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VARIATIONAL THEOREMS IN GNOSTICAL THEORY OF UNCERTAIN DATA

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Gnostical theory of uncertain data (GT) is a new approach to the processing of data influenced by uncertainty. For GT, as for any theory of data processing, the problem of characterizing optimality principles leading to estimators is of primary interest. Solving of the problem is the main topic of this paper. The optimality principles are formulated as specific variational theorems.

INTRODUCTION

Mai ı objection of any theory dealing with estimation is to find "good" estimators. But how to justify that estimators derived by means of the theory are really good? One of the most popular methods is to choose est:mators according to some optimality principle. Examples of optimality principles commonly used in statistics are maximum likelihood principle or minimum distance principle.

Let us turn to GT. Estimators are defined in GT also by means of specific optimality principles. Particular cases of such principles were introduced by the author of GT [6]. *Our goal is to state a substantially more general version of the principles covering, of course, all the particular ones. These principles will be formulated as variational theorems of a specific nature.*

GT has some inspirations in measurement theory [2], relativistic mechanics and geometry [4]. It is worth mentioning that both quantification and estimation procedures can be modelled – or interpreted – using GT. Namely, a quantification procedure can be interpreted in GT as a motion along a path in the Minkowskian plane, while an estimation procedure can be interpreted as a motion along a path in the Euclidean plane.

GT introduces various characteristics of an individual observed datum as well as of data samples ([2, 6]). Examples of such characteristics are entropy, information and irrelevance. Consider for instance the gnostical approach to entropy. In contrast with classical information theory, entropy is not related with any probabilistic model

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in GT. Instead, it is related with quantification (estimation resp.) procedure (see [5, 6]). It results that an essential feature of GT is that the entropy depends on specific paths in the Minkowskian (Euclidean resp.) plane. We shall show that this feature of the entropy is, in fact, common to majority of characteristics of data samples introduced in GT. Namely, such a characteristic can be viewed as a functional depending on paths in the plane. For cases of entropy and information this dependence was established by the author of GT (see [2, 5, 6]).

An optimality principle of GT regarding the entropy can be formulated as follows: find such paths in the plane that the entropy corresponding to these paths is minimal (maximal resp.). Optimality principle concerning other characteristics of data samples can be formulated in the same way.

Paths optimal for the cases of entropy and information were found by the author of GT ([2, 5, 6]); they were shown to be of the same type for both cases (ibid). *The aim of the paper is to show that, under assumptions commonly applied in GT, the same paths are optimal for all real-valued characteristics of data samples appropriate for GT.*

The paper is organized as follows.

Sections 1, 3.2 and 5.1 are devoted to modelling of quantification and estimation procedures in GT. *Ideas of the sections are used to motivate assumptions under which the variational theorem and other results are derived in the paper.*

Section 2 is devoted to **gnostical characteristics**. Important characteristics of individual data and of data samples have been introduced in GT. Examples are entropy, information and irrelevance (see [2, 3, 5, 6]). We shall call such characteristics *gnostical characteristics.* Features common to a majority of gnostical characteristics are extracted in Section 2. Considerations of the rest of the paper are based on these features. Therefore, the results obtained in the paper cover all gnostical characteristics at once.

A type of paths, called **gnostical paths** in the paper, plays an important role in GT. We shall show that these paths are extremals of functionals typical for GT. Topics related with gnostical paths are considered in Section 3.

Variational theorem of GT proved in the paper deals with **gnostical functionals** introduced in Section 4.2. *A one-to-one relationship between gnostical characteristics and gnostical functionals is established.*

Variational theorem of GT is stated and proved in Section 5. *The theorem states that gnostical paths are local extremals of gnostical functionals.* The value of a gnostical functional over a path can be interpreted as the overall change of gnostical characteristic during a procedure modelled by means of the path, i. e. during the quantification or the estimation procedure. *Therefore, under the interpretation, gnostical paths are optimal models of quantification and estimation procedures.*

Section 6 is devoted to a notion of **residuum**. A *residuum* is introduced to characterize the overall change of a gnostical characteristic during a process consisting of both quantification and estimation procedures. *It is shown that the residuum takes on its local extremal value when both procedures are modelled by gnostical paths.*

Main results of the paper concern extracting of fundamental features of gnostical characteristics of data samples (Section 2), stating and proving variational theorems of GT (Sections 5 and 6).

NOTATION

The following notation is used throughout this paper.

The symbol R denotes the field of real numbers endowed with the topology of the real line, the symbol R^+ denotes the set of positive real numbers. The symbols x, y, z, r, Ω denote real numbers.

Let f and g be functions, M and M' be sets. Then $\mathrm{Dom}\, f$ and $\mathrm{Ran}\, f$ are the domain and the range of the function f. The expression $f : M \to M'$ means that f is a total mapping (hence $\mathrm{Dom}\, f = M$, $\mathrm{Ran}\, f \subseteq M'$), while $f : M - \to M'$ means that f is a partial mapping (hence $\mathrm{Dom}\, f \subseteq M$, $\mathrm{Ran}\, f \subseteq M'$). The symbol $f \restriction M$ denotes the restriction of the function f to the set M; hence $f \restriction M = f \cap [M \times \mathrm{Ran}\, f]$. Composition of the functions f and g is denoted by $g \circ f$; hence $(g \circ f)(m) = f(g(m))$.

Let f be a real-valued function. We put $\sup |f| := \sup\{\, |f(m)| \,|\, m \in \mathrm{Dom}\, f\}$ and $\inf |f| := \inf\{\, |f(m)| \,|\, m \in \mathrm{Dom}\, f\}$. The expression $\mathrm{sg}\,(f)$ denotes the "sign" of the function f; we put $\mathrm{sg}\,(f) = 1$ if f is nonnegative and takes on a positive value, $\mathrm{sg}\,(f) = -1$ if f is not positive and takes on a negative value, $\mathrm{sg}\,(f) = 0$ otherwise.

Fundamental considerations of GT are related to two varieties – the Minkowskian and the Euclidean plane represented by two algebraic structures – the algebra R_j of double numbers [11, 8] and the field R_i of complex numbers. The **indeterminate** will be denoted by j in the case of double numbers and by i in the case of complex numbers. (Thus we have $j^2 = 1$, $i^2 = -1$.)

Both the algebra R_j of double numbers and the field R_i of complex numbers have many properties in common. The indeterminate will be denoted by s when dealing with such properties. Thus in the following we have $s \in \{j, i\}$.

We shall assume that the algebra R_s is endowed with the **Euclidean topology**. The **modulus** of $x + ys \in R_s$ will be denoted by $|x + ys|_s$. Hence we have $|x+ys|_s = \sqrt{x^2 - s^2y^2}$. For $z \in R$ negative we put $\sqrt{z} := i\sqrt{|z|}$.

Exponential function $\exp_s$, **hyperbolic sine** $\sinh_s$, **cosine** $\cosh_s$ and **tangent** $\tanh_s$ are defined in the customary way. **Polar coordinates** are used in GT as a tool. Any $x + ys \in \exp_s(R_s)$ can be rewritten in the form $x + ys = r \cdot \exp_s(\Omega s)$, where $r \in R^+$ and $\Omega \in R$ are polar coordinates of $x + ys$. If moreover $x \neq 0$, then

$$\frac{y}{x} = \frac{1}{s} \tanh_s(\Omega s). \tag{0.1}$$

1. MODELLING OF QUANTIFICATION AND ESTIMATION IN GT

As stated in the introduction, quantification and estimation procedures are modelled in GT as motions along paths. Namely, a quantification procedure is modelled by a motion along a path in the Minkowskian plane represented by the algebra R_j of double numbers. An estimation procedure is modelled as a motion along a path in the Euclidean plane represented by the field R_i of complex numbers. More precisely, the procedures are modelled by paths in specific parts of the planes [6]. The aim of the section is to *define* these parts.

Notation. We put
$$U_{0s} := \{ x + ys \mid x > |y| \}.$$

A *quantification procedure* is modelled as a motion along a path in U_{0j}. An *estimation procedure* is modelled as a motion along a path in U_{0i}.

Each point in U_{0s} can be expressed by means of polar coordinates, because $U_{0s} \subseteq \exp_s(R_s)$. It is a matter of an easy calculation that we have
$$U_{0s} = \{ r \exp_s(\Omega s) \mid r \in R^+, \Omega \in I_{0s} \},$$
where
$$I_{0j} = R,$$
$$I_{0i} = \left(-\frac{\pi}{4}, \frac{\pi}{4} \right).$$

2. GNOSTICAL CHARACTERISTICS

Various characteristics of individual data and of data samples have been introduced in GT. Examples are entropy, information and irrelevance (see [2, 3, 5, 6]). We shall call such characteristics **gnostical characteristics** below. Some of them are summarized in Table 1.

Properties of particular gnostical characteristics were thoroughly investigated by the author of GT (ibid). *The aim of the section is to extract features common to a majority of gnostical characteristics and relevant from our point of view. Considerations of the rest of the paper are based on the features just mentioned. Therefore, instead of dealing with particular gnostical characteristics, theorems stated below cover all gnostical characteristics at once.*

A gnostical characteristic will be denoted by $\mathcal{G}_s$.

The first property mentioned above reads

(a) *A gnostical characteristic is a mapping from the set U_{0s} into the set of real numbers.*

The second property relevant from our point of view is the following.

(b) *A gnostical characteristic is a homogeneous function of the order* 0.

It means that the value of a gnostical characteristic at a point $x + ys \in U_{0s}$ depends *only* on the value of the ratio $\frac{y}{x}$.

When the polar coordinates are considered, then the value of a gnostical characteristic at the point $r \exp_s(\Omega s)$ depends *only* on the value of Ω. This property is equivalent with (b), as follows from (0.1). Hence there is a function $G_s : I_{0s} \to R$ such that
$$\forall x + ys = r \exp_s(\Omega s) \in U_{0s} : \quad \mathcal{G}_s(x + ys) = G_s(\Omega). \tag{2.1}$$

If (2.1) is true, then we say that the function G_s **represents** the **gnostical characteristic** $\mathcal{G}_s$.

Table 1. Gnostical characteristics

gnostical characteristic	The value of Gnostical characteristic at $x + ys$ (Cartesian coordinates)	at $r \exp_s(\Omega s)$ (polar coordinates)
Entropy	$\dfrac{2s^2y^2}{x^2 - s^2y^2}$	$\cosh_s(2\Omega s) - 1$
Information	$\dfrac{2xy}{x^2+y^2} \ln \dfrac{x+y}{x-y} - \ln \dfrac{x^2+y^2}{x^2-y^2}$ (a) $-\dfrac{4xy}{x^2-y^2} \arctan \dfrac{y}{x} + \ln \dfrac{x^2+y^2}{x^2-y^2}$ (b)	$2s\Omega \tanh_s(2\Omega s) - \ln(\cosh_s(2\Omega s))$
Irrelevance	$\dfrac{2xy}{x^2 - s^2y^2}$	$\dfrac{1}{s} \sinh_s(2\Omega s)$
$p_j - \frac{1}{2}$	$\dfrac{xy}{x^2 + y^2}$ (c)	$\dfrac{1}{2j} \tanh_j(2\Omega j)$
Kernel	$S^{-1} \cdot \left(\dfrac{x^2 - y^2}{x^2 + y^2}\right)^2$ (d)	$S^{-1} \cdot \cosh_j^{-2}(2\Omega j)$

(a) Double case.
(b) Complex case.
(c) Double case only; p_j is the gnostical distribution function (see [5]).
(d) Double case only; S is a positive constant called parameter of scale (ibid).

Assume that a function G_s represents a gnostical characteristic. The third of the properties mentioned above reads

(c) *The function G_s is continuously differentiable on I_{0s}.*

The last property is given by

(d) *The function G_s is strictly monotone on each of the intervals $I_{0s} \cap (-\infty, 0]$ and $I_{0s} \cap [0, \infty)$.*

Moreover, we shall consider gnostical characteristics satisfying

(e) $G_s(0) = 0$.

If G_s^1 represents a gnostical characteristic $\mathcal{G}_s^1$ and $G_s^1(0) \neq 0$, we can consider the function $\mathcal{G}_s$ given by

$$\mathcal{G}_s(\cdot) := \mathcal{G}_s^1(\cdot) - \mathcal{G}_s^1(1).$$

Clearly, $\mathcal{G}_s$ is a gnostical characteristic differing from $\mathcal{G}_s^1$ up to an additive constant and satisfying (a) and (b). Moreover, the conditions (c), (d) and (e) are valid for the function G_s representing $\mathcal{G}_s$.

3. GNOSTICAL PATHS

The section is related to gnostical paths playing an important role in the text.

Problem regarding the optimality principles of GT is the main topic of the paper. Optimality principles of GT will be formulated as variational theorems. The corresponding functional operates on paths which are characterized in this section. Specific paths shown to be local extremals of the functional (Section 5.2) will be called *gnostical paths.*

The section is organized as follows. Gnostical paths are defined in Section 3.1, their basic properties are stated. Quantification and estimation procedures can be modelled (or interpreted) in GT as motions over specific paths. This topic is considered in Section 3.2. A topology on a set of such paths is introduced in Section 3.3. To be able to state variational theorems of GT in Section 5.2, paths from a neighbourhood of gnostical paths are analyzed in Section 3.4.

3.1. Gnostical paths

Paths[1] of special type, called gnostical paths below, play an important role in GT. We shall show that gnostical paths are optimal trajectories from the viewpoint of quantification and estimation procedures (see Section 5.2).

Consider a path C in U_{0s} expressed in polar coordinates. It means that there are two continuously differentiable[2] functions

$$r_C : [0,\, 1] \to R^+, \quad \Omega_C : [0,\, 1] \to R \tag{3.1}$$

such that

$$C(t) \;=\; r_C(t) \exp_s(\Omega_C(t)s) \tag{3.2}$$

is true for all $t \in [0,\, 1]$. Moreover, the function r_C and the derivatives $\dot{r}_C$, $\dot{\Omega}_C$ are determined unambiguously.

Notation. Gnostical paths introduced below are parametrized by a monotone and continuously differentiable[2] function

$$\eta_s : [0,\, 1] \to I_{0s}.$$

The function η_s is assumed to have a nonzero derivative at each point in $[0,\, 1]$, hence

$$\eta_0 \;:=\; \inf |\dot{\eta}_s| \;>\; 0.$$

Let us proceed to a definition of a gnostical path.

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[1] A path is usually defined as a piecewise continuously differentiable curve (see [9], p. 202). We shall limit our considerations to continuously differentiable curves only. See Footnote 1 on p. 74 for details.

[2] One-side derivatives are considered at the end-points of an interval here and in the foregoing text.

Definition 3.1. Let $r \in R^+$. The symbol D_{rs} denotes a path called a **gnostical path** defined by

$$\forall t \in [0, 1] : \quad D_{rs}(t) := r \exp_s(\eta_s(t)s). \tag{3.3}$$

Clearly, a gnostical path D_{rs} is a path in U_{0s}. If polar coordinates are considered, then

$$r_{D_{rs}}(t) = r, \quad \Omega_{D_{rs}}(t) = \eta_s(t) \tag{3.4}$$

is true for all $t \in [0, 1]$.

3.2. Modelling of quantification and estimation in GT

Considerations on modelling of quantification and estimation procedures in GT started in Section 1 are continued in this section.

Each of quantification and estimation procedures is modelled in GT as a motion along a path (cf. [2, 5]). Let us start with modelling of quantification. The motion starts at some point in U_{0j} representing the precise value of the measured quantity. This point *always* lies on the real axis, i.e. it has the form $r_j + 0j$. The motion finishes at some point $u \in U_{0j}$ characterizing, in a sense, the particular observed value of the measured quantity, when observation may be influenced by uncertainty. The point u has the form $r_j \exp_j(\Omega_j j)$, where $\Omega_j \in I_{0j}$.

Therefore the quantification procedure can be modelled in GT by means of a class of paths in U_{0j} having the same "end points" (denoted $r_j + 0j$ and u above).

Assume that $\Omega_j \neq 0$. Hence the point u does not lie on the real axis. It results that there is *just one* gnostical path $D_{r_j j}$ having $r_j + 0j$ and $r_j \exp_j(\Omega_j j)$ as starting and ending points[1]. The path $D_j(r_j, \Omega_j)$ defined by

$$\forall t \in [0, 1] : \quad D_j(r_j, \Omega_j)(t) := r_j \exp_j(\Omega_j \cdot tj) \tag{3.5}$$

is of this type.

The estimation procedure is modelled as a motion along a path in U_{0i}. The motion starts at some point in U_{0i}; let us denote it by $r_i \exp_i(\Omega_i i)$. It finishes at a point in U_{0i}, which *always* lies on the real axis, namely at the point $r_i + 0i$. It follows that an estimation procedure may be modelled by means of paths in U_{0i} having the same end points. If $\Omega_i \neq 0$, then there is *just one* gnostical path having $r_i \exp_i(\Omega_i i)$ and $r_i + 0i$ as starting and ending points[1]. The path $D_i(r_i, \Omega_i)$ defined by

$$\forall t \in [0, 1] : \quad D_i(r_i, \Omega_i)(t) := r_i \exp_i(\Omega_i \cdot (1 - t)i) \tag{3.6}$$

is of this type.

[1] More precisely, the mentioned gnostical path is unique up to a parametrization by the function $\eta_\bullet$; on the other hand reparametrization of a gnostical path does not affect our results, as it does not affect results on integrals over paths in complex analysis (see [9], p. 202).

3.3. A topology on a set of paths

Optimality principle of GT can be interpreted as a characterization of paths modelling quantification and estimation procedures in an optimal manner. To be able to state this principle we need a topology on a set of paths in U_{0s} having the same end points. We shall consider the $\mathcal{C}_1$-topology. The aim of the section is to state a metric producing the topology.

Notation. Consider a fixed path D in U_{0s}. We put

$$\mathcal{U}_s(D)$$

equal to the set of all paths C in U_{0s} satisfying

$$r_C(0) = r_D(0),\ r_C(1) = r_D(1),\ \Omega_C(0) = \Omega_D(0),\ \Omega_C(1) = \Omega_D(1).$$

Hence $\mathcal{U}_s(D)$ is the class of all paths in U_{0s} having the same "end points" as D.

Assume that the symbols C and C' denote paths in $\mathcal{U}_s(D)$. Clearly, $\mathcal{C}_1$-topology on the set $\mathcal{U}_s(D)$ of paths is produced by the metric

$$\rho(C,\, C') := \sup |r_C - r_{C'}| + \sup |\dot{r}_C - \dot{r}_{C'}| + \sup |\dot{\Omega}_C - \dot{\Omega}_{C'}|.$$

Notation. For any path D in U_{0s} and $0 < \delta \in R$ we put

$$\mathcal{U}_s(D,\, \delta) := \{C \in \mathcal{U}_s(D) \,|\, \rho(C,\, D) < \delta\}.$$

Hence $\mathcal{U}_s(D,\, \delta)$ is a neighbourhood of the path D in $\mathcal{U}_s(D)$.

3.4. On neighbourhoods of gnostical paths

We shall show in the foregoing text that gnostical paths are local extremals of functionals typical for GT. For this reason we derive properties of paths lying in a neighbourhood of a gnostical path.

Recall that the symbol $\mathrm{sg}\,(f)$ denotes a "sign" of a real-valued function f (see section Notation).

Lemma 3.1. Let $r \in R^+$. Consider

$$\delta_{1s} := \min\left\{\frac{\eta_0}{2},\, \frac{\eta_0 r}{2+\eta_0}\right\} \tag{3.7}$$

and a path $C \in \mathcal{U}_s(D_{rs},\, \delta_{1s})$.

Then Ω_C is strictly monotone on $[0,\, 1]$, $\mathrm{Ran}\,\Omega_C = \mathrm{Ran}\,\Omega_{D_{rs}}$ and

$$\mathrm{sg}\,(\dot{\Omega}_C) = \mathrm{sg}\,(\dot{\eta}_s),$$

$$\mathrm{sg}\left(|\dot{\Omega}_C| - \frac{|\dot{r}_C|}{r_C}\right) = 1.$$

Proof. Consider a path $C \in \mathcal{U}_s(D_{rs}, \delta_1)$. Let us denote $D := D_{rs}$, $\delta := \delta_{1s}$ and put

$$w := r_C - r_D, \quad \omega := \Omega_C - \Omega_D. \tag{3.8}$$

Then w and ω are continuously differentiable on [0, 1]. We have

$$\sup|w| < \delta, \quad \sup|\dot{w}| < \delta, \quad \sup|\dot{\omega}| < \delta, \tag{3.9}$$

because $C \in \mathcal{U}_s(D, \delta)$.

It holds $r_D(t) = r$ for all $t \in [0, 1]$ by (3.4). Moreover $r > \delta$ according to (3.7), so that

$$r_C(t) = r + w(t) > r - \delta > 0. \tag{3.10}$$

For all $t \in [0, 1]$ we have

$$|\dot{r}_C(t)| = |\dot{w}(t)| < \delta \tag{3.11}$$

by (3.10) and (3.9).

It holds $\dot{\Omega}_D = \dot{\eta}_s$ according to (3.4), hence

$$\dot{\Omega}_C = \dot{\eta}_s + \dot{\omega}. \tag{3.12}$$

by (3.8). Moreover for all $t \in [0, 1]$ we have $|\dot{\omega}(t)| < \delta < \eta_0$ and $\eta_0 \le |\dot{\eta}_s(t)|$, so that the function Ω_C is strictly monotone and $\operatorname{sg}(\dot{\Omega}_C) = \operatorname{sg}(\dot{\eta}_s)$, as follows from (3.12). Therefore also $\operatorname{Ran}\Omega_C = \operatorname{Ran}\Omega_{D_{rs}}$, because C has the same "end" points as D_{rs}.

Further on, for all $t \in [0, 1]$ we have $|\dot{\Omega}_C(t)| \ge |\dot{\eta}_s(t)| - |\dot{\omega}(t)| > \eta_0 - \delta$ by (3.12) and (3.9), hence

$$|\dot{\Omega}_C(t)| > \frac{\eta_0}{2},$$

by (3.7), so that

$$\begin{aligned} |\dot{\Omega}_C(t)| - \frac{|\dot{r}_C(t)|}{r_C(t)} &> \frac{\eta_0}{2} - \frac{\delta}{r-\delta} \\ &= \frac{\frac{\eta_0 r}{2+\eta_0} - \delta_{1s}}{2(r - \delta_{1s})} \cdot (2 + \eta_0) \\ &\ge 0 \end{aligned}$$

is true, where the former inequality follows from (3.10) and (3.11), the latter one from (3.7) and (3.10). Therefore Lemma 3.1 is valid. □

4. GNOSTICAL FUNCTIONALS

We shall show that a value of a gnostical characteristic can be interpreted as a value of a specific functional on a gnostical path. Such a functional is called a gnostical functional below. The aim of the section is to define gnostical functionals and show their basic properties. The functionals play an important role in the paper – variational theorems stated in the next two Sections 5 and 6 are based on gnostical functionals.

The section is organized as follows. An auxiliary functional E is introduced in Section 4.1, its basic properties are stated. A gnostical functional is defined in Section 4.2 as a specific type of the functional E. A one-to-one relationship between gnostical functionals and gnostical characteristics is established in the section.

4.1. Functional E

We introduce a functional E, which is a slight modification of the integral over a path[2]. The reason of the modification is to overcome problems raised by complex values of integral over a path in the double case.

Definition 4.1. Assume that C is a path in R_s. Let $g_s : R_s - \longrightarrow R$ be continuous on $\operatorname{Ran} C$. We put

$$E(g_s, C) := |s|_s \cdot \int_C g_s(l)\mathrm{d}l := |s|_s \cdot \int_0^1 g_s(C(t)) \cdot |\dot{C}(t)|_s \mathrm{d}t. \tag{4.1}$$

We shall view $E(g_s, \cdot)$ as a functional operating on paths C in U_{0s}.

Notation. To give some formulas stated below more readable, we sometimes omit arguments of functions under an integral sign.

Applying this convention to (4.1) we obtain

$$E(g_s, C) = \int_0^1 C \circ g_s \cdot |s\dot{C}|_s \mathrm{d}t.$$

Finally, we express the functional $E(g_s, \cdot)$ via the polar coordinates and state the values of the functional on the gnostical paths. These results are applied below as a tool.

Lemma 4.1. Assume that C is a path in U_{0s} given by (3.1) and (3.2). Let $g_s : R_s - \longrightarrow R$ be continuous on $\operatorname{Ran} C$. Then

$$E(g_s, C) = \int_0^1 C \circ g_s \cdot \sqrt{r_C^2 \dot{\Omega}_C^2 - s^2 \dot{r}_C^2}\mathrm{d}t.$$

Recall that D_{rs} denotes a gnostical path given by (3.3), while $D_s(r, \Omega)$ denotes the specific gnostical path given by (3.5) and (3.6).

[2] An integral of a continuous function over a piecewise continuously differentiable curve is studied in complex analysis (see [9]). We shall limit ourselves to integrals of continuous functions over paths (i.e. over continuously differentiable curves) only. The reason is that obtained formulas are of simpler form in this case. Nevertheless extension of our results to a more general case of piecewise continuity is quite simple, as it is in complex analysis, too.

Lemma 4.2. Consider a function $g_s : R_s - \rightarrow R$. Let $r \in R^+$ and $\Omega \in I_{0s}$.
a) If the function g_s is continuous on $\operatorname{Ran} D_{rs}$, then

$$E(g_s, D_{rs}) = r \cdot \operatorname{sg}(\dot{\eta}_s) \cdot \int_{\Omega_{D_{rs}}(0)}^{\Omega_{D_{rs}}(1)} g_s(r \exp_s(zs)) \mathrm{d}z$$

and $E(g_s, D_{rs})$ is a real number.
b) If the function g_s is continuous on $\operatorname{Ran} D(r, \Omega)$, then

$$E(g_s, D_s(r, \Omega)) = r_s \cdot \operatorname{sign}(\Omega) \cdot \int_0^{\Omega} g_s(r \exp_s(zs)) \mathrm{d}z$$

and $E(g_s, D_s(r, \Omega))$ is a real number.
c) If $r \in \operatorname{Dom} g_s$, then

$$E(g_s, D_s(r, 0)) = 0.$$

4.2. Gnostical functionals

A gnostical functional is defined in the section as a specific type of the functional E. A one-to-one relationship between gnostical characteristics and gnostical functionals is established.

We start with the definition of a gnostical functional.

Definition 4.2. Suppose that $g_s : R_s - \rightarrow R$ is continuous on U_{0s}. The functional $E(g_s, \cdot)$ is called **gnostical** *iff* the value of

$$E(g_s, D_s(r, \Omega))$$

does not depend on r, i.e. *iff* $E(g_s, D_s(r, \Omega)) = E(g_s, D_s(r', \Omega))$ is true for all $r, r' \in R^+$ and $\Omega \in I_{0s}$.

A relationship between gnostical functionals and gnostical characteristics is established in

Theorem 4.1. Consider a function $g_s : R_s - \rightarrow R$.
a) Assume that the function g_s is continuous on U_{0s} and that the functional $E(g_s, \cdot)$ is gnostical.

Then there is a function $G_s : I_{0s} \rightarrow R$ such that

a1 $G_s(0) = 0$

a2 G_s is continuously differentiable on I_{0s}

a3 It holds

$$\forall r \in R^+, \Omega \in I_{0s} : \quad g_s(r \exp_s(\Omega)) = \frac{1}{r} \cdot \dot{G}_s(\Omega) \tag{4.2}$$

a4 For all $r \in R^+$ and $\Omega \in I_{0s}$ we have

$$E(g_s, D_s(r, \Omega)) = \operatorname{sign}(\Omega) \cdot G_s(\Omega). \tag{4.3}$$

b) Assume that a function $G_s : I_{0s} \to R$ satisfies a1 and a2, the mapping g_s satisfies (4.2).

Then g_s is defined and continuous on U_{0s}. Moreover the functional $E(g_s, \cdot)$ is gnostical and a4 takes place.

Proof. a) Assume that the functional $E(g_s, \cdot)$ is gnostical. Hence, as follows from Lemma 4.2b, c there is a function

$$G_s : I_{0s} \to R$$

such that for all $r \in R^+$ and $\Omega \in I_{0s}$ we have (4.3). Clearly, without a loss of generality we can assume that

$$G_s(0) = 0. \tag{4.4}$$

Let us fix some $r \in R^+$ and $\Omega \in I_{0s}$. We have

$$\operatorname{sign}(\Omega) \cdot G_s(\Omega) = r \cdot \operatorname{sign}(\Omega) \cdot \int_0^\Omega g_s(r \exp_s(zs))\mathrm{d}z \in R, \tag{4.5}$$

as follows from (4.3), (4.4) and Lemma 4.2b. Therefore

$$G_s(\Omega) = r \cdot \int_0^\Omega g_s(r \exp_s(zs))\mathrm{d}z$$

is true by (4.5) and (4.4). So that

$$\dot{G}_s(\Omega) = r \cdot g_s(r \exp(\Omega s))$$

takes place, which proves (4.2). Hence also $\dot{G}_s(\Omega) = g_s(\exp_s(\Omega s))$. Now $g_s(\exp_s(\bullet s))$ is continuous on I_{0s}, therefore G_s is continuously differentiable on I_{0s}.

b) Suppose that the assumptions of Theorem 4.1b are fulfilled, $r \in R^+$ and $\Omega \in I_{0s}$. Hence

$$\begin{aligned} E(g_s, D_s(r, \Omega)) &= r \cdot \operatorname{sign}(\Omega) \cdot \int_0^\Omega \frac{1}{r} \cdot \dot{G}_s(z)\mathrm{d}z \\ &= \operatorname{sign}(\Omega) \cdot G_s(\Omega) \end{aligned}$$

is true according to Lemma 4.2b, (4.2) and a1, so that a4 is valid. Now a4 implies that the functional $E(g_s, \cdot)$ is gnostical. □

Theorem 4.1 can be interpreted as follows. Each gnostical functional determines a gnostical characteristic via a function G_s, where G_s is continuously differentiable and $G_s(0) = 0$. Conversely, any gnostical characteristic represented by a continuously differentiable function G_s satisfying $G_s(0) = 0$ determines unambiguously a gnostical functional. Keeping in mind the conditions (c) and (e) from Section 2 *a one-to-one relationship between gnostical characteristics and gnostical functionals is established.*

We express the value of the gnostical functional $E(g_s, \cdot)$ using the function G_s in the following corollary. It is an immediate consequence of Lemma 4.2 and Theorem 4.1.

Corollary 4.1. Consider a function $g_s : R_s - \to R$ continuous on U_{0s}. Assume that the functional $E(g_s, \cdot)$ is gnostical. Moreover let G_s be defined as in Theorem 4.1a.

If C is a path in U_{0s}, then

$$E(g_s, C) = \int_0^1 \Omega_C \circ \dot{G}_s \cdot \sqrt{\dot{\Omega}_C^2 - s^2 \cdot \frac{\dot{r}_C^2}{r_C^2}} \mathrm{d}t. \tag{4.6}$$

If $r \in R^+$, then

$$E(g_s, D_{rs}) = \mathrm{sg}\,(\dot{\eta}_s) \cdot [G_s(\Omega_{D_{rs}}(1)) - G_s(\Omega_{D_{rs}}(0))].$$

5. VARIATIONAL THEOREM OF GT

The aim of the section is to state and prove basic variational theorem of GT.

Considerations on modelling of quantification and estimation procedures started in Sections 1 and 3.2 are continued in Section 5.1. They are used to motivate assumptions under which a variational theorem of GT and related results will be stated and proved.

Variational theorem of GT proved in Section 5.2 shows that gnostical paths are extremals of gnostical functionals. Interpretation of this fact is stated at the end of the section.

5.1. Modelling of quantification and estimation in GT

We shall continue considerations on modelling of quantification and estimation procedures in GT started in Sections 1 and 3.2.

Recall that quantification (estimation resp.) procedure is modelled in GT as a motion along a path $D_{r_s s}$ starting at some point $r_s + 0s$ representing the precise value of a measured quantity; it finishes at some point $r_s \exp_s(\Omega_s s) \in U_{0s}$ characterizing the observed value of the measured quantity. Assume for the sake of simplicity that $\Omega_s \neq 0$.

Consider a gnostical characteristic $\mathcal{G}_s$ and function G_s representing it. The function G_s is continuously differentiable and strictly monotone on each of the intervals $I_{0s} \cap [0, \infty)$ and $I_{0s} \cap (-\infty, 0]$, as follows from conditions (c) and (d) of Section 2. One of these intervals containing Ω_s will be denoted as I_s.

Hence G_s is continuously differentiable and strictly monotone on the interval I_s. Moreover we put

$$U_s := \{r \exp_s(\Omega s) \,|\, r \in R^+, \Omega \in I_s\}.$$

It follows that the whole image of $D_{r_s s}$ lies in U_s, i.e. that $D_{r_s s}$ is a path in U_s.

Finally, assume that g_s is defined by (4.2). Then g_s is continuous on U_s and

$$\mathrm{sg}\,(g_s \restriction U_s) \in \{-1, 1\}. \tag{5.1}$$

The relations just derived are stated as assumptions in the next two theorems, which are the basic results of the paper.

5.2. Extremals of gnostical functionals

Variational theorem of GT is stated and proved in this section.

The condition (5.1) stated and reasoned in the previous section is in a good accordance with basic principles of GT. If the function g_s satisfies the condition and the functional $E(g_s, \cdot)$ is gnostical, then gnostical paths are local extremals of the functional, as stated in

Theorem 5.1. *Variational theorem of GT.*
Consider a function $g_s : R_s - \to R$ continuous on U_{0s}.

Suppose that sg $(g_s \restriction U_s) \in \{-1, 1\}$ and that the functional $E(g_s, \cdot)$ is gnostical.

Let $r \in R^+$. Assume that the gnostical path D_{rs} is a path in U_s, δ_{1s} is defined by (3.7).

a) If $C \in \mathcal{U}_s(D_{rs}, \delta_{1s})$, then C is a path in U_s.

b) The path D_{rs} is a local extremal of the functional $E(g_s, \cdot)$ on $\mathcal{U}_s(D_{rs}, \delta_{1s})$.

c) For all $C \in \mathcal{U}_s(D_{rs}, \delta_{1s})$ we have

$$s^2 \cdot q_s \cdot E(g_s, C) \leq s^2 \cdot q_s \cdot E(g_s, D_{rs}),$$

where $q_s :=$ sg $(g_s \restriction U_s)$.

Proof. 1. Consider the notation of Theorem 5.1. We denote $D := D_{rs}$. Let $C \in \mathcal{U}_s(D, \delta_{1s})$. Then $\operatorname{Ran}\Omega_C = \operatorname{Ran}\Omega_D$ according to Lemma 3.1, so that C is a path in U_s, hence $E(g_s, C)$ is defined and Theorem 5.1a is true.

Theorem 5.1b is an immediate consequence of Theorem 5.1c. Let us proceed to proof of the latter one.

2. Let G_s be the mapping defined by the function g_s according to Theorem 4.1a restricted on the set I_s. We have $q_s \in \{-1, 1\}$ and sg $\left(\dot{G}_s\right) = q_s$ according to the assumptions of the theorem being proved. Hence

$$|\dot{G}_s| = q_s \cdot \dot{G}_s \,. \tag{5.2}$$

3. For each path $C \in \mathcal{U}_s(D, \delta_{1s})$ we shall consider the following integral

$$\begin{aligned} J_s(C) &:= \int_0^1 \left(\Omega_C \circ \dot{G}_s\right) \cdot \dot{\Omega}_C \mathrm{d}t \\ &= \int_0^1 [G_s(\Omega_C(t))]' \,\mathrm{d}t \\ &= G_s(\Omega_C(1)) - G_s(\Omega_C(0)) \,. \end{aligned}$$

Therefore we have

$$J_s(C) = G_s(\Omega_D(1)) - G_s(\Omega_D(0)) = J_s(D), \tag{5.3}$$

because $\Omega_C(0) = \Omega_D(0)$ and $\Omega_C(1) = \Omega_D(1)$ according to the definition of $\mathcal{U}_s(D, \delta_{1s})$.

4. It holds

$$E(g_s, D) = \operatorname{sg}(\dot{\eta}_s) \cdot J_s(D)$$

according to Corollary 4.1 and (5.3), so that

$$q_s \cdot \operatorname{sg}(\dot{\eta}_s) \cdot J_s(C) = q_s \cdot E(g_s, D) \tag{5.4}$$

by (5.3).

5. Let us consider the double case first. It holds

$$q_j \cdot E(g_j, C) = \int_0^1 \left|\Omega_C \circ \dot{G}_j\right| \cdot \sqrt{\dot{\Omega}_C^2 - \frac{\dot{r}_C^2}{r_C^2}} dt$$

by (4.6) and (5.2). Moreover

$$\frac{|\dot{r}_C(t)|}{r_C(t)} < \left|\dot{\Omega}_C(t)\right|, \quad \left|\dot{\Omega}_C(t)\right| = \operatorname{sg}(\dot{\eta}_j) \cdot \dot{\Omega}_C(t)$$

is true for all $t \in [0, 1]$ according to Lemma 3.1, hence

$$\begin{aligned} q_j \cdot E(g_j, C) &\leq \int_0^1 \left|\Omega_C \circ \dot{G}_j\right| \cdot \left|\dot{\Omega}_C\right| dt \\ &= \operatorname{sg}\left(\dot{G}_j\right) \cdot \operatorname{sg}(\dot{\eta}_j) \cdot \int_0^1 \left(\Omega_C \circ \dot{G}_j\right) \cdot \dot{\Omega}_C dt \\ &= q_j \cdot \operatorname{sg}(\dot{\eta}_j) \cdot J_j(C). \end{aligned}$$

Therefore

$$q_j \cdot E(g_j, C) \leq q_j \cdot E(g_j, D)$$

by (5.4), which together with $j^2 = 1$ gives

$$j^2 \cdot q_j \cdot E(g_j, C) \leq j^2 \cdot q_j \cdot E(g_j, D).$$

6. Consider the complex case now. We have

$$\begin{aligned} q_i \cdot E(g_i, C) &= \int_0^1 \left|\Omega_C \circ \dot{G}_i\right| \cdot \sqrt{\dot{\Omega}_C^2 + \frac{\dot{r}_C^2}{r_C^2}} dt \\ &\geq \int_0^1 \left|\Omega_C \circ \dot{G}_i\right| \cdot \left|\dot{\Omega}_C\right| dt \end{aligned} \tag{5.5}$$

by (4.6) and (5.2). Moreover $|\dot{\Omega}_C| = \operatorname{sg}(\eta_i) \cdot \dot{\Omega}_C$ according to Lemma 3.1, so that

$$\begin{aligned} \int_0^1 \left|\Omega_C \circ \dot{G}_i\right| \cdot \left|\dot{\Omega}_C\right| dt &= \operatorname{sg}\left(\dot{G}_i\right) \cdot \operatorname{sg}(\dot{\eta}_i) \cdot \int_0^1 \left(\Omega_C \circ \dot{G}_i\right) \cdot \dot{\Omega}_C dt \\ &= q_i \cdot \operatorname{sg}(\dot{\eta}_i) \cdot J_i(C). \end{aligned}$$

Hence, keeping in mind (5.4) and (5.5), we obtain

$$q_i \cdot E(g_i, C) \geq q_i \cdot E(g_i, D),$$

so that

$$i^2 \cdot q_i \cdot E(g_i, C) \leq i^2 \cdot q_i \cdot E(g_i, D).$$

□

Let us proceed to an interpretation of the results obtained in Theorem 5.1. It will be based on the ideas and notation introduced in Section 5.1.

Consider a gnostical characteristic $\mathcal{G}_s$ and the corresponding function g_s. As follows from the considerations of Section 5.1, the function g_s is continuous on U_{0s} and $\operatorname{sg}(g_s \restriction U_s) \in \{-1, 1\}$. Hence Theorem 5.1 can be applied.

We can interpret the value of $E(g_s, C_s)$ as the overall change of the gnostical characteristic during the quantification (estimation resp.) procedure. The change of the gnostical characteristic is extremal if C_s is a gnostical path $D_{r_s s}$, as follows from Theorem 5.1. As a rule, the "worst" type of quantification procedure is modelled by the gnostical path $D_{r_j j}$; the "best" type of the estimation procedure is modelled by the gnostical path $D_{r_i i}$. It means that the game of the nature is considered resulting in maximal damage of observed value of the measured quantity. The strategy of an estimator applying GT is such that it uses an optimal (i.e. the "best") available type of estimation.

6. RESIDUUM IN GNOSTICAL THEORY

The concept of residuum is introduced in GT [6] to characterize the overall change of a gnostical characteristic during a process consisting of both the quantification and the estimation procedures. We show that the residuum reaches its local extremal value when both the quantification and estimation procedures are modelled by gnostical paths.

Definition 6.1. Consider a path C_s in U_s; let $g_s : R_s - \longrightarrow R$ be continuous on $\operatorname{Ran} C_s$.

The sum

$$\operatorname{Res}(g_j, g_i, C_j, C_i) := E(g_j, C_j) + E(g_i, C_i)$$

is called a **residuum of the pair $\langle g_j, g_i \rangle$ of functions over the pair $\langle C_j, C_i \rangle$ of paths**.

Notation. For each $\delta > 0$ we define the neighbourhood $\mathcal{U}(C_j, C_i, \delta)$ of the pair $\langle C_j, C_i \rangle$ of paths by

$$\mathcal{U}(C_j, C_i, \delta) := \mathcal{U}_j(C_j, \delta) \times \mathcal{U}_i(C_i, \delta).$$

The following residuum theorem is a consequence of Theorem 5.1.

Theorem 6.1. Consider a function $g_s : R_s - \to R$ continuous on U_{0s}.

Suppose that $q_s := \mathrm{sg}\,(g_s \restriction U_s) \in \{-1, 1\}$, $q_j \neq q_i$ and that the functional $E(g_s, \cdot)$ is gnostical.

Assume that $r_s \in R^+$. Finally, let the gnostical path $D_{r_s s}$ be a path in U_s, δ_{1s} defined by (3.7), $\delta = \min(\delta_{1j}, \delta_{1i})$.

a) If $\langle C_j, C_i \rangle \in \mathcal{U}\left(D_{r_j j}, D_{r_i i}, \delta\right)$, then C_s is a path in U_s.

b) The pair $\left\langle D_{r_j j}, D_{r_i i} \right\rangle$ is an extremal of $\mathrm{Res}\,(g_j, g_i, \cdot, \cdot)$ on the set $\mathcal{U}\left(D_{r_j j}, D_{r_i i}, \delta\right)$.

c) For any pair $\langle C_j, C_i \rangle \in \mathcal{U}\left(D_{r_j j}, D_{r_i i}, \delta\right)$ we have

$$q_j \cdot \mathrm{Res}\,\left(g_j, g_i, C_j, C_i\right) \le q_j \cdot \mathrm{Res}\,\left(g_j, g_i, D_{r_j j}, D_{r_i i}\right).$$

P r o o f. We shall prove Theorem 6.1c only.

We have $q_s \in \{-1, 1\}$ and $q_i \neq q_j$, so that $q_j = s^2 q_s$. Moreover

$$\begin{aligned} q_j E(g_s, C_s) &= s^2 q_s E(g_s, C_s) \\ &\le s^2 q_s E(g_s, D_{r_s s}) \\ &= q_j E(g_s, D_{r_s s}) \end{aligned}$$

according to Theorem 5.1, so that

$$\begin{aligned} q_j \cdot \mathrm{Res}\,(g_j, g_i, C_j, C_i) &= j^2 q_j E(D_j, C_j) + i^2 q_i E(g_i, C_i) \\ &\le j^2 q_j E(g_j, D_{r_j j}) + i^2 q_i E(g_i, D_{r_i i}) \\ &= q_j \mathrm{Res}\,(g_j, g_i, D_{r_j j}, D_{r_i i}). \end{aligned}$$

□

Consider a gnostical characteristic represented by a function G_s. If G_j increases on I_j and G_i decreases on I_i, then the residuum $\mathrm{Res}\,(g_j, g_i, \cdot, \cdot)$ takes on its local maximum on the pair $\left\langle D_{r_j j}, D_{r_i i} \right\rangle$ of gnostical paths. If G_j decreases on I_j and G_i increases on I_i, then the residuum takes on its local minimum on the pair.

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