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KYBERNETIKA — VOLUME 7 (1971), NUMBER 4

To the Generation of Two-Dimensional Probabilistic Images and the Determination of their Statistical Characteristics

Valerij Božkov, Tomáš Radil-Weiss

The paper deals with the theory and practice of generating Two-Dimensional Probabilistic Images (TDPI) (characterized by the statistical distribution of brightness over their surface only) used in psychophysiological experiments. The TDPI is considered a quadratic matrix constructed according to a certain law of probability. Different realizations of the TDPI-s generated by means of the LINC computer and photographed from its digital oscilloscope are demonstrated.

From the point of view of experimental analysis of visual functions our paper is of methodological nature. We shall deal with the theory and practice of generation of Two-Dimensional Probabilistic Images (TDPI) (first introduced by Julesz [5] and used by Picket [8]) characterized by the statistical distribution of brightness over its discrete surface only, and not by other features like color, Gestalt etc. In the simplest case any real object may be represented as a distribution of a variable B on a limited part of a plane or in other words as a continuous function $B(x, y)$ in which the independent variables x, y may change, within certain limits. We shall interpret the value B as brightness. By representing B by a finite number of its values $B_1 \ldots B_m$ and by determining suitable steps of quantization Δx and Δy along the axis x and y, any continuous image may be approximated by a finite number of discrete elements of equal magnitude $\Delta x \,.\, \Delta y$. Any one of these elements may attain one of the m possible levels of brightness B_i. In this case a rectangular matrix $\mathbf{A} = (a_{ij})$ may be considered a suitable model of the discrete representation of the image. Any element a_{ij} of the matrix corresponds to a certain element of the image with similar coordinates. The value of this element of the matrix represents the level of brightness of the corresponding element of the image.

As we are interested for experimental reasons in the statistical features of two dimensional images, we shall consider two images identical when their statistical features (see below) are identical, irrespectively of their semantic content. We use therefore in our psychophysiological experiments as visual stimuli TDPI-s generated by computers the statistical features of which are determined in advance. It will be demonstrated that different classes of TDPI-s may be generated according to the statistical procedure used. In dependence on the degree of statistical interrelationships between elements of the TDPI, images with lower or higher level of statistical similarity with real two dimensional images may be generated.

The generation of TDPI-s considered in this paper is demonstrated in general form in Fig. 1. Let $t = 1, 2, \ldots, N$ be subsequently generated by block 1. Each of

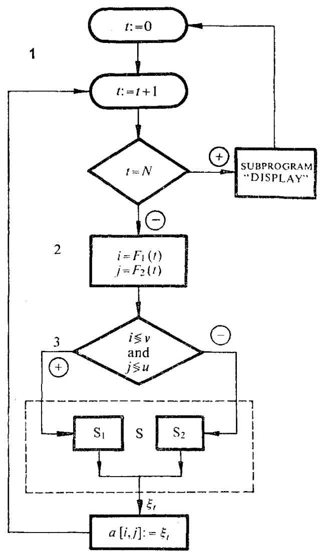


Fig. 1. Flow chart of the generation of the two dimensional probabilistic image (TDPI) For description see text.

them corresponds to the number of trial performed by a discrete random source S. $S = (B, P)$ specified by its alphabet $B = \{B_1, B_2, \ldots, B_m\}$ and a fixed probability law P according to which the successive symbols $\{\xi_t\}$, $t = 1, 2, \ldots, N$ are selected from the alphabet B. Only one of the symbols of B occurs in any trial t. Block 2 determines the values of a pair of symbols (i, j) corresponding unambiguously to any trial t in the following way:

$$F_1 : i = \begin{cases} 1 & \text{if} \quad t \leqq n\,, \\ \left[\dfrac{t-1}{n}\right] + 1 & \text{if} \quad t > n\,, \end{cases} \tag{1}$$

$$F_2 : j = \begin{cases} t & \text{if} \quad t \leqq n\,, \\ t - \left[\dfrac{t-1}{n}\right] . \, n & \text{if} \quad t > n\,, \end{cases}$$

so that

(2) $$t = n(i - 1) + j\,,$$

where by $[x]$ we denote the greatest integer less than or equal to x and n determines the size of the matrix $\boldsymbol{A}$ represented in the memory of computer by a two-dimensional array. Obviously n must be related to N by $N = n^2$.

If ξ_t is the random result corresponding to the t-th trial and $i = F_1(t)$, $j = F_2(t)$, then the element a_{ij} of the array $\boldsymbol{A}$ is a realisation of a random variable $X_{ij} = \xi_t$, i.e. we define

$$X_{F_1(t)F_2(t)} = \xi_t\,.$$

In some cases source S contains two subsources S_1, S_2 activated according to the value of indexes i and j (block 3).

After $N = n^2$ trials have been performed by S any element a_{ij} of array $\boldsymbol{A}$ represents in correspondence to its value a certain level of brightness at a determined point of a cathode ray tube (CRT) of the computer with coordinates $x = j$ and $y = i$. The photograph of the CRT represents the TDPI.

The features of the TDPI-s are in our cases determined by the parameters of the discrete random source S. Changing the value of parameters of S (probabilistic law P), different realizations of TDPI-s of the same class, changing the S itself, different classes of TDPI-s are produced.

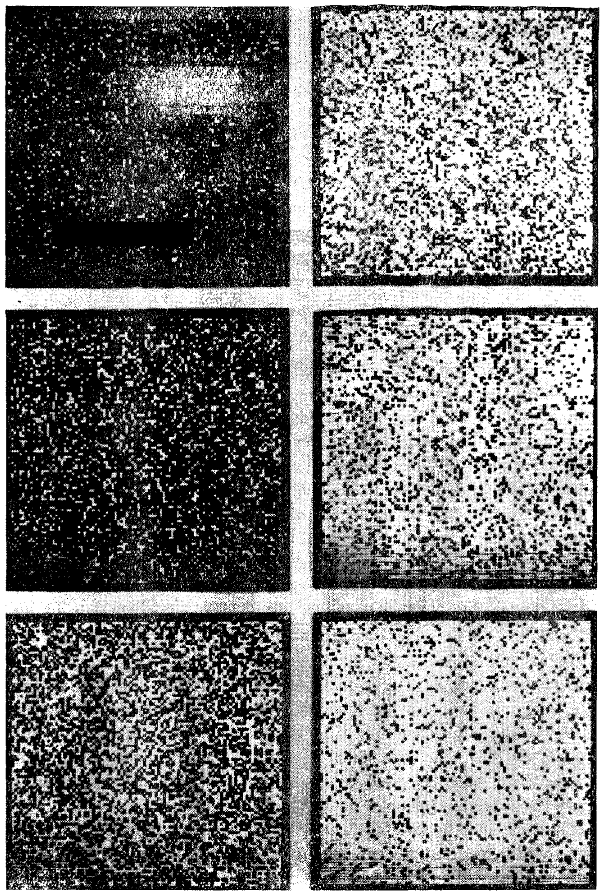
We shall deal with four different classes of the TDPI-s. For the sake of simplicity we shall consider only the two-symbols alphabet $B = \{0, 1\}$ corresponding physically to two brightness levels black and white.

The simplest class of TDPI-s is characterized by mutual statistical independence of $\xi_1, \xi_2, \ldots, \xi_N$. In this case the probabilities of X_{ij} are mutually independent for different i, j so that X_{ij} is determined by the absolute probabilities of occurrence of brightness levels only. The corresponding S will be a zero-memory source the probability law P of which is given by the first order probability distribution $P = \{p(0), p(1)\} = \{p, 1 - p\}$ on $B = \{0, 1\}$ where $p \in (0, 1)$.

Modeling such an S by means of the LINC computer a pseudorandom generator of uniformly distributed numbers within the interval (0, 1) has been used. Examples of TDPI-s with the different values of p are in Fig. 2. The average amount of information corresponding to one symbol of physical alphabet (black and white elements) evidently equals to the entropy H of the random source S:

(3) $$H = H_1 = -\sum_i p(i) \log p(i) = -p \log p - (1 - p) \log (1 - p)\,.$$

Index 1 attached to H has been used in this case as a sign of order of probability



$p_\alpha = 0.11$; $H = 0.5$
$p_\alpha = 0.215$; $H = 0.75$
$p_\alpha = 0.5$; $H = 1$

$p_\alpha = 0.68$; $H = 0.9$
$p_\alpha = 0.81$; $H = 0.75$
$p_\alpha = 0.89$; $H = 0.5$

Fig. 2. Examples of TDPI-s in which the elements are mutually independent. The corresponding values of p and H are also given. For details see text.

distribution according to which H has been computed. In cases the message is characterized by statistical relationships between its elements H_1 is usually called the first approximation to the real value of the entropy [4, 7].

Theoretical and computed values of H_1 are shown in Fig. 3 (the computed values being averages of 20 realizations of TDPI-s generated with determined values of p). The computation has been performed always after the generation of TDPI-s by means of a special program. The value of redundancy which we consider a useful experimental variable in this case equals

$$R_1 = 1 - \frac{H_1}{H_{\max}} \tag{4}$$

where $H_{\max} = \log m$; R_1 may be considered a measure of the degree of uncertainity removed, taking in account the first order probability distribution of the brightness levels.

It is evident that the TDPI-s of this class with respect to their statistical characteristics are fare from real images. In real images a considerable statistical dependence exists between neighbouring elements [3, 6]. The incidence of pairs of elements of equal brightness is much higher than of those with different brightness, the absolute

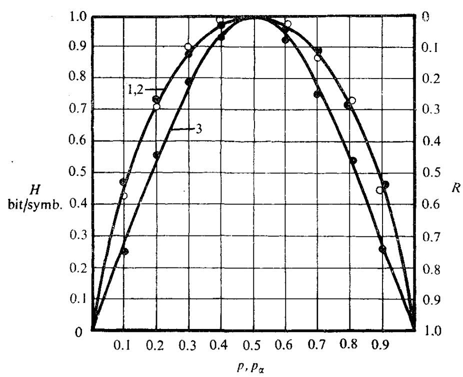


Fig. 3. Graph of dependence between entropy (H in bit/symb) resp. redundancy (R) of the TDPI and parameters of the generator (p, p_α). Curve 1 represents the theoretical function $H = -p \log p - (1 - p) \log (1 - p)$, black dots the computed values. Curve 2 (identical with curve 1) represents the theoretical function $H = -p_\alpha \log p_\alpha - (1 - p_\alpha) \log (1 - p_\alpha)$, white dots the computed values. Curve 3 represents the theoretical function $H = [-r \log r - (1 - r) \log (1 - r)] p_\varepsilon + (1 - p_\varepsilon)$ corresponding dots the computed values. For details see text.

probability of the occurrence of black and white elements being equal, however. Knowing the brightness of any element, the brightness of the neighbouring one may be predicted with a high degree of probability, i.e. information about the brightness of a certain element considerably decreases when the brightness of the preceding element is already known. Therefore the redundancy of such images is determined by the degree of mutual interrelationships between their elements.

The following symbols have to be introduced before other types of TDPI-s will be described. The elements $a_{i,j-1}, a_{i-1,j-1}, a_{i-1,j}$ of TDPI (see Fig. 4) will be considered neighbouring with respect to an arbitrarly chosen element a_{ij}. The formal symbols $\alpha, \beta, \varepsilon, \pi$ for different directions in the array $\boldsymbol{A}$ are also shown.

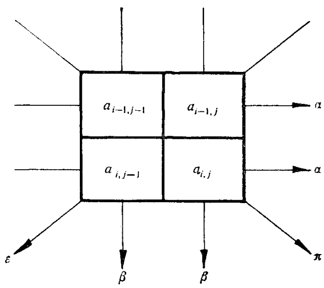


Fig. 4. Schematic representation of the directions of considered dependencies between neighbouring elements of the TDPI. (For details see text.)

The conditional probabilities of the occurrence of any one of the brightness levels in a given element in case the value of brightness of the neighbouring one is already known, are the following

$$(5)\quad \begin{aligned} P_\alpha(i \mid j) &= \mathsf{P}[X_{rs} = i \mid X_{r,s-1} = j]\,, & r &= 1, 2, \ldots, n\,, \quad s = 2, \ldots, n\,,\\ P_\beta(i \mid j) &= \mathsf{P}[X_{rs} = i \mid X_{r-1,s} = j]\,, & r &= 2, \ldots, n\,, \quad s = 1, \ldots, n\,,\\ P_\varepsilon(i \mid j) &= \mathsf{P}[X_{rs-1} = i \mid X_{r-1,s} = j]\,, & r, s &= 2, \ldots, n\,,\\ P_\pi(i \mid j) &= \mathsf{P}[X_{rs} = i \mid X_{r-1,s-1} = j]\,, & r, s &= 2, \ldots, n \end{aligned}$$

where we suppose here and in the sequel, that the conditional probabilities above do not depend on r, s. The parameters $\alpha, \beta, \varepsilon, \pi$ we used to characterize directions of dependence considered, and $i, j \in B = \{0, 1\}$.

Further we shall deal with such TDPI-s only in which

$$(6)\qquad P_\lambda(i \mid i) = P_\lambda(j \mid j)\,, \quad i \neq j\,, \quad \lambda = \alpha, \beta, \varepsilon, \pi\,.$$

In this case the probabilities $p_\alpha, p_\beta, p_\varepsilon, p_\pi$ that two randomly chosen neighbouring elements are of the same brightness values (0 or 1), irrespectively whether they are white or black can be considered as a measure of mutual statistical relationships between neighbouring elements of $\boldsymbol{A}$

(7)
$$\begin{aligned} p_\alpha &= \mathsf{P}[X_{rs} = X_{r,s-1}] &= p_\alpha(i \mid i)\,, \quad i = 0, 1\,,\\ p_\beta &= \mathsf{P}[X_{rs} = X_{r-1,s}] &= p_\beta(i \mid i)\,, \quad i = 0, 1\,,\\ p_\varepsilon &= \mathsf{P}[X_{r,s-1} = X_{r-1,s}] &= p_\varepsilon(i \mid i)\,, \quad i = 0, 1\,,\\ p_\pi &= \mathsf{P}[X_{rs} = X_{r-1,s-1}] &= p_\pi(i \mid i)\,, \quad i = 0, 1\,. \end{aligned}$$

The numbers p_λ, $\lambda = \alpha, \beta, \varepsilon, \pi$, satisfy the following relations

(8)
$$p_\lambda = \sum_i p(i)\, p_\lambda(i \mid i) = p_\lambda(i \mid i) \sum_i p(i) = p_\lambda(i \mid i)\,, \quad i = 0, 1\,.$$

Now we shall deal with the class of TDPI-s with statistical interrelationships between neighbouring elements in direction α as described above and without any statistical dependence between other pairs of elements. For this purpose a Markov

Table 1.

Matrix of transitional probabilities for TDPI-s with dependencies between the neighbouring elements in direction α

$0 \leqq b \leqq 1$

	$E_1(0)$	$E_2(1)$
$E_1(0)$	b	$1 - b$
$E_2(1)$	$1 - b$	b

(For details see text.)

discrete random source S_1 is adopted. It posseses two states $\{E_1, E_2\} = \{0, 1\}$ and its matrix of transition probabilities $\boldsymbol{P} = (p_{ij})$, $i, j = 1, 2$, are demonstrated in Tab. 1. Accordingly to this matrix we may write

(9)
$$\begin{aligned} p_\alpha &= b\,,\\ p_\beta = p_\varepsilon = p_\pi &= 0{\cdot}5\,. \end{aligned}$$

This S_1 is functioning in trials only, the number t of which corresponds to $i = 1, 2, \ldots, n, j > 1$ and X_{i1} is to be generated randomly by S_2 as an initial state for S_1. We suppose $\mathsf{P}[X_{i1} = 1] = 0{\cdot}5$, $i = 1, 2, \ldots, n$. Examples of TDPI-s with different values of p_α are shown in Fig. 5.

The sequence of n^2 events $X_{11}, \ldots, X_{1n}, \ldots, X_{nn}$ is composed of equivalent (with respect to their statistical features), n independent subsequences, each one of which may be considered a Markov chain of the length n. Therefore considering the S_1 being stationary ergodic source the average amount of information per one symbol of the physical alphabet (black and white elements) of the TDPI-s of this class equals to the entropy H of source S_1:

$$H = H_2 = -\sum_i p(E_i) \sum_j p_{ij} \log p_{ij}, \quad i, j = 1, 2, \tag{10}$$

where $p(E_i)$ is the unconditional probability of its state E_i. As in our case $\mathbf{P}$ is a double stochastic matrix than it follows [2] that $\{p(E_1), p(E_2)\} = \{0{\cdot}5, 0{\cdot}5\}$. Introducing values of $p(E_i)$ in expression and considering (8) we have

$$H = H_2 = -p_\alpha \log p_\alpha - (1 - p_\alpha) \log (1 - p_\alpha). \tag{11}$$

As in these case $p(0) = p(1) = 0{\cdot}5$ it follows that

$$H_1 = H_{\max} = \log 2 = 1,$$
$$R_1 = 0,$$

and the real value of redundancy will be determined by statistical interrelationships between the elements of the TDPI only

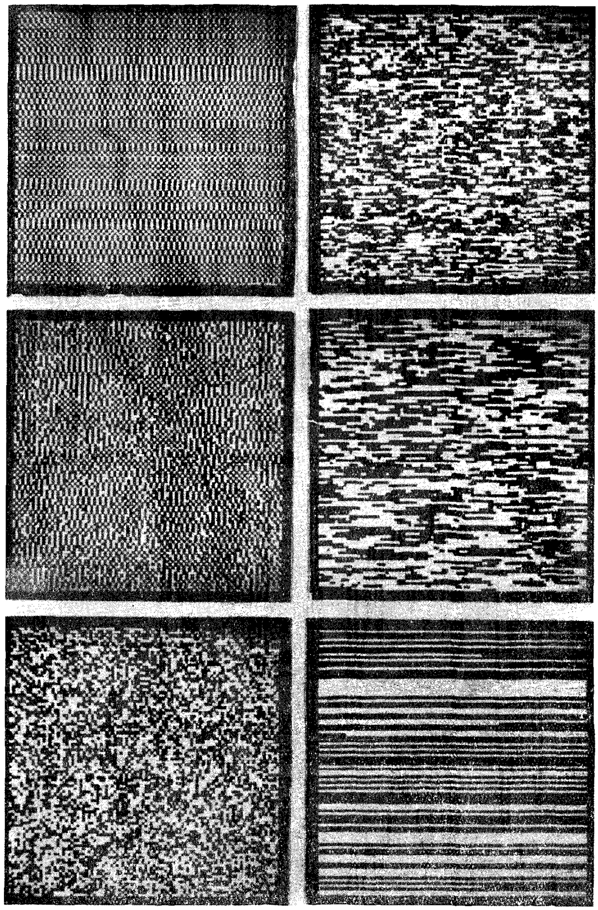
$$R_2 = 1 - \frac{H_2}{H_1} = 1 - H_2. \tag{12}$$

Let us consider now any useful statistical characteristics of TDPI-s relatively to some direction λ.

At first we shall deal with the "frequency" characteristics determining the "contourness" of the TDPI in any direction λ. For this purpose the number of elements in an uninterrupted sequence of elements of equal brightness in any of the described directions is considered a random variable ω_λ (which we may call "the length of uninterrupted sequence of elements with equal brightness in direction λ"). Its mean value is marked $\bar{\omega}_\lambda$. For computing $\bar{\omega}_\lambda$ the probability diagram of ω_λ is constructed:

$\omega_\lambda = i$	1	2	3	...	i	...
$p_i = \mathrm{P}[\omega_\lambda = i]$	$\bar{p}_\lambda$	$p_\lambda \cdot \bar{p}_\lambda$	$p_\lambda^2 \cdot \bar{p}_\lambda$	...	$p_\lambda^{i-1} \cdot \bar{p}_\lambda$	

where $\bar{p}_\lambda = 1 - p_\lambda$.



$p_\alpha = 0.01$; $H = 0.08$
$p_\alpha = 0.11$; $H = 0.5$
$p_\alpha = 0.5$; $H = 0.99$

$p_\alpha = 0.76$; $H = 0.70$
$p_\alpha = 0.89$; $H = 0.5$
$p_\alpha = 0.99$; $H = 0.08$

Fig. 5. Examples of TDPI-s in which neighbouring elements are interdependent in direction α, with the corresponding values of H and p_α; $H_1 = 1$.

From this diagram we have

$$\sum_{i=1}^{\infty} p_i = \bar{p}_\lambda(1 + p_\lambda + p_\lambda^2 + \ldots + p_\lambda^{i-1} + \ldots)\,. \tag{13}$$

The sum of the series in parentheses equals $1/(1 - p_\lambda)$ and therefore

$$\sum_{i=1}^{\infty} p_i = 1$$

and it is evident that

$$\mathsf{P}[\omega_\lambda = i] = p_\lambda^{i-1}(1 - p_\lambda)\,, \quad i = 1, 2, 3, \ldots, \tag{14}$$

is distribution law of ω_λ.

Let us consider the mathematical expectation of ω_λ

$$\bar{\omega}_\lambda = \mathsf{M}[\omega_\lambda] = \sum_{i=1}^{\infty} i \,.\, p_i = \bar{p}_\lambda(1 + 2p_\lambda + 3p_\lambda^2 + \ldots + i \,.\, p_\lambda^{i-1} + \ldots)\,. \tag{15}$$

It is evident that the series in parentheses represents the result of differentiation of the geometrical progression

$$p_\lambda + p_\lambda^2 + p_\lambda^3 + \ldots + p_\lambda^i + \ldots = \frac{p_\lambda}{1 - p_\lambda}\,.$$

Then considering that

$$\frac{\mathrm{d}}{\mathrm{d}p_\lambda} \frac{p_\lambda}{1 - p_\lambda} = \frac{1}{(1 - p_\lambda)^2}$$

it follows from (15) that

$$\bar{\omega}_\lambda = \frac{1}{1 - p_\lambda} \tag{16}$$

whence it follows also that

$$p_\lambda = \frac{\bar{\omega}_\lambda - 1}{\bar{\omega}_\lambda}\,. \tag{17}$$

This expression gives the possibility of computation the value of p_λ for desirable $\bar{\omega}_\lambda$ of the TDPI.

The quantity

$$f_\lambda = K_\lambda \frac{1}{\bar{\omega}_\lambda} = K_\lambda(1 - p_\lambda) \tag{18}$$

may be considered as "an average number of black → white and white → black transitions corresponding to a conditional unit of length in direction λ of the surface of the TDPI-s and K_λ is a scale factor the value of which is dependent from size of the element Δx, Δy of the TDPI. From this point of view f_λ describes the "contourness" of the TDPI in direction λ.

It is useful to know in some cases the dispersion of the random variable ω_λ. We determine it according to the expression

$$(18^*) \qquad D_\lambda = \mathsf{D}[\omega_\lambda] = \nu_2 - \bar{\omega}_2$$

where ν_2 is the second order initial moment of the quantity ω_λ, which is given by

$$(19) \qquad \nu_2 = \mathsf{M}[\omega_\lambda^2] = \sum_{i=1}^{\infty} i^2 \cdot p_i = \bar{p}_\lambda(1^2 + 2^2 \cdot p_\lambda + \ldots + i^2 \cdot p_\lambda^{i-1} + \ldots)\,.$$

For computing the sum of the series in parentheses we multiply by p_λ the elements of the series

$$1 + 2p_\lambda + 3p_\lambda^2 + \ldots + i \cdot p_\lambda^{i-1} + \ldots = \frac{1}{(1 - p_\lambda)^2}$$

and by differentiating this result with respect p_λ we have

$$\frac{\mathrm{d}}{\mathrm{d}p_\lambda} \frac{p_\lambda}{(1 - p_\lambda)^2} = \frac{1 + p_\lambda}{(1 - p_\lambda)^3}\,.$$

Multiplying this result by $\bar{p}_\lambda = 1 - p_\lambda$ results

$$(20) \qquad \nu_2 = \frac{1 + p_\lambda}{(1 - p_\lambda)^2}\,.$$

We find the expression for D_λ according to (18*)

$$(21) \qquad D_\lambda = \nu_2 - \bar{\omega}_\lambda^2 = \frac{p_\lambda}{(1 - p_\lambda)^2} = \bar{\omega}_\lambda(\bar{\omega}_\lambda - 1)\,.$$

The corresponding standard deviation is

$$(22) \qquad \sigma_\lambda = \bar{\omega}_\lambda \sqrt{p_\lambda}\,.$$

As the Markov source described above is an stationary ergodic source $(\lim_{n\to\infty} p_{ik}^{(n)} \to u_k = 0{\cdot}5 > 0)$ it is evident that a sequence of events $\xi_t, \xi_{t+1}, \xi_{t+2}, \ldots, \xi_{t+r}, \ldots$ generated by this source in successive trials $t, t + 1, \ldots, t + r, \ldots, t = 1, 2, \ldots, r = 1, 2, \ldots$ is a stationary ergodic Markov process, which we define by $\{\xi^{(r)}, r = 1, 2, \ldots\}$. Therefore the correlation function of $\xi^{(r)}$ may be expressed by

$$(23) \quad K_\xi(r, r + n) = K_\xi(n) = \sum_{ij} i \cdot j \cdot \mathsf{P}[\xi^{(r)} = i, \xi^{(r+n)} = j]\,, \quad n = 0, 1, 2, \ldots,$$

for any arbitrary value of r where i and j are considered being $\pm a$ $(a > 0)$ and $\mathsf{M}[\xi^{(r)}] = 0$.

The joint probability equals in our case

$$\mathsf{P}[\xi^{(r)} = i,\ \xi^{(r+n)} = j] = \pi_i p_{ij}^{(n)},$$

where

$$p_{ij}^{(n)} = \mathsf{P}[\xi^{(r+n)} = j \mid \xi^{(r)} = i],$$
$$\pi_i = \mathsf{P}[\xi^{(r)} = i].$$

It is known [2] that for matrix

$$\boldsymbol{P} = \begin{pmatrix} 1-p & p \\ a & 1-a \end{pmatrix}, \quad 0 \leqq p \leqq 1, \quad 0 \leqq a \leqq 1$$

the matrix $\boldsymbol{P}^{(n)}$ may be expressed as

$$\boldsymbol{P}^{(n)} = \frac{1}{a+p}\begin{pmatrix} a & p \\ a & p \end{pmatrix} + \frac{(1-a-p)^n}{a+p}\begin{pmatrix} p & -p \\ -a & a \end{pmatrix}.$$

Then it is simple to derive that for our matrix $\boldsymbol{P}$ (see Table 1) $\boldsymbol{P}^{(n)}$ equals

$$\boldsymbol{P}^{(n)} = \begin{pmatrix} \dfrac{1+(2b-1)^n}{2} & \dfrac{1+(2b-1)^n(b-1)}{2(1-b)} \\ \dfrac{1+(2b-1)^n(b-1)}{2(1-b)} & \dfrac{1+(2b-1)^n}{2} \end{pmatrix}. \tag{24}$$

Substituting the elements of $\boldsymbol{P}^{(n)}$ to expression (23) and considering that $\pi_i = 0{\cdot}5$ we have definitively that

$$K_\xi(n) = a^2(2b-1)^n = a^2(2p_\alpha - 1)^n. \tag{25}$$

For the class of TDPI-s described above the values of $\bar{\omega}_\lambda, f_\lambda, D_\lambda, \sigma_\lambda$ and $K_\xi(n)$ may be found when real values of p_λ from (9) are substituted to the expressions (16), (18), (21), (25).

In the third class of TDPI-s the value of X_{rs} is generated in dependence upon the values of $X_{r-1,s}$ and $X_{r,s-1}$, $r, s = 2, \ldots, n$ only. There are no dependencies between other triplets of X_{ij} in this case. X_{rs} is generated by the source S_1 which is a discrete random source with a "memory" for n symbols preceding the symbol actually generated. S_1 has four states $\{E_1, \ldots, E_4\} = \{00, \ldots, 11\}$ and its probability law P is considered by Table 2. The state of S_1 in the t-th trial $(E_k^t,\ k = 1, \ldots, 4)$ is determined by symbols generated in trials $(t-1)$ and $(t-n)$. $\{X_{1j}\}$ and $\{X_{i1}\}$, $i, j = 1, 2, \ldots, n$ are generated by the Markov source S_2 described above and represent the initial state of S_1.

Table 2.

Probabilistic law of generation of the class of TDPI-s with dependencies between the neighbouring elements in directions α and β

$0 \leqq r \leqq 1$

	$X_{r-1,s}$	$X_{r,s-1}$	X_{rs}		$\{P(E_i)\}$
			0	1	
E_1	0	0	r	$1 - r$	$\frac{p_\varepsilon}{2}$
E_2	0	1	0·5	0·5	$\frac{1 - p_\varepsilon}{2}$
E_3	1	0	0·5	0·5	$\frac{1 - p_\varepsilon}{2}$
E_4	1	1	$1 - r$	r	$\frac{p_\varepsilon}{2}$

E_1, E_2, E_3, E_4 represent different states of the random source. $P(E_i)$ corresponds to the probability of the state E_i.

The following relations are considered for further purposes

$$X_{rs} = \xi_t\,, \quad t = 1, 2, \ldots, N\,,$$
$$X_{r,s-1} = \xi_{t-1}\,, \quad t = 2, 3, \ldots, N\,,$$
$$X_{r-1,s} = \xi_{t-n}\,, \quad t = n + 1, \ldots, N\,,$$

where $r, s = F_1(t), F_2(t)$.

Considering these relations the probability of the state of source S_1 in the t-th trial may be represented by

$$(26) \qquad P[E^t = ij] = P[\xi_{t-n} = i,\ \xi_{t-1} = j]\,, \quad i, j = 0, 1\,.$$

Let us demonstrate now that the sequence of symbols $\{\xi_t\}$ at the output of S_1 is characterized by the same first order probability distribution $\{p(i)\} = \{0{\cdot}5, 0{\cdot}5\}$, $i = 0, 1$, like the sequence generated by S_2.

Let us consider the sequence trial $t = n + 2$. The whole first row (i.e. the events $\xi_1, \ldots, \xi_n$) and the first element of the second row (i.e. the event ξ_{n+1}) of array $\boldsymbol{A}$ are generated by S_2 by the moment $t = n + 2$. The event ξ_{n+2} is the first one generated by S_1. Considering the parameters of S_2 known, the probabilities of events $(\xi_1 = i \cap \xi_2 = j)$ and $(\xi_1 = i \cap \xi_{n+1} = j)$, $i, j = 0, 1$, are determined.

On the basis of these known values and of the probability low P of the source S_1 we shall demonstrate that $P[\xi_{n+2} = 0] = 0{\cdot}5$.

According to expression of full probability we may write that

$$(27) \qquad \mathsf{P}[\xi_t = 0] = \sum_{ij} \mathsf{P}[\xi_{t-n} = i,\ \xi_{t-1} = j]\ \mathsf{P}[\xi_t = 0 \mid \xi_{t-n} = i,\ \xi_{t-1} = j]$$

where $t = n + 2$ and $i, j = 0, 1$.

In (27) the probability of the event $\xi_{t-n} = i \cap \xi_{t-1} = j$ may be expressed by

$$(28) \qquad \mathsf{P}[\xi_{t-n} = i,\ \xi_{t-1} = j] = \sum_k \mathsf{P}[\xi_{t-n-1} = k,\ \xi_{t-n} = i,\ \xi_{t-1} = j]\,,$$

$$i, j, k = 0, 1\,.$$

Substituting this result in (27) we have

$$(29) \qquad \mathsf{P}[\xi_t = 0] = \sum_{ij} \mathsf{P}[\xi_t = 0 \mid \xi_{t-n} = i,\ \xi_{t-1} = j] \sum_k \mathsf{P}[\xi_{t-n-1} = k]\,.$$

$$.\ \mathsf{P}[\xi_{t-n} = i \mid \xi_{t-n-1} = k]\,.$$

Considering that for $t = n + 2$ $\mathsf{P}[\xi_{t-n} = i \mid \xi_{t-n-1} = k] = p(i \mid k)$ and $\mathsf{P}[\xi_{t-1} = i \mid \xi_{t-n-1} = k] = p(i \mid k)$ are the elements of the matrix $\boldsymbol{P}$ of source S_2 and substituting for each pair of (i, j) in $\mathsf{P}[\xi_t = 0 \mid \xi_{t-n} = i,\ \xi_{t-1} = j]$ the corresponding values from probability low P of S_1 (Table 2) we have by simple computations that

$$\mathsf{P}[\xi_{n+2} = 0] = 0{\cdot}5\,.$$

Repeating the same procedure sequentially for $t = n + 3,\ n + 4,\ n + p, \ldots$ and considering at each step the result of the previous one it may be shown that the first order probability distribution of the resulting sequence $\xi_{n+2}, \xi_{n+3}, \ldots$ does not change and equals $\{p(0), p(1)\} = \{0{\cdot}5, 0{\cdot}5\}$ after applying of the probability law P given by Table 2 to the sequence $\xi_1, \xi_2, \ldots, \xi_{n+1}$ generated by S_2. It may be also demonstrated that quantity p_α of a sequence $\xi_{n+2}, \xi_{n+3}, \xi_{n+4}, \ldots$ equals to the initial quantity. Let again determine $t = n + 2$ and demonstrate that

$$(30) \qquad \mathsf{P}[\xi_{t-1} = 0,\ \xi_t = 0] = 0{\cdot}5b$$

in this condition. The expression of full probability is used again

$$(31) \qquad \mathsf{P}[\xi_{t-1} = 0,\ \xi_t = 0] = \sum_i \mathsf{P}[\xi_{t-n} = i,\ \xi_{t-1} = 0]\,.$$

$$.\ \mathsf{P}[\xi_t = 0 \mid \xi_{t-n} = i,\ \xi_{t-1} = 0] = \sum_{ik} \mathsf{P}[\xi_t = 0 \mid \xi_{t-2} = i,\ \xi_{t-1} = 0]\,.$$

$$.\ \mathsf{P}[\xi_{t-n-1} = k]\,.\ \mathsf{P}[\xi_{t-n} = i \mid \xi_{t-n-1} = k]\,.\ \mathsf{P}[\xi_{t-1} = 0 \mid \xi_{t-n-1} = k]\,.$$

First we substitute in this expression the values corresponding to $\mathsf{P}[\xi_t = 0 \mid \xi_{t-n} = i,\ \xi_{t-1} = 0]$ from Table 2. The remaining values in (31) are parameters of source S_2 as the result we have (30). Repeating the same reasoning for $t = n + 3,\ n + 4, \ldots, n +$

+ p, ... it may be shown that for any arbitrarily selected value of $t \geqq n + 2$ it is true that

$$\mathsf{P}[\xi_t = i,\ \xi_{t+1} = i] = 0{\cdot}5b\,, \quad i = 0, 1\,.$$

It may be shown [1] that the probability distribution of states of S_1 equals

$$\{p(E_1), p(E_2), \ldots, p(E_4)\} = \left\{\frac{p_\varepsilon}{2}, \frac{1 - p_\varepsilon}{2}, \frac{1 - p_\varepsilon}{2}, \frac{p_\varepsilon}{2}\right\}. \tag{31*}$$

A good correspondence has been found between these theoretical results and the results of computation of these probability distributions for computer realisations of the TDPI-s of this class.

For having the possibility to use the relationships (16), (18), (21), (25) derived above for one dimensional case, we determine now the corresponding expressions for the quantities p_α, p_β, p_ε and p_π.

According to the Table 2 and taking in account the above relationship (31*) we may write

$$p_\alpha = p_\beta = r \,.\, p_\varepsilon + 0{\cdot}5 \,.\, (1 - p_\varepsilon)\,, \tag{32}$$

$$p_\varepsilon = p_\alpha \cdot p_\beta + (1 - p_\alpha)(1 - p_\beta) = p_\alpha^2 + (1 - p_\alpha)^2\,, \tag{33}$$

$$p_\pi = r p_\alpha^2 + (1 - r)(1 - p_\alpha)^2 + p_\alpha(1 - p_\alpha) = \\ = (1 - p_\alpha) - r(1 - 2p_\alpha)\,. \tag{34}$$

Substituting real values of the above determined quantities into the expressions (16), (18), (21), (25) the values of $\bar{\omega}_\lambda, f_\lambda$, D_λ and $K_\xi(n)$ for any direction $\lambda = \alpha, \beta, \varepsilon, \pi$ in the TDPI-s of this class may be established. As all quantities we are interested in are expressed through p_α, it may be considered a suitable parameter of this class of TDPI-s. Let us express parameter r from Table 2 through p_α. For this purpose first we express p_ε through p_α and r using the expression (32)

$$p_\varepsilon = \frac{p_\alpha - 0{\cdot}5}{r - 0{\cdot}5}\,.$$

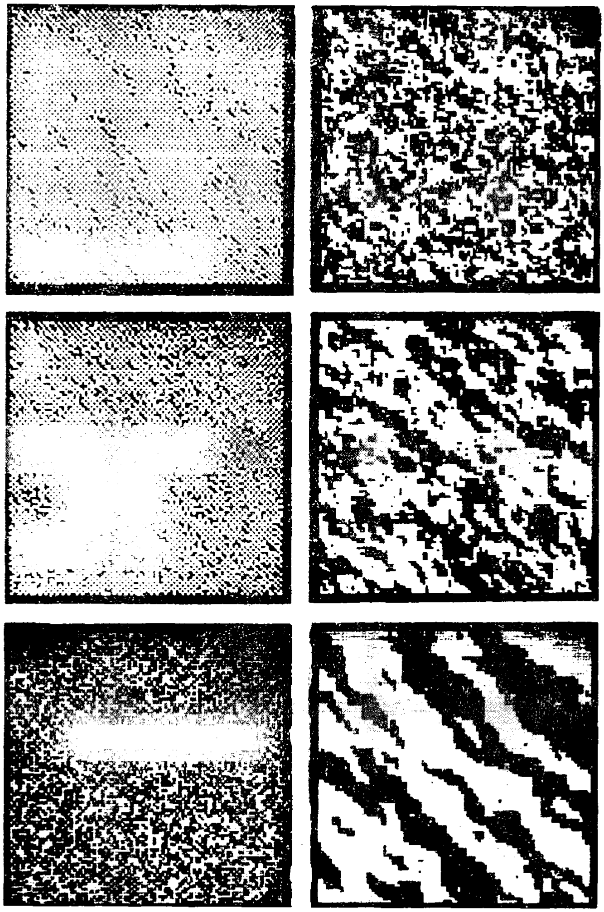
Than substituting this expression instead of p_ε in the expression (33) we have

$$r = \frac{p_\alpha^2}{p_\alpha^2 + (1 - p_\alpha)^2} = \frac{p_\alpha^2}{p_\varepsilon} \tag{35}$$

Examples of TDPI-s with different values of p_α are in Fig. 6.

The average amount of information per symbol of TDPI-s of this class equals evidently to the entropy of the source S_2

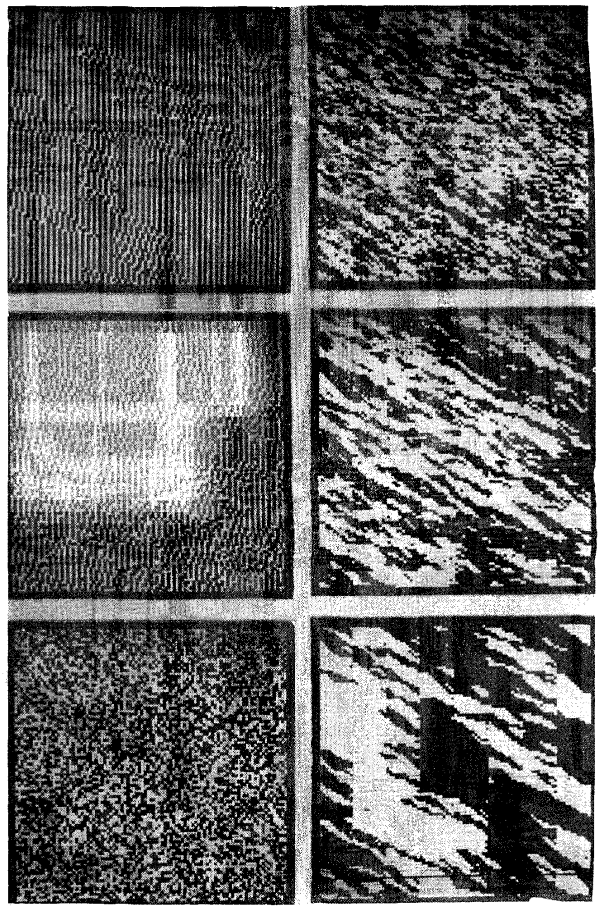
$$H = H_3 = \sum_i \sum_j p(E_i)\, p(j \mid E_i) \log p(j \mid E_i) = p_\varepsilon H^r + (1 - p_\varepsilon) \tag{36}$$



$p_\alpha = 0.08$; $H = 0.21$
$p_\alpha = 0.2$; $H = 0.54$
$h_\alpha = 0.5$, $H = 0.997$

$p_\alpha = 0.69$; $H = 0.79$
$p_\alpha = 0.8$; $H = 0.54$
$p_\alpha = 0.92$; $H = 0.21$

Fig. 6. Examples of TDPI-s in which neighbouring elements are interdependent in directions α and β with the corresponding values of H and p_α; $H_1 = 1$.



$p_\alpha = 0.05$; $H = 0.13$ | $p_\alpha = 0.72$; $H = 0.71$
$p_\alpha = 0.175$; $H = 0.47$ | $p_\alpha = 0.83$; $H = 0.42$
$p_\alpha = 0.5$, $H = 0.99$ | $p_\alpha = 0.94$; $H = 0.14$

Fig. 7. Examples of TDPI-s in which neighbouring elements are interdependent in directions α and π with the corresponding values of H and p_α; $H_1 = 1$.

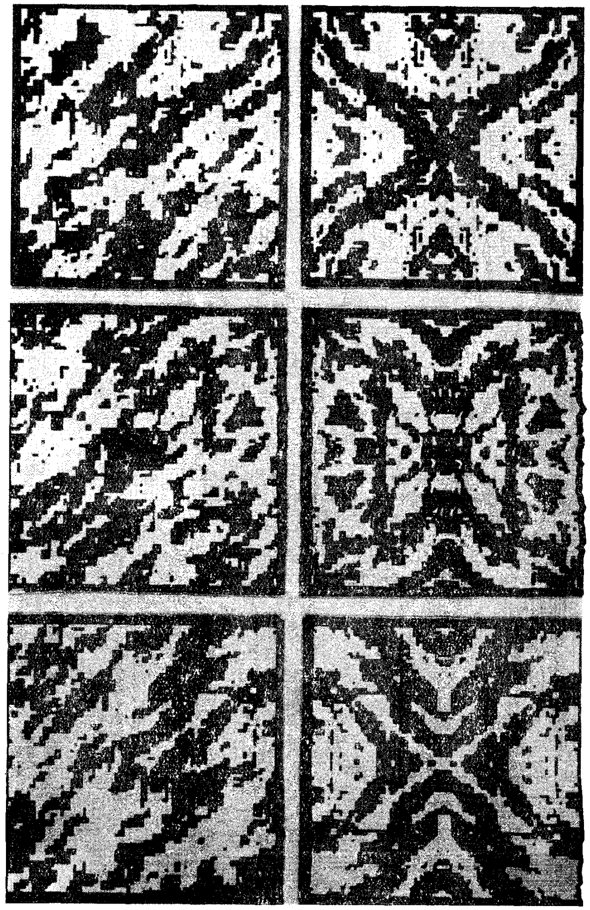


Fig. 8. Examples of two dimensional random figures suitable for psychophysiological experiments. At the right symmetrical transformation of the TDPI, shown in the same row at the left, with respect to two axes.

Table 3.

Probabilistic law of generation the class of TDPI-s with dependencies between the neighbouring elements in directions α and π

$0 \leqq r \leqq 1$

	$X_{r-1,s-1}$	$X_{r,s-1}$	X_{rs}		$\{P(E_i)\}$
			0	1	
E_1	0	0	r	$1 - r$	$\frac{p_\beta}{2}$
E_2	0	1	0·5	0·5	$\frac{1 - p_\beta}{2}$
E_3	1	0	0·5	0·5	$\frac{1 - p_\beta}{2}$
E_4	1	1	$1 - r$	r	$\frac{p_\beta}{2}$

(For details see text.)

where $H^r = -r \log r - (1 - r) \log (1 - r)$, $i = 1, \ldots, 4$, $j = 0, 1$. The graph of the theoretical function $H = f(p_\alpha)$ and the average empirical values computed for different realizations of TDPI-s are in Fig. 3.

The fourth class of TDPI-s differs from the third class described above only with respect to the direction of dependences of neighbouring elements (in the fourth class α and π). The probability law of the source S_2 is given in Tab. 3. The mathematics described for the previous class of TDPI is valid for this case. Examples of TDPI-s are shown in Fig. 7.

For the sake of completness different two dimensional random figures are shown in Fig. 8 which are suitable for psychophysiological experiments. Figures of this type may be constructed from TDPI-s of any of the four classes mentioned by symmetrical transformation of the elements of any quadrant (or other substructure) of the TDPI (in the demonstrated case of the third class) according to its two axes.

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REFERENCES

[1] V. Božkov, T. Radil-Weiss (with the technical collaboration of J. Kolář): To the information content of two-dimensional probabilistic figures. Biokybernetik (Band III). G. Fischer Verlag, Jena 1971, 278—285.

[2] W. Feller: An introduction to probability theory and its applications. (Vol. I). John Viley, New York 1952.

[3] В. Д. Глезер, И. И. Пуккерман: Информация и зрение. Изд. Ак. Наук СССР, Москва 1961.

[4] А. М. Яглом, И. М. Яглом: Вероятность и информация. Гостехиздат, Москва 1960.

[5] B. Julesz: Visual pattern discrimination. Trans. IRE *T-8* (1962), 2.

[6] Д. С. Лебедев: Статистическая модель изображения. Пространственная фильтрация изображений. Изд. Наука, Москва 1970, 53—65.

[7] Д. С. Лебедев: Исследования возможностей статистического кодирования в телевидении. Диссертация МЭИС, Москва 1958.

[8] R. M. Pickett: The perception of a visual texture. J. Exp. Psych. *68* (1964), 13—20.

VÝTAH

Ke generování dvojrozměrných pravděpodobnostních obrazců a určení jejich statistických charakteristik

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Práce se zabývá teorií a praxí generování dvojrozměrných pravděpodobnostních obrazců (charakterizovaných výlučně statistickým rozložením stupně světlosti jejich povrchu). Tyto obrazce se používají v psychofyziologických pokusech vzhledem k tomu, že jsou přesně kvantitativně popsány a neobsahují sémantickou informaci související s předchozí zkušeností vyšetřovaných. Obrazce jsou tvořeny čtercovou maticí konstruovanou podle určitých pravděpodobnostních pravidel. Jsou popsány čtyři různé typy obrazců, lišících se stupněm statistické závislosti mezi sousedními elementy (z nichž jsou obrazce vytvářeny). V práci je též podána formulace hodnocení některých statistických vlastností vhodných při psychofyziologických aplikacích. Jsou popsány algoritmy generování těchto obrazců samočinným počítačem LINC a ukázány jejich příklady.

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