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Optimization Problem with Parameter and Its Application to the Problems of Two-Stage Stochastic Nonlinear Programming

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If there is an unknown parameter in an optimization problem then the optimum is a function of this parameter. Surely, the study of this function is very important. In this paper, firstly, we shall consider a deterministic problem with an unknown vector parameter. We shall try to find conditions under which the optimum is a continuous, concave and Lipschitz function of the parameter. This at the same time also yields sufficient conditions for stability of the deterministic nonlinear optimization problems.

Secondly, we shall use the mentioned results for some types of stochastic models. It is easy to see that, generally, the optimalized function in two-stage stochastic programming problems is the mathematical expectation of the optimal value of the optimalized function in a deterministic optimization problem with the parameter. We shall introduce conditions under which the optimalized function in a two-stage stochastic nonlinear programming problem is continuous and differentiable. At the end of this paper some examples of two-stage stochastic nonlinear programming problems fulfilling this conditions will be given.

I. Introduction

In the first part of this paper we shall consider a deterministic optimization problem with an unknown parameter. We shall investigate the dependence of the optimal value on the parameter. From the practical point of view, it is very important that "small" variations of the parameter caused "small" variations of the optimal value too. It is easy to see (cf. [1]) that this property need not be fulfilled even in a very simple case. In this paper we shall try to find a class of deterministic parameteric problems for which the optimum is a continuous, concave and Lipschitz function of the parameter.

We shall apply the results of the first part of this paper to derive the properties of two-stage stochastic nonlinear programming problems. We shall find conditions under which the optimalized function in these problems is continuous and differentiable. These conditions apply directly to functions figuring in the setting of the problem itself. The results of this paper complete those of the results of [4] where the continuity of the optimalized function follows from the continuity of a set mapping. At the end of this paper some examples of two-stage stochastic nonlinear programming problems fulfilling our conditions will be given.

It remains to remark that the corresponding results for linear case has been proved in [2].

II. Deterministic Models

1. FORMULATION OF THE PROBLEM

Let $X \subset E_n$, $Y \subset E_m$ be non-empty sets, $F(\mathbf{x}, \mathbf{y})$, $F_i(\mathbf{x}, \mathbf{y})$, $i = 1, 2, \ldots, l$ be real valued functions defined on $E_n \times E_m$. (E_n, $n \geqq 1$ denotes n-dimensional Euclidean space).

If

$$(1) \qquad K(\mathbf{y}) = \{\mathbf{x} \in X : F_i(\mathbf{x}, \mathbf{y}) \geqq 0, i = 1, 2, \ldots, l\} \quad \text{for} \quad \mathbf{y} \in E_m ,$$

then the general deterministic optimization problem with the vector parameter $\mathbf{y}$ can be introduced as a problem to find

$$(2) \qquad \sup \{F(\mathbf{x}, \mathbf{y}) : \mathbf{x} \in K(\mathbf{y})\} = \varphi(\mathbf{y}) \quad \text{for} \quad \mathbf{y} \in Y .$$

2. SOME AUXILIARY ASSERTIONS

If $Y \neq \emptyset$ then (1) defines a mapping of Y into the space of subsets of X. If, further, X is a compact set and $F_i(\mathbf{x}, \mathbf{y})$, $i = 1, 2, \ldots, l$ are continuous functions then (1) defines a mapping of Y into the space of compact subsets of X. Since it is easy seen from [4] that the continuity of $\varphi(\mathbf{y})$ follows (under general conditions) from the uniform continuity of $K(\mathbf{y})$; so to obtain conditions under which $\varphi(\mathbf{y})$ is a continuous function it is enough to find such properties of $F_i(\mathbf{x}, \mathbf{y})$, $i = 1, 2, \ldots, l$ that $K(\mathbf{y})$ is a uniformly continuous mapping.

We shall give some definitions.

The Hausdorff distance between two subsets in E_n is defined in the following way.

Definition 1. If $X', X'' \subset E_n$, $n \geqq 1$ are two non-empty sets then the Hausdorff distance of these sets $\Delta_n(X', X'')$ is defined by

$$(3) \qquad \Delta_n(X', X'') = \max [\delta_n(X', X''), \delta_n(X'', X')]$$

$$\delta_n(X', X'') = \sup_{\mathbf{x}' \in X'}, \inf_{\mathbf{x}'' \in X''} \varrho_n(\mathbf{x}', \mathbf{x}'')$$

where ϱ_n denotes the Euclidean metric in E_n. (We usually leave the subscripts in symbols Δ_n, ϱ_n, δ_n.)

Now, we can already give the definition of the mapping semicontinuous from below and the definition of the uniformly continuous mapping.

Definition 2. $K(\mathbf{y})$ is a mapping semicontinuous from below in the point $\mathbf{y}_0 \in Y$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that the implication

$$\varrho_m(\mathbf{y}, \mathbf{y}_0) < \delta \,, \quad \mathbf{y} \in Y \Rightarrow \delta_n[K(\mathbf{y}_0), K(\mathbf{y})] < \varepsilon$$

is valid.

Definition 3. $K(\mathbf{y})$ is a uniformly continuous mapping on Y if for every $\varepsilon > 0$ there exists $\delta > 0$ such that the implication

$$\mathbf{y}, \mathbf{y}' \in Y \,, \quad \varrho_m(\mathbf{y}, \mathbf{y}') < \delta \Rightarrow \Delta[K(\mathbf{y}), K(\mathbf{y}')] < \varepsilon$$

is valid.

Now we can formulate our first auxiliary assertion.

Lemma 1. Let $X \neq \emptyset$, $Y \neq \emptyset$ be compact sets, $K(\mathbf{y}) \neq \emptyset$ for $\mathbf{y} \in Y$. Let, further, $F_i(\mathbf{x}, \mathbf{y})$, $i = 1, 2, \ldots, l$ be continuous functions on $X \times Y$. If $K(\mathbf{y})$ is a mapping semicontinuous from below in every $\mathbf{y} \in Y$ then $K(\mathbf{y})$ is a uniformly continuous mapping on Y.

Proof. We shall give the proof of this statement by contradiction. Let us assume that, under the assumptions of Lemma, there exists $\varepsilon > 0$ such that for every natural number r there exist $\mathbf{y}_r, \mathbf{y}'_r \in Y$ for which

$$\varrho_m(\mathbf{y}_r, \mathbf{y}'_r) < 1/r \,, \quad \Delta[K(\mathbf{y}_r), K(\mathbf{y}'_r)] > \varepsilon \,.$$

But this assumption is equivalent to the following one: There exists $\varepsilon > 0$ such that for every natural number r there exist $\mathbf{y}_r, \mathbf{y}'_r \in Y$ and $\mathbf{x}_r \in K(\mathbf{y}_r)$ for which

$$\varrho_m(\mathbf{y}_r, \mathbf{y}'_r) < 1/r \,, \quad \varrho_n(\mathbf{x}_r, \mathbf{x}'_r) > \varepsilon \,, \quad \text{for every} \quad \mathbf{x}'_r \in K(\mathbf{y}'_r) \,. \tag{4}$$

As the sets X, Y are compact and the functions $F_i(\mathbf{x}, \mathbf{y})$, $i = 1, 2, \ldots, l$ continuous, it is easy to see that there exist subsequences $\{\mathbf{y}_{r_k}\}$, $\{\mathbf{x}_{r_k}\}$ of the sequences $\{\mathbf{y}_r\}$, $\{\mathbf{x}_r\}$ and points $\mathbf{y}_0 \in Y$, $\mathbf{x}_0 \in X$ such that $\mathbf{y}_{r_k} \to \mathbf{y}_0$, $\mathbf{x}_{r_k} \to \mathbf{x}_0 \in K(\mathbf{y}_0)$.

Further, we obtain from (4) that there exists $\varepsilon > 0$ such that for every $\delta > 0$ there exists $\mathbf{y}' \in Y$ for which

$$\varrho_m(\mathbf{y}_0, \mathbf{y}') < \delta \,, \quad \varrho_n(\mathbf{x}_0, \mathbf{x}') > \varepsilon \quad \text{for every} \quad \mathbf{x}' \in K(\mathbf{y}') \,.$$

But this contradicts to the assumption that $K(\mathbf{y})$ is a mapping semicontinuous from below in every $\mathbf{y} \in Y$. $\square$

Let for every $\varepsilon > 0$, $X(\varepsilon)$ be defined by

$$X(\varepsilon) = X + B(\varepsilon) = \{\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2 : \mathbf{x}_1 \in X, \mathbf{x}_2 \in B(\varepsilon)\},$$

where $B(\varepsilon)$ denotes ε-surroundings of $0 \in E_n$.

If we shall assume that X fulfils condition

$$(5) \qquad X \supset \{\mathbf{x} \in E_n : F_i(\mathbf{x}, \mathbf{y}) \geqq 0,\ i = 1, 2, \ldots, l \text{ for an } \mathbf{y} \in Y\}$$

and if

(i) $X \neq \emptyset$ is a compact, convex set, $Y \neq \emptyset$ is a compact set;

(ii) $F_i(\mathbf{x}, \mathbf{y})$, $i = 1, 2, \ldots, l$ are continuous functions on $X \times Y$ and, further, there exist on $X(\varepsilon_0)$ the continuous partial derivatives of the functions $F_i(\mathbf{x}, \mathbf{y})$ for every $\mathbf{y} \in Y$ and an $\varepsilon_0 > 0$;

(iii) $K(\mathbf{y}) \neq \emptyset$ for every $\mathbf{y} \in Y$;

then we can introduce conditions under which $K(\mathbf{y})$ is a uniformly continuous mapping.

Lemma 2. Let the conditions (i), (ii), (iii) and the relation (5) be fulfilled. If the vectors of partial derivatives of the functions $F_i(\mathbf{x}, \mathbf{y})$, $i = 1, 2, \ldots, l$ with respect to the components of vector x are linearly independent at all points $(\mathbf{x}, \mathbf{y})$, $\mathbf{x} \in X$, $\mathbf{y} \in Y$ such that $F_i(\mathbf{x}, \mathbf{y}) = 0$ at least for one $i \in \{1, 2, \ldots, l\}$ then $K(\mathbf{y})$ is a uniformly continuous mapping on Y.

Proof. In according to Lemma 1 to prove Lemma 2 it is enough to prove that $K(\mathbf{y})$ is a mapping semicontinuous from below in every $\mathbf{y} \in Y$. Since X is a compact set the sets $K(\mathbf{y})$ for every $\mathbf{y} \in Y$ are compact too. But from this it is easy to see that the assertion of Lemma 2 will be proved if we shall prove the following:

For every $\mathbf{y}_0 \in Y$, $\mathbf{x}_0 \in K(\mathbf{y}_0)$, $\varepsilon > 0$, there exists $\delta > 0$ such that

$$(6) \qquad \varrho_m(\mathbf{y}, \mathbf{y}_0) < \delta\,, \quad \mathbf{y} \in Y \Rightarrow \mathbf{x} \in K(\mathbf{y}) \quad \text{exists such that} \quad \varrho_n(\mathbf{x}, \mathbf{x}_0) < \varepsilon\,.$$

Let $\mathbf{y}_0 \in Y$, $\mathbf{x}_0 \in K(\mathbf{y}_0)$, $\varepsilon > 0$ be arbitrary. Then either

a) $F_i(\mathbf{x}_0, \mathbf{y}_0) > 0$ for every $i \in \{1, 2, \ldots, l\}$, or

b) $F_i(\mathbf{x}_0, \mathbf{y}_0) = 0$ at least for one $i \in \{1, 2, \ldots, l\}$.

If a), then the validity of (6) follows from the continuity of the functions $F_i(\mathbf{x}, \mathbf{y})$, $i = 1, 2, \ldots, l$. So it remains to prove (6) in the case b). In this case we divide the set $1, 2, \ldots, l$ into two parts I_1, I_2 by

$$i \in I_1 \Leftrightarrow F_i(\mathbf{x}_0, \mathbf{y}_0) > 0$$
$$i \in I_2 \Leftrightarrow F_i(\mathbf{x}_0, \mathbf{y}_0) = 0\,.$$

From the continuity of the functions $F_i(\mathbf{x}, \mathbf{y})$, $i = 1, 2, \ldots, l$ follows the existence of an $\delta_1 > 0$ such that, for every $\mathbf{y} \in Y$ such that $\varrho_m(\mathbf{y}, \mathbf{y}_0) < \delta_1$, there exists $\mathbf{x} \in K(\mathbf{y})$ such that $\varrho_n(\mathbf{x}, \mathbf{x}_0) < \varepsilon$ and $F_i(\mathbf{x}, \mathbf{y}) \geqq 0$ for $i \in I_1$.

It remains to consider the case $i \in I_2$. We denote the vector of partial derivatives $\partial F_i / \partial x_j$, $j = 1, 2, \ldots, n$ in the point $(\mathbf{x}_0, \mathbf{y}_0)$ by $\nabla F_i(\mathbf{x}_0, \mathbf{y}_0) = \nabla F_i$ for $i \in I_2$. From the assumptions of Lemma 2 follows the existence of $\mathbf{x}'_0 \in E_n$ such that $(\nabla F_i, \mathbf{x}'_0) > 0$, for every $i \in I_2$ (symbol $(\;, \;)$ denotes the scalar product in E_n). But, as the vectors of the partial derivatives are continuous, we can easy see that there exists also $\mathbf{x}' \in X(\varepsilon_0)$ such that $\varrho_n(\mathbf{x}', \mathbf{x}_0) < \varepsilon$ and $F_i(\mathbf{x}', \mathbf{y}_0) > 0$ for $i \in I_2$ simultaneously.

Further, the validity of the implication

$$\varrho_m(\mathbf{y}, \mathbf{y}_0) < \delta_2 \,, \quad \mathbf{y} \in Y \Rightarrow \text{the existence of } \mathbf{x} \in K(\mathbf{y}) \text{ such that}$$

$$F_i(\mathbf{x}, \mathbf{y}) \geqq 0 \,, \quad i \in I_2 \,, \quad \varrho_n(\mathbf{x}, \mathbf{x}_0) < \varepsilon \quad \text{simultaneously}$$

for an $\delta_2 > 0$, follows from the continuity of the functions $F_i(\mathbf{x}, \mathbf{y})$, $i = 1, 2, \ldots, l$.

Now it is easy to see that, for $\delta = \min(\delta_1, \delta_2)$, the relation (6) is valid. $\square$

Remark 1. If we assume the independence of the vectors of partial derivatives only for $i \in \{1, 2, \ldots, l\}$ such that $F_i = 0$ in the corresponding points instead of for every $i = 1, 2, \ldots, l$, then the assertion of Lemma 2 is valid too. (This follows immediately from the proof of Lemma 2.)

The conditions under which the mapping $K(\mathbf{y})$ is uniformly continuous are given in Lemma 2. The assumption of the existence of the partial derivatives of $F_i(\mathbf{x}, \mathbf{y})$, $i = 1, 2, \ldots, l$ with respect to the components of vector $\mathbf{x}$ occurs in these conditions. Now we shall introduce some other conditions. There the demand of the existence of the partial derivatives will be replaced by a concavity.

If

(ii') $F_i(\mathbf{x}, \mathbf{y})$, $i = 1, 2, \ldots, l$ are continuous functions on $X(\varepsilon_0) \times Y$ for an $\varepsilon_0 > 0$ and if they are, for every $\mathbf{y} \in Y$, concave functions of $\mathbf{x}$ on $X(\varepsilon_0)$;

(iii') $\{\mathbf{x} \in X : F_i(\mathbf{x}, \mathbf{y}) > 0, \; i = 1, 2, \ldots, l\} \neq \emptyset$ for every $\mathbf{y} \in Y$, then the following statement is valid:

Lemma 3. If the conditions (i), (ii'), (iii') are fulfilled then $K(\mathbf{y})$ is uniformly continuous mapping on Y.

Proof. Obviously, to prove this Lemma it is enough to prove (6) for all $\mathbf{y}_0 \in Y$, $\mathbf{x}_0 \in K(\mathbf{y}_0)$ and $\varepsilon > 0$.

Let $\mathbf{y}_0 \in Y$, $\mathbf{x}_0 \in K(\mathbf{y}_0)$, $\varepsilon > 0$ be arbitrary. We shall define the subsets I_1, I_2 by

$$i \in I_1 \Leftrightarrow F_i(\mathbf{x}_0, \mathbf{y}_0) > 0$$

$$i \in I_2 \Leftrightarrow F_i(\mathbf{x}_0, \mathbf{y}_0) = 0$$

Continuity of the functions $F_i(\mathbf{x}, \mathbf{y})$, $i = 1, 2, \ldots, l$ yields the existence of an $\delta_1 > 0$ such that for every $\mathbf{y} \in Y$ for that $\varrho(\mathbf{y}, \mathbf{y}_0) < \delta_1$ there exists $\mathbf{x} \in K(\mathbf{y})$ such that $\varrho_n(\mathbf{x}, \mathbf{x}_0) < \varepsilon$ and also $F_i(\mathbf{x}, \mathbf{y}) \geqq 0$ for $i \in I_1$.

Now, we shall consider the case $i \in I_2$. It is easy to see that, under the conditions of Lemma, there exist $\mathbf{x}_1 \in K(\mathbf{y}_0)$, $\mathbf{x}_2 \in X(\varepsilon_0)$ such that the conditions $\varrho_n(\mathbf{x}_1, \mathbf{x}_0) < \varepsilon$, $\varrho_n(\mathbf{x}_2, \mathbf{x}_0) < \varepsilon$, $F_i(\mathbf{x}_1, \mathbf{y}_0) > 0$, $F_i(\mathbf{x}_2, \mathbf{y}_0) < 0$ are fulfilled for $i \in I_2$. Further, the existence of an $\delta_2 > 0$ for which the implication

$$\varrho_m(\mathbf{y}, \mathbf{y}_0) < \delta_2\,, \quad \mathbf{y} \in Y \Rightarrow F_i(\mathbf{x}_1, \mathbf{y}) > 0\,, \quad F_i(\mathbf{x}_2, \mathbf{y}) < 0 \quad \text{for every}$$
$$i \in I_2 \quad \text{and some} \quad \mathbf{x}_1 \in X\,, \quad \mathbf{x}_2 \in X(\varepsilon_0)$$

is valid, follows from the continuity of functions $F_i(\mathbf{x}_1, \mathbf{y})$, $i = 1, 2, \ldots, l$.

But now we have, for $i \in I_2$,

$$\varrho_m(\mathbf{y}, \mathbf{y}_0) < \delta_2\,, \quad \mathbf{y} \in Y \Rightarrow \text{there exists } \mathbf{x} \in K(\mathbf{y}) \text{ such that } F_i(\mathbf{x}, \mathbf{y}) \geqq 0$$
$$\text{for every} \quad i \in I_2 \text{ and } \varrho_n(\mathbf{x}, \mathbf{x}_0) < \varepsilon_0 \text{ simultaneously.}$$

It is easy to see that for $\delta = \min(\delta_1, \delta_2)$ the condition (6) is valid. □

3. ASSERTIONS

In the previous part we have dealt with auxiliary assertions. Now we shall utilize them to get the properties of the functions $\varphi(\mathbf{y})$. However to derive these we shall use the results of the paper [4] too. There are given conditions under which the uniform continuity of $K(\mathbf{y})$ yields the continuity of the function $\varphi(\mathbf{y})$.

For a reference we now present Lemma 1 of [4].

Lemma 4. Let $K(\mathbf{y})$ be uniformly continuous mapping of Y into a space of non-empty and closed subsets of X. Let, further, $F(\mathbf{x}, \mathbf{y})$ is a uniformly continuous function on $X \times Y$. If

$$(7) \qquad \varphi(\mathbf{y}) = \sup \{F(\mathbf{x}, \mathbf{y}) : \mathbf{x} \in K(\mathbf{y})\} < +\infty \quad \text{for every} \quad \mathbf{y} \in Y$$

then the function $\varphi(y)$ is continuous on $\mathbf{Y}$.

Theorem 1. Let the conditions (i), (ii), (iii) and the relation (5) be fulfilled. If the vectors of partial derivatives of the functions $F_i(\mathbf{x}, \mathbf{y})$, $i = 1, 2, \ldots, l$ with respect to the components of vector $\mathbf{x}$ are linearly independent at all points $(\mathbf{x}, \mathbf{y})$, $\mathbf{x} \in X$, $\mathbf{y} \in Y$ such that $F_i(\mathbf{x}, \mathbf{y}) = 0$ for at least one $i \in \{1, 2, \ldots, l\}$ and if $F(\mathbf{x}, \mathbf{y})$ is a continuous function on $X \times Y$ then $\varphi(\mathbf{y})$ is a uniformly continuous function on Y.

Proof. As X, Y are compact sets, it follows from Lemma 2 and Lemma 4 that to prove the Theorem it is enough to prove the validity of (7). But this follows immediately from the condition (i), conditions (ii), (iii) and the assumption of the continuity of the function $F(\mathbf{x}, \mathbf{y})$. □

Remark 2. If we assume the independence of the vectors of partial derivatives of the functions $F_i(\mathbf{x}, \mathbf{y})$ only for $i \in \{1, 2, \ldots, l\}$ such that $F_i = 0$ in the corresponding points instead of for every $i = 1, 2, \ldots, l$ then the assertion of Theorem holds too. (This follows immediately from Remark 1.)

Theorem 2. If the conditions (i), (ii'), (iii') are fulfilled and if $F(\mathbf{x}, \mathbf{y})$ is a continuous function on $X \times Y$ then $\varphi(\mathbf{y})$ is a uniformly continuous function on Y.

Proof. The assertion of Theorem 2 follows from Lemma 3 and Lemma 4. □

Theorem 3. Let X, Y be convex and non-empty sets.

1. If $F(\mathbf{x}, \mathbf{y})$, $F_i(\mathbf{x}, \mathbf{y})$, $i = 1, 2, \ldots, l$ are concave functions on $X \times Y$ and if $K(\mathbf{y}) \neq \emptyset$ for every $\mathbf{y} \in Y$, then $\varphi(\mathbf{y})$ is a concave function on Y;
2. If Y is a compact set and if there exists $\varepsilon_0 > 0$ such that
   a) $F(\mathbf{x}, \mathbf{y})$ is a concave, bounded function on $X \times Y(\varepsilon_0)$,
   b) $F_i(\mathbf{x}, \mathbf{y})$, $i = 1, 2, \ldots, l$ are concave functions on $X \times Y(\varepsilon_0)$, $K(\mathbf{y}) \neq \emptyset$ for every $\mathbf{y} \in Y$;
   then $\varphi(\mathbf{y})$ is a Lipschitz function on Y.

($Y(\varepsilon)$ for $\varepsilon > 0$ is defined in the same way as $X(\varepsilon)$.)

Proof. The assertion 1 follows from Lemma 1 of [3].

Further, it follows from 1. that under the assumptions of 2. the function $\varphi(\mathbf{y})$ is bounded and concave on $Y(\varepsilon_0)$. Thus we can utilize Theorem 10.2 of [8]. The assertion 2 follows immediately from this Theorem. □

It is easy to see that, generally, $\varphi(\mathbf{y})$ can be infinite. But under the assumptions of this paper the function $\varphi(\mathbf{y})$ (or its stochastic equivalent) is bounded.

At the end of this part we shall note that the conditions of the stability of deterministic optimization problems are given in the theorems we proved too.

Some special models of two-stage stochastic nonlinear programming problems are given at the end of this paper. The stable deterministic optimization problems can be found from this special stochastic models.

III. Stochastic Models

Now we shall try to use the results of the previous parts for two-stage stochastic nonlinear programming problems.

1. FORMULATION OF TWO-STAGE STOCHASTIC NONLINEAR PROGRAMMING PROBLEMS

First we note that in this part we will try to preserve the notation employed in the previous part. Hence to define two-stage stochastic nonlinear programming problems we divide vector $\mathbf{y}$ into two parts $\mathbf{y} = (\mathbf{z}, \mathbf{u})$, where $\mathbf{z} \in E_s$, $\mathbf{u} \in E_r$, $r + s = m$.

Let now $X \subset E_n$, $Z \subset E_s$, $U \subset E_r$ be non-empty sets, $(\Omega, \mathcal{S}, P)$ be probability space, $\xi(\omega)$ be s-dimensional random vector defined on $(\Omega, \mathcal{S}, P)$.

Let, further, the vector $\xi(\omega)$ fulfil the condition

$$P\{\omega : \xi(\omega) \in Z\} = 1 . \tag{8}$$

(in (8) it is assumed that $\{\omega : \xi(\omega) \in Z\} \in \mathcal{S}$).

If $F(\mathbf{x}, \mathbf{z}, \mathbf{u})$, $F_i(\mathbf{x}, \mathbf{z}, \mathbf{u})$, $i = 1, 2, \ldots, l$ are real valued continuous functions defined on $E_n \times E_s \times E_r$ and if the mapping $K(\mathbf{u}, \mathbf{z})$ and the function $\varphi(\mathbf{u}, \mathbf{z})$ are defined by

$$\begin{aligned} K(\mathbf{u}, \mathbf{z}) &= \{\mathbf{x} \in X : F_i(\mathbf{x}, \mathbf{z}, \mathbf{u}) \geqq 0 , \quad i = 1, 2, \ldots, l\} \\ \varphi(\mathbf{u}, \mathbf{z}) &= \sup \{F(\mathbf{x}, \mathbf{z}, \mathbf{u}) : \mathbf{x} \in K(\mathbf{u}, \mathbf{z})\} \quad \text{for} \quad \mathbf{u} \in E_r , \quad \mathbf{z} \in E_s , \end{aligned} \tag{9}$$

then we can introduce the general problem of two-stage stochastic nonlinear programming as a problem to find

$$\sup \{\mathbf{E}\, \varphi(\mathbf{u}, \xi(\omega)) : \mathbf{u} \in U\} ,$$

where $\mathbf{E}$ denotes the operator of the mathematical expectation. (In this paper we shall assume such conditions that all symbols in the definition of two-stage stochastic programming problems are meaningful.)

The aim of this part is to find conditions under which the function $\mathbf{E}\, \varphi(\mathbf{u}, \xi(\omega))$ is continuous and differentiable.

2. CONTINUITY OF THE FUNCTION $\mathbf{E}\, \varphi(\mathbf{u}, \xi(\omega))$

Theorem 4. Let $F(\mathbf{x}, \mathbf{z}, \mathbf{u})$ be a continuous function on $X \times Z \times U$ and let

1. $Z \neq \emptyset$, $U \neq \emptyset$ be compact sets,
   $X \neq \emptyset$ be a compact, convex set such that the condition
   $X \supset \{\mathbf{x} \in E_n : F_i(\mathbf{x}, \mathbf{z}, \mathbf{u}) \geqq 0$ for every $i = 1, 2, \ldots, l$ and an $\mathbf{u} \in U$, $\mathbf{z} \in Z\}$
   holds;
2. $F_i(\mathbf{x}, \mathbf{z}, \mathbf{u})$, $i = 1, 2, \ldots, l$ be continuous functions on $X \times Z \times U$ and let there exist on $X(\varepsilon_0)$ continuous partial derivatives of the functions $F_i(\mathbf{x}, \mathbf{z}, \mathbf{u})$, $i = 1, 2, \ldots, l$ for every $\mathbf{u} \in U$, $\mathbf{z} \in Z$ and an $\varepsilon_0 > 0$;

3. $K(\boldsymbol{u}, \mathbf{z}) \neq \emptyset$ for every $\boldsymbol{u} \in U$, $\mathbf{z} \in Z$.

If the vectors of partial derivatives of the functions $F_i(\mathbf{x}, \mathbf{z}, \boldsymbol{u})$, $i = 1, 2, \ldots, l$ with respect to the components of vector $\mathbf{x}$ are linearly independent at all points $(\mathbf{x}, \mathbf{z}, \boldsymbol{u})$, $\mathbf{x} \in X$, $\mathbf{z} \in Z$, $\boldsymbol{u} \in U$ such that $F_i(\mathbf{x}, \mathbf{z}, \boldsymbol{u}) = 0$ at least for one $i \in \{1, 2 \ldots, l\}$ and if the condition (8) is fulfilled then $\mathsf{E}\, \varphi(\boldsymbol{u}, \xi(\omega))$ is a continuous function on U.

Proof. The assertion of Theorem 4 follows from Theorem 1 and properties of integral. □

Theorem 5. Let $F(\mathbf{x}, \mathbf{z}, \boldsymbol{u})$ be a continuous function on $X \times Z \times U$ and let

1. $X \neq \emptyset$ be a compact, convex set,
   $Z \neq \emptyset$, $U \neq \emptyset$ be compact sets;
2. $F_i(\mathbf{x}, \mathbf{z}, \boldsymbol{u})$, $i = 1, 2, \ldots, l$ be continuous functions on $X(\varepsilon_0) \times Z \times U$ for an $\varepsilon_0 > 0$ and $F_i(\mathbf{x}, \mathbf{z}, \boldsymbol{u})$, $i = 1, 2, \ldots, l$ be for every $\mathbf{z} \in Z$, $\boldsymbol{u} \in U$ concave functions of $\mathbf{x}$ on $X(\varepsilon_0)$;
3. $\{\mathbf{x} \in X : F_i(\mathbf{x}, \mathbf{z}, \boldsymbol{u}) > 0,\ i = 1, 2, \ldots, l\} \neq \emptyset$ for every $\mathbf{z} \in Z$, $\boldsymbol{u} \in U$.

If the condition (8) is fulfilled then $\mathsf{E}\, \varphi(\boldsymbol{u}, \xi(\omega))$ is a continuous function on U.

Proof. The assertion of Theorem 5 follows from Theorem 2 and properties of integral. □

Theorem 6. Let the assumptions 1, 3 of Theorem 5 be fulfilled. Let, further, the functions $F(\mathbf{x}, \mathbf{z}, \boldsymbol{u})$, $F_i(\mathbf{x}, \mathbf{z}, \boldsymbol{u})$, $i = 1,2, \ldots, l$, be concave on $X(\varepsilon_0) \times U(\varepsilon_0)$ for an $\varepsilon_0 > 0$ and every $\mathbf{z} \in Z$. If $F(\mathbf{x}, \mathbf{z}, \boldsymbol{u})$, $F_i(\mathbf{x}, \mathbf{z}, \boldsymbol{u})$, $i = 1, 2, \ldots, l$ are continuous functions on $X(\varepsilon_0) \times Z \times U(\varepsilon_0)$ and if the condition (8) is fulfilled then

1. $\mathsf{E}\, \varphi(\boldsymbol{u}, \xi(\omega))$ is a continuous and concave function on U;
2. $\mathsf{E}\, \varphi(\boldsymbol{u}, \xi(\omega))$ is a Lipschitz function on U.

Proof. Since it follows from Theorem 2 that $\varphi(\boldsymbol{u}, \mathbf{z})$ is a continuous function on $U \times Z$ and since, further, it follows from Theorem 3 that $\varphi(\boldsymbol{u}, \mathbf{z})$ is a concave function for every $\mathbf{z} \in Z$ on U, the assertion 1 follows from the properties of integral.

Further, since X, Z, U are compact sets and $F(\mathbf{x}, \mathbf{z}, \boldsymbol{u})$ is a continuous function, we can easy see that $F(\mathbf{x}, \mathbf{z}, \boldsymbol{u})$ is a bounded function on $X \times Z \times U$. However, under the condition that $F(\mathbf{x}, \mathbf{z}, \boldsymbol{u})$ is a bounded function so that $\varphi(\boldsymbol{u}, \mathbf{z})$ must be bounded function too. Now, the assertion 2 follows from [5] (condition (10)) and the properties of integral). □

Until now we have dealt with the continuity of the function $\mathsf{E}\, \varphi(\boldsymbol{u}, \xi(\omega))$. In the sequel we shall try to find conditions under which $\mathsf{E}\, \varphi(\boldsymbol{u}, \xi(\omega))$ is a differentiable function.

The case of discrete random variables is discussed in [3]. For discrete case, the result has been obtained by simple generalization of the corresponding result in linear case. The continuous case is complicated. We shall deal with special cases only. However these cases are quite important from the practical point of view.

Let $Y_t \subset E_{m_t}$, $t = 1, 2$ be convex, compact sets for which int $Y_t \neq \emptyset$; X, U be convex sets; $\mathbf{h}_t(u, z) = [h_{t1}(u, z), \ldots, h_{tm_t}(\mathbf{u}, \mathbf{z})]$, $t = 1, 2$ be vector functions defined on $E_r \times E_s$ mapping $U \times Z$ into int Y_t, $t = 1, 2$.

Let, further, the functions $F(\mathbf{x}, \mathbf{z}, \mathbf{u})$, $F_i(\mathbf{x}, \mathbf{z}, \mathbf{u})$, $i = 1, 2, \ldots, l$ fulfil the conditions

$$F(\mathbf{x}, \mathbf{z}, \mathbf{u}) = \bar{F}(\mathbf{x}, \mathbf{h}_1(\mathbf{u}, \mathbf{z})) = \bar{F}(\mathbf{x}, \mathbf{y}_1),$$

$$F_i(\mathbf{x}, \mathbf{z}, \mathbf{u}) = \bar{F}_i(\mathbf{x}, \mathbf{h}_2(\mathbf{u}, \mathbf{z})) = \bar{F}_i(\mathbf{x}, \mathbf{y}_2), \quad i = 1, 2. \ldots, l,$$

where $\bar{F}(\mathbf{x}, \mathbf{y}_1)$ or $\bar{F}_i(\mathbf{x}, \mathbf{y}_2)$, $i = 1, 2, \ldots, l$ are real valued functions defined on $E_n \times \times E_{m_1}$ or $E_n \times E_{m_2}$ respectively.

If $\bar{K}(\mathbf{y}_2)$, $\bar{\varphi}(\mathbf{y}_1, \mathbf{y}_2)$, $\mathbf{y}_1 \in Y_1$, $\mathbf{y}_2 \in Y_2$ are defined by

$$\bar{K}(\mathbf{y}_2) = \{\mathbf{x} \in X : \bar{F}_i(\mathbf{x}, \mathbf{y}_2) \geqq 0, \quad i = 1, 2, \ldots, l\}, \quad \mathbf{y}_2 \in E_{m_2},$$

$$\bar{\varphi}(\mathbf{y}_1, \mathbf{y}_2) = \sup \{\bar{F}(\mathbf{x}, \mathbf{y}_1) : \mathbf{x} \in \bar{K}(\mathbf{y}_2)\}, \quad \mathbf{y}_1 \in E_{m_1}, \quad \mathbf{y}_2 \in E_{m_2},$$

then

$$K(\mathbf{u}, \mathbf{z}) = \bar{K}(\mathbf{h}_2(\mathbf{u}, \mathbf{z})),$$

$$\varphi(\mathbf{u}, \mathbf{z}) = \bar{\varphi}(\mathbf{h}_1(\mathbf{u}, \mathbf{z}), \mathbf{h}_2(\mathbf{u}, \mathbf{z})) \quad \text{for every} \quad \mathbf{u} \in E_r, \quad \mathbf{z} \in E_s.$$

Now we shall introduce the conditions of differentiability of the function $\mathbf{E}\,\varphi(\mathbf{u}, \xi(\omega))$.

Theorem 7. Let U, X be convex sets and $\bar{F}(\mathbf{x}, \mathbf{y}_1)$, $\bar{F}_i(\mathbf{x}, \mathbf{y}_2)$ $i = 1, 2, \ldots, l$ be concave functions defined on $X \times Y_1(\varepsilon_0)$, $X \times Y_2(\varepsilon_0)$ for an $\varepsilon_0 > 0$ respectively. Let, further, $h_{tj}(\mathbf{u}, \mathbf{z})$, $t = 1, 2, j = 1, 2, \ldots, m_t$ be continuous functions on $U \times Z$ such that for every $\mathbf{z} \in Z$ they are differentiable on U. Then if the condition (8) is fulfilled and if

1. $[\mathbf{h}_t(\mathbf{u}, \xi(\omega)), t = 1, 2]$ are for every $\mathbf{u} \in U$ random vectors such that their probability measure are absolute continuous with respect to the Lebesque measure in $E_{m_1} \times E_{m_2}$;
2. the condition

$$\varrho_{m_t}[\mathbf{h}_t(\mathbf{u}, \mathbf{z}), \mathbf{h}_t(\mathbf{u}', \mathbf{z})] \leqq g_t(\mathbf{z})\, \varrho_r(\mathbf{u}, \mathbf{u}')$$

is fulfilled for $t = 1, 2$ and every $\mathbf{u}, \mathbf{u}' \in U$, $\mathbf{z} \in Z$ and, further, there exists finite $\mathbf{E}\,\mathbf{g}_t(\xi(\omega))$ for $t = 1, 2$;

3. $\bar{F}(\mathbf{x}, \mathbf{y}_1)$ is a bounded function on $X \times Y_1(\varepsilon_0)$, and $K(\mathbf{y}_2) \neq \emptyset$ for every $\mathbf{y}_2 \in Y_2(\varepsilon_0)$, then $\mathbf{E}\, \varphi(\mathbf{u}, \xi(\omega))$ is a differentiable function on int U. (Symbols $Y_1(\varepsilon_0)$, $Y_2(\varepsilon_0)$ are defined in the same way as $X(\varepsilon_0)$, $Y(\varepsilon)$ in the previous part.)

The assumption 2 is rather simple in the case when the functions $g_t(\mathbf{z})$, $t = 1, 2$ do not depend on $\mathbf{z}$. This happens, for example, if $h_{ti}(\mathbf{u}, \mathbf{z})$, $t = 1, 2$, $i = 1, 2, \ldots, m_t$ are bounded and concave (or convex) function on $U(\varepsilon_0)$ (cf. (10) of [5]).

Proof. It follows from Theorem 3 that $\bar{\varphi}(\mathbf{y}_1, \mathbf{y}_2)$ is a concave and Lipschitz function on $Y_1 \times Y_2$.

Let, now, $\mathbf{u}_0 \in \text{int}\, \mathbf{U}$ be arbitrary. According to [7], there exists a set $\mathcal{N} \subset \subset \text{int}\,[Y_1 \times Y_2]$ of the Lebesque's measure 0 such that $\bar{\varphi}(\mathbf{y}_1, \mathbf{y}_2)$ is a differentiable function on $[\text{int}\, Y_1 \times \text{int}\, Y_2 - \mathcal{N}]$. If

$$Z(\mathbf{u}_0) = \{\mathbf{z} \in E_s : [\mathbf{h}_1(\mathbf{u}, \mathbf{z}), \mathbf{h}_2(\mathbf{u}, \mathbf{z})] \in \mathcal{N}\}\,,$$

then it is easy to see that

$$P\{\omega : \xi(\omega) \in Z(\mathbf{u}_0)\} = 0\,.$$

Further, as $\mathbf{h}_{ti}(\mathbf{u}, \mathbf{z})$, $t = 1, 2$, $i = 1, 2, \ldots, m_t$ are for every $\mathbf{z} \in Z$ differentiable on U it is easy to see that $\bar{\varphi}(\mathbf{h}_1(\mathbf{u}, \mathbf{z}), \mathbf{h}_2(\mathbf{u}, \mathbf{z}))$ is a differentiable function in the point $\mathbf{u}_0$ for every $\mathbf{z} \notin Z(\mathbf{u}_0)$.

Since we have proved that $\bar{\varphi}(\mathbf{y}_1, \mathbf{y}_2)$ is a Lipschitz function on $Y_1 \times Y_2$, we can easily see from the assumption 2 that $\varphi(\mathbf{u}, \mathbf{z}) = \bar{\varphi}(\mathbf{h}_1(\mathbf{u}, \mathbf{z}), \mathbf{h}_2(\mathbf{u}, \mathbf{z}))$ is for every $\mathbf{z} \in Z$ a Lipschitz function on U. If we denote the Lipschitz constant of $\varphi(\mathbf{u}, \mathbf{z})$ by $H(\mathbf{z})$, it follows from the assumptions that there exists finite $\mathbf{E}\, H(\xi(\omega))$. Since, further, it follows from the assumption 3 that $\bar{\varphi}(\mathbf{y}_1, \mathbf{y}_2)$ is a bounded function we get that there exists a finite $\mathbf{E}\, \varphi(\mathbf{u}, \xi(\omega))$ for every $\mathbf{u} \in U$.

But this already verifies the Lebesque's Theorem assumptions from which the differentiability of $\mathbf{E}\, \varphi(\mathbf{u}, \xi(\omega))$ in the point $\mathbf{u}_0$ directly follows. As $\mathbf{u}_0 \in \text{int}\, U$ has been arbitrary the proof of Theorem is finished. □

Corollary 1. Let U, X be a convex set and, for an $\varepsilon_0 > 0$, $\bar{F}(\mathbf{x}, \mathbf{y}_1)$, $\bar{F}_i(\mathbf{x}, \mathbf{y}_2)$, $i = 1, 2, \ldots, l$ be concave functions on $X \times Y_1(\varepsilon_0)$, $X \times Y_2(\varepsilon_0)$ respectively. Let, further, $h_{tj}(\mathbf{u}, \mathbf{z})$, $t = 1, 2$, $j = 1, 2, \ldots, m_t$ be continuous functions on $U \times Z$ and let there exist on U partial derivatives of the functions $h_{tj}(\mathbf{u}, \mathbf{z})$, $t = 1, 2, j = 1, 2, \ldots, m_t$ for every $\mathbf{z} \in Z$. If

a) $h_{tj}(\mathbf{u}, \mathbf{z})$, $t = 1, 2, j = 1, 2, \ldots, m_t$ are for every $\mathbf{z} \in Z$ concave functions on $U(\varepsilon_0)$;

b) $h_{jt}(\mathbf{u}, \mathbf{z})$, $t = 1, 2, j = 1, 2, \ldots, m_t$ are on $U(\varepsilon_0)$ bounded functions by constant not depending on $\mathbf{z} \in Z$;

and if the assumptions 1, 3 of Theorem 7 and the condition (8) are fulfilled, then $\mathbf{E}\, \varphi(\mathbf{u}, \xi(\omega))$ is a differentiable function on int U.

Proof. Obviously to prove Corollary 1 it is enough to prove that the assumption 2 of Theorem 7 follows from the assumptions a), b). Further, to prove this it suffices to prove $h_{tj}(\boldsymbol{u}, \boldsymbol{z})$, $t = 1, 2, j = 1, 2, \ldots, m_t$ are Lipschitz functions of $\boldsymbol{u}$, $\boldsymbol{u} \in U$ with Lipschitz constant not depending on $\boldsymbol{z}$, $\boldsymbol{z} \in Z$. But this statement has been proved in [5] (relation (10)). □

If $F(\mathbf{x}, \mathbf{z}, \boldsymbol{u})$, $F_i(\mathbf{x}, \mathbf{z}, \boldsymbol{u})$, $i = 1, 2, \ldots, l$ are concave functions the following result takes place.

Corollary 2. If the assumptions of Theorem 7 are fulfilled and if $F(\mathbf{x}, \mathbf{z}, \boldsymbol{u})$, $F_i(\mathbf{x}, \mathbf{z}, \boldsymbol{u})$, $i = 1, 2, \ldots, l$ are for every $\mathbf{z} \in Z$ concave functions on $X \times U$ then $\mathbf{E}\, \varphi(\boldsymbol{u}, \xi(\omega))$ is a concave, differentiable function on int U.

Proof. According to Theorem 3 $\varphi(\boldsymbol{u}, \mathbf{z})$ is for every $\mathbf{z} \in Z$ a concave function on U. From this it follows that $\mathbf{E}\, \varphi(\boldsymbol{u}, \xi(\omega))$ is a concave function on U too. The differentiability of $\mathbf{E}\, \varphi(\boldsymbol{u}, \xi(\omega))$ on int U follows from Theorem 7. □

Now, we shall introduce another conditions for the differentiability of the function $\mathbf{E}\, \varphi(\boldsymbol{u}, \xi(\omega))$.

Theorem 8. Let $Z \in E_s$ be a compact set and let for every $\mathbf{z} \in Z$, $h_{tj}(\boldsymbol{u}, \mathbf{z})$, $t = 1, 2$, $j = 1, 2, \ldots, m_t$ be differentiable functions of $\boldsymbol{u}$. If the relation (8) is fulfilled and if

1. $\boldsymbol{h}_t(\boldsymbol{u}, \xi(\omega))$, $t = 1, 2$ are for every $\boldsymbol{u} \in U$ random vectors such that their probability measures are absolute continuous with respect to the Lebesque's measure in $E_{m_1} \times E_{m_2}$;
2. the condition

$$\varrho_{mt}[\boldsymbol{h}_t(\boldsymbol{u}, \mathbf{z}), \boldsymbol{h}_t(\boldsymbol{u}', \mathbf{z})] \leqq g_t(\mathbf{z})\, \varrho_r(\boldsymbol{u}, \boldsymbol{u}')$$

   is fulfilled for $t = 1, 2$ and every $\boldsymbol{u}, \boldsymbol{u}' \in U$, $\mathbf{z} \in Z$, and further, there exists finite $\mathbf{E}\, g_t(\xi(\omega))$ for $t = 1, 2$;
3. $\mathbf{y}_2 \in Y_2 \Rightarrow K(\mathbf{y}_2) \neq \emptyset$ and the fulfilment at least one of two conditions
   a) $\bar{K}(\mathbf{y}_2)$ is a compact set,
   b) $\bar{F}(\mathbf{x}, \mathbf{y}_1)$ is a bounded function on $\bar{K}(\mathbf{y}_2)$;
4. $\bar{F}(\mathbf{x},\mathbf{y}_1)$ is a Lipschitz function on $E_n \times Y_1$ with Lipschitz constant C_1,
5. $\Delta[\bar{K}(\mathbf{y}_2), \bar{K}(\mathbf{y}_2')] \leqq C_2\, \varrho_{m_2}(\mathbf{y}_2, \mathbf{y}_2')$ for every $\mathbf{y}_2, \mathbf{y}_2' \in Y_2$, where C_2 is a constant,

then $\mathbf{E}\, \varphi(\boldsymbol{u}, \xi(\omega))$ is a differentiable function on int U.

6. $\bar{F}(\mathbf{x}, \mathbf{y}_1)$, $\bar{F}_i(\mathbf{x}, \mathbf{y}_2)$, $i = 1, 2, \ldots, l$ be concave functions on $E_n \times Y_1$, $E_n \times Y_2$ respectively.

Proof. The assertion of Theorem 8 follows from Theorem 2 and Remark 4 of [3]. □

(Some conditions under which the assumption 5 of Theorem 8 is fulfilled are given in [3] and [6].)

Till now we have dealt with theoretical problems only. First we found conditions under which the optimum in a parametric deterministic optimization problem is a continuous, concave and Lipschitz function of the parameter. Further, we used the results for deterministic parametric problems to gain the properties of the optimalized function in two-stage stochastic nonlinear programming problems.

In this last part we shall try to introduce special cases of functions $F(\mathbf{x}, \mathbf{z}, \boldsymbol{u})$, $F_i(\mathbf{x}, \mathbf{z}, \boldsymbol{u})$, $i = 1, 2, \ldots, l$ fulfilling the conditions under which $\varphi(\boldsymbol{u}, \mathbf{z})$ and $\mathbf{E}\ \varphi(\boldsymbol{u}, \xi(\omega))$ are continuous, concave and differentiable functions. At this place we can note that these stochastic problems easily yield examples of stable deterministic problems.

In this part of the paper we shall assume that

(a) the condition (8) is fulfilled;

(b) the functions $F(\mathbf{x}, \mathbf{z}, \boldsymbol{u})$, $F_i(\mathbf{x}, \mathbf{z}, \boldsymbol{u})$, $i = 1, 2, \ldots, l$ fulfil all conditions that have been introduced in the beginning of Part 3.

Example 1. In this first case we shall assume

$$F_i(\mathbf{x}, \mathbf{z}, \boldsymbol{u}) = h_i(\boldsymbol{u}, \mathbf{z}) - g_i(\mathbf{x}, \mathbf{z}) \quad \text{for every} \quad i = 1, 2, \ldots, l\,.$$

It follows from Theorem 5 that if

a) X is a compact, convex set and U, Z are compact sets;

b) $g_i(\mathbf{x}, \mathbf{z})$, $h_i(\boldsymbol{u}, \mathbf{z})$, $i = 1, 2, \ldots, l$ are continuous functions on $X(\varepsilon_0) \times U \times Z$ for an $\varepsilon_0 > 0$, and further, $g_i(\mathbf{x}, \mathbf{z})$ is for every $\mathbf{z} \in Z$ a convex function on $X(\varepsilon_0)$;

c) $\{\mathbf{x} \in X : g_i(\mathbf{x}, \mathbf{z}) < h_i(\boldsymbol{u}, \mathbf{z}), i = 1, 2, \ldots, l\} \neq \emptyset$ for every $\mathbf{z} \in Z$, $\mathbf{u} \in U$;

and if $F(\mathbf{x}, \mathbf{z}, \boldsymbol{u})$ is a continuous function on $X \times Z \times U$ then $\varphi(\boldsymbol{u}, \mathbf{z})$ and $\mathbf{E}\ \varphi(\boldsymbol{u}, \xi(\omega))$ are continuous functions on $U \times Z$ too. (Condition c) is fulfilled if, for example, there exists for every $\mathbf{z} \in Z$ a point $\boldsymbol{x} = \boldsymbol{x}(\mathbf{z})$ fulfilling the conditions $g_i(\mathbf{x}, \mathbf{z}) \leqq 0$, $i = 1, 2, \ldots, l$ and if at the same $h_i(u, z) > 0$, $i = 1, 2, \ldots, l$ for every $\mathbf{u} \in U$, $z \in Z$.)

If we assume

b′) $g_i(\mathbf{x}, \mathbf{z})$, $h_i(\boldsymbol{u}, \mathbf{z})$, $i = 1, 2, \ldots, l$ are continuous functions on $X(\varepsilon_0) \times Z \times U$ for an $\varepsilon_0 > 0$ and if there exist on $X(\varepsilon_0)$ continuous partial derivatives of the functions $g_i(\mathbf{x}, \mathbf{z})$, $i = 1, 2, \ldots, l$ for every $\mathbf{z} \in Z$ and an $\varepsilon_0 > 0$, and the vectors of partial derivatives of the functions $g_i(\mathbf{x}, \mathbf{z})$, $i = 1, 2, \ldots, l$ with respect to the components of vector $\mathbf{x}$ are linearly independent at all points $(\mathbf{x}, \mathbf{y})$, $\mathbf{x} \in X$, $\mathbf{y} \in Y$,

c′) $\{\mathbf{x} \in E_n : g_i(\mathbf{x}, \mathbf{z}) \leqq h_i(\boldsymbol{u}, \mathbf{z}),\ i = 1, 2, \ldots, l\} = \{\mathbf{x} \in X : g_i(\mathbf{x}, \mathbf{z}) \leqq h_i(\boldsymbol{u}, \mathbf{z}),\ i = 1, 2, \ldots, l\} \neq \emptyset$ for every $\mathbf{z} \in Z$, $\mathbf{u} \in U$,

instead of b, c then $\varphi(\mathbf{u}, \mathbf{z})$ and $\mathbf{E}\ \varphi(\mathbf{u}, \xi(\omega))$ are continuous functions too.

We have introduced examples of two-stage stochastic nonlinear programming problem in which $\mathbf{E}\ \varphi(\boldsymbol{u}, \xi(\omega))$ is a continuous function of $\boldsymbol{u}$. Now we will find conditions under which $\mathbf{E}\ \varphi(\boldsymbol{u}, \xi(\omega))$ is a differentiable function.

Example 2. Consider the following rather simple case of two-stage stochastic nonlinear programming problem in which

$$F_i(\mathbf{x}, \mathbf{z}, \mathbf{u}) = h_i(\mathbf{u}, \mathbf{z}) - f_i(\mathbf{x}), \quad i = 1, 2, \ldots, l$$

$$F(\mathbf{x}, \mathbf{z}, \mathbf{u}) = f(\mathbf{x}),$$

where $h_i(\mathbf{u}, \mathbf{z})$, $i = 1, 2, \ldots, l, f(\mathbf{x})$ are for every $\mathbf{z} \in Z$ concave functions on $X(\varepsilon_0) \times U(\varepsilon_0)$ (for an $\varepsilon_0 > 0$), $f_i(x)$, $i = 1, 2, \ldots, l$ are convex functions on $X(\varepsilon_0)$.

If $h_i(\mathbf{u}, \mathbf{z}), f_i(\mathbf{x})$, $i = 1, 2, \ldots, l$, $f(x)$ are continuous functions on $X(\varepsilon_0) \times U(\varepsilon_0) \times Z$ and if

a) X, Z, U are convex, compact and non-empty sets,
b) the probability measure of the random vector $[h_1(\mathbf{u}, \xi(\omega)), \ldots, h_l(\mathbf{u}, \xi(\omega))]$ is for every $\mathbf{u} \in U$ absolute continuous with respect to the Lebesgue's measure in E_l,
c) $\{\mathbf{x} \in X : f_i(\mathbf{x}) \leqq h_i(\mathbf{u}, \mathbf{z}),\ i = 1, 2, \ldots, l\} \neq \emptyset$ for every $\mathbf{u} \in U$, $\mathbf{z} \in Z$;
d) $h_i(\mathbf{u}, \mathbf{z})$, $i = 1, 2, \ldots, l$ are for every $\mathbf{z} \in Z$ differentiable functions on U;

then $\mathbf{E}\ \varphi(\mathbf{u}, \xi(\omega))$ is a continuous, concave and differentiable function on int U.

The statement of this example follows from Theorem 3, Theorem 7 and Corollary 1.

Certainly, there exist a possibility to find the other examples of two-stage stochastic nonlinear programming problems in which $\mathbf{E}\ \varphi(\mathbf{u}, \xi(\omega))$ is a continuous and differentiable function. Now, we shall introduce the last example.

Example 3. Let $\mathbf{A}(\mathbf{u}, \mathbf{z})$ be for every $\mathbf{u} \in E_r$, $\mathbf{z} \in E_s$ an $(m \times n)$ matrix with non-zero columns, $\mathbf{h}(\mathbf{u}, \mathbf{z})$ be for every $\mathbf{u} \in E_r$, $\mathbf{z} \in E_s$ an $(m \times 1)$ vector function.

Let, further, $K(\mathbf{u}, \mathbf{z})$ be defined by

$$K(\mathbf{u}, \mathbf{z}) = \{\mathbf{x} \in E_n : \mathbf{A}(\mathbf{u}, \mathbf{z})\, \mathbf{x} \leqq \mathbf{h}(\mathbf{u}, \mathbf{z}), \quad \mathbf{x} \geqq 0\} \tag{10}$$

If

a) U, Z are non-empty compact sets;
b) there exists $\varepsilon_0 > 0$ such that the elements of $\mathbf{A}(\mathbf{u}, \mathbf{z})$ and $\mathbf{h}(\mathbf{u}, \mathbf{z})$ are a non-negative and continuous function on $U(\varepsilon_0) \times Z$. Further, the rank of matrix $\mathbf{A}(\mathbf{u}, \mathbf{z})$ is for every $\mathbf{u} \in U$, $\mathbf{z} \in Z$ equal m;
c) $\inf\{\mathbf{h}(\mathbf{u}, \mathbf{z}) : \mathbf{u} \in U, \mathbf{z} \in Z\} > 0$;
d) $F(\mathbf{x}, \mathbf{z}, \mathbf{u})$ is a continuous function on $X \times Z \times U$;

then $\mathbf{E}\ \varphi(\mathbf{u}, \xi(\omega))$ is a continuous function on U.

The statement of this examples follows from Theorem 5. (The existence of a compact, convex set X follows from the asumptions.)

The case $\mathbf{A}(\mathbf{u}, \mathbf{z}) \equiv \mathbf{A}$ for every $\mathbf{u} \in U$, $\mathbf{z} \in Z$ (where $\mathbf{A}$ is a constant matrix) is introduced in [3]. Under general conditions $\mathbf{E}\ \varphi(\mathbf{u}, \xi(\omega))$ is in this case a differentiable function on int U.

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