmatrix} (f_k)^T \\ \vdots \\ (f_1)^T \\ (f_0)^T \end{bmatrix} = (\overline{\overline{f}}_k)^T. \qquad (A.5)$$

Then (A.5) has a solution if and only if:

$$\operatorname{Im}(\overline{\overline{f}}_k)^T \subset \operatorname{Im}[\,(M_0)^T\ (M_1)^T\ \cdots\ (M_k)^T\,].$$

As $\operatorname{rank} M^i(\nabla) = r_i$, for $i = 0, 1, \ldots, k$, then,

$$\operatorname{rank}[\,(M_0)^T\ (M_1)^T\ \cdots\ (M_k)^T\,]$$
$$= \begin{bmatrix} M_0 \\ M_1 \\ \vdots \\ M_k \end{bmatrix} = \operatorname{rank}[\,M_0 + \nabla M_1 + \cdots + \nabla M_k\,] = r_k.$$

Hence we can choose, at most, r_k arbitrary coefficients in $(\overline{\overline{f}}_k)^T$, so that there exists a solution to (A.5). Then, by the previous steps, there exists a solution to (A.4).

Now we have already assigned r_0, $r_1, \ldots$, and r_{k-1} coefficients in $\overline{\overline{f}}_0^T, \overline{\overline{f}}_1^T, \ldots, \overline{\overline{f}}_k^T$ and so in $\overline{f}_0^T, \overline{f}_1^T, \ldots$, and $\overline{f}_{k-1}^T$ at the previous steps. Then, by Lemma A.1, we can assign r_k arbitrary coefficients in $(\overline{f}_k)^T$.

We so assign:

$r_k = n_1^k$ arbitrary coefficients of $\{\overline{a_{ik}}\}_{i\in\underline{n}}$ (the $n - r_k$ others being non-arbitrary), *for all* $k \in I\!N$.

Hence the step k allows us to choose the coefficients and any set $\{j_k \in I\!N\}_{k\in I\!N}$ so that:

$j_k \leq r_k - j_{k-1}$ coefficients, different from those chosen at the steps $0, 1, \ldots, k-1$, have an arbitrary term of degree k in ∇, (without arbitrary ones of degrees $0, 1, \ldots, n-1$ in ∇), for $k \in I\!N$, $k \geq 2$.

Of course the $j - j_0 - j_1 - \cdots - j_{k-1}$ other terms of degree k in ∇ of the coefficients are non-arbitrary.

We so can assign $r_k = n_1^k$ coefficients of $\{\overline{a_{ik}}\}_{i\in\underline{n}}$.

Of course we can precise these coefficients by the condition of solution of (A.4).

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