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Multivariable Deadbeat Servo Problem

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The problem of tracking a reference vector variable from a given class is considered for discrete-time linear multi-input-output plants. The plant and the reference are both described by an input/output relation and the objective is to track exactly in a minimum number of steps. The optimal control law is shown to contain both feedback and feedforward terms and it is obtained by solving only three linear matrix polynomial equations. The design procedure is remarkably simple and efficient.

1. INTRODUCTION

The problem of signal tracking, known also as a servo problem, has found various applications in practice in both continuous-time and discrete-time cases. In discrete-time and/or sampled-data systems, there is a possibility to achieve the exact tracking after a finite period of time. Such a system is said to exhibit a deadbeat response. Especially, we can compensate the plant so that the tracking error and the control sequence both vanish in a shortest time possible.

This deadbeat servo problem was considered by many authors using the transfer function approach for the single-input-output systems. Here the plant is often required to be initially at rest. Moreover, they all postulated the structure of the control system as shown in Fig. 1, i.e., they forced the control law to operate on the tracking error only. In this paper it is shown that such a control law is only sub-optimal.

The state space theory brought a new structure of controllers, see Fig. 2. This structure consists of the feedback part, containing an observer for the plant, and the feedforward part, containing an observer for the reference generator. Thus the tracking error is no longer used to compensate the plant. In the state space approach, servo problems are solved by combining the plant with the reference generator into an augmented system and so reducing the servo problem to a regulator problem of higher dimension. This approach is well developed for LQG tracking, but the author is not aware of any related result on deadbeat tracking. This situation arises because the problem of driving a given linear combination of state variables to zero in a finite time subject to a stability constraint has yet to be solved.

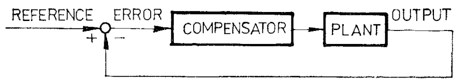

Fig. 1. Classical error-actuated servo system

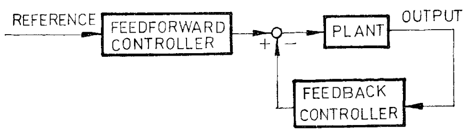

Fig. 2. Modern servo system.

A polynomial algebra proves to be a convenient tool for such a problem. The single-input-output case was well formulated and solved in a novel way in [3]. The design procedure described there consists in solving only two linear equations in polynomials.

The aim of this paper is to extend the deadbeat servo problem formulation and solution given in [3] to multi-input-output plants. The plant and the reference are both described by input/output relations and the control law is assumed to operate on the reference and the plant output. The design procedure consists in solving three linear equations in polynomial matrices. This proves to be simple and computationally efficient.

The deadbeat servo systems will be designed so as to track any sequence from a prespecified class, not only one particular sequence. Moreover, the optimal performance will be obtained for every initial state of the plant. These features are of considerable practical significance. Finally, the results advanced in this paper are not restricted to reversible systems and remain intact for plants and references defined over an arbitrary but fixed field.

2. PRELIMINARIES

Polynomial matrices will play a prominent role throughout the paper. That is why we first recall some mathematical concepts. Precise definitions can be found, for example, in [1].

A square polynomial matrix U is called *unimodular* if and only if $\det U$ is a non-zero real number. Polynomial matrices A and B are *equivalent* (we write $A \sim B$) if and only if there are such unimodular matrices U_1, U_2 that $A = U_1 B U_2$.

Consider polynomial matrices A, B and C such that $A = BC$. Then B is a *left divisor* of A and A is a *right multiple* of B, while C is a *right divisor* of A and A is a *left multiple* of C.

Consider again some polynomial matrices A and B. A square polynomial matrix G is termed a *common right divisor* of A and B if and only if G is a right divisor of both A and B; if, furthermore, G is a left multiple of every common right divisor of A and B, then G is a *greatest common right divisor* of A and B. A *common left divisor* and a *greatest common left divisor* can be defined by an analogic manner.

The polynomial matrices A, B, having the some number of rows, are said to be *left coprime* if and only if their only common left divisors are unimodular matrices. The polynomial matrices A_2, B_2, having the same number of columns, are said to be *right coprime* if and only if their only common right divisors are unimodular.

Let a rational matrix S be given. Then the pair A, B of polynomial matrices is a *left matrix fraction representation* of S if and only if $S = A^{-1}B$; this fraction is said to be *left coprime* whenever A and B are left coprime. The pair A_2, B_2 of polynomial matrices is a *right matrix fraction representation* of S if and only if $S = B_2 A_2^{-1}$; this fraction is said to be *right coprime* whenever A_2 and B_2 are right coprime.

Finally, the *degree of a column* of a polynomial matrix is defined as the highest degree occuring in this column.

3. PROBLEM FORMULATION

Consider a discrete-time plant governed by the equation

$$y = A^{-1}Bu + A^{-1}C \tag{1}$$

where u is the m-vector input sequence and y is the l-vector output sequence. The A, B and C are respectively $l \times l$, $l \times m$ and $l \times 1$ polynomial matrices in the delay operator d (often denoted by z^{-1}), with arbitrary relative degrees, such that $B(0) = 0$ and $A(0)$ is invertible. Clearly, $A^{-1}B$ is a left matrix fraction representation of the transfer matrix and reflects the input/output properties of the plant while $A^{-1}C$ represents the effect of the initial state on the plant output.

Further consider a reference l-vector sequence r modelled by the equation

$$r = F^{-1}G \tag{2}$$

where F and G are $l \times l$ and $l \times 1$ polynomial matrices in d, respectively, with arbitrary relative degrees, such that $F(0)$ is invertible. Thus r can be thought of as a free motion of a reference generator.

Observe that C in (1) and G in (2) depend solely on the initial state of the plant and the generator respectively. Thus letting these polynomial vectors vary we can account for any initial state in the plant, or generate a whole class of reference sequences.

A general linear control law which generates the vector sequence u from y and r can be represented as

$$u = -P^{-1}Qy + P^{-1}Rr \tag{3}$$

where P, Q and R are respectively $m \times m$, $m \times l$ and $m \times l$ polynomial matrices in d with $P(0)$ invertible. A block diagram of the resulting control system is shown in Fig. 3.

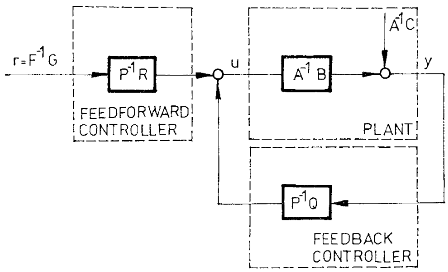

Fig. 3. Block diagram of servo system.

Given the plant and the reference, it is desired to find P, Q and R so as to make the control sequence u and the tracking error $e = r - y$ both vanish in a minimum number of steps, i.e., the $(m + l)$-vector $\begin{bmatrix} e \\ u \end{bmatrix}$ should be a polynomial vector of minimal degree.

Moreover, the control law (3) is to be independent of C and G in order to ensure the optimal performance for any initial state of the plant and the entire class of references.

4. PROBLEM SOLUTION

Theorem. The deadbeat servo problem has a solution if and only if

(i) A and B are left coprime matrices and

(ii) F is a right divisor of A, i.e., there is an $l \times l$ polynomial matrix K such that $A = KF$.

 Then P and Q is a left coprime polynomial decomposition for $Q_2 P_2^{-1}$,

$$P^{-1}Q = Q_2 P_2^{-1} \tag{4}$$

where $m \times l$ polynomial matrix Q_2 and $l \times l$ polynomial matrix P_2 are such a solution of the linear Diophantine equation

$$AP_2 + BQ_2 = I_l \tag{5}$$

that the degree of every column of the matrix $\begin{bmatrix} P_2 \\ Q_2 \end{bmatrix}$ is minimal.

The R is such a solution of the linear Diophantine equation

$$\begin{bmatrix} S \\ H \end{bmatrix} F + \begin{bmatrix} B_2 \\ -A_2 \end{bmatrix} R = \begin{bmatrix} I_l \\ 0 \end{bmatrix} \tag{6}$$

that the degree of every column of $\begin{bmatrix} S \\ H \end{bmatrix}$ minimal. Here B_2 and A_2 are respectively $l \times m$ and $m \times m$ polynomial matrices defined as the right coprime fraction of $A^{-1}B$,

$$B_2 A_2^{-1} = A^{-1}B \tag{7}$$

for which the equation

$$PA_2 + QB_2 = I_m \tag{8}$$

is satisfied.

Proof. The proof will be devided into two parts. First, we shall construct the optimal controller provided it exists and, second, we shall prove the problem solvability.

Concerning the first part, the relations (1) through (4) and (7) yield

$$u = -Q_2(AP_2 + BQ_2)^{-1} C + A_2(PA_2 + QB_2)^{-1} RF^{-1}G \tag{9}$$

$$e = -P_2(AP_2 + BQ_2)^{-1} C + (I_l - B_2(PA_2 + QB_2)^{-1} R) F^{-1}G \tag{10}$$

Now the first terms of u and e are polynomial vectors, independently of C, if and only if the equation (5) is satisfied. For A, B, P_2 and Q_2 satisfying (5) and some P, Q from (4) there are unique matrices A_2, B_2 satisfying (7) and (8) because

$$\begin{bmatrix} A & B \\ Q & -P \end{bmatrix} \begin{bmatrix} P_2 & B_2 \\ Q_2 & -A_2 \end{bmatrix} = \begin{bmatrix} I_l & 0 \\ 0 & I_m \end{bmatrix}$$

Hence due to (5) and (8)

$$u = -Q_2C + A_2RF^{-1}G \tag{11}$$

$$e = -P_2C + (I_l - B_2R)\,F^{-1}G \tag{12}$$

For the second term in (11) to be a polynomial vector independently of G, there must be some $m \times l$ polynomial matrix H such that

$$A_2R = HF$$

or, equivalently,

$$HF - A_2R = 0 \tag{13}$$

Analogically, for the second member in (12) to be a polynomial independently of G, there must be some $l \times l$ polynomial matrix S such that

$$I_l - B_2R = SF$$

or equivalently,

$$SF + B_2R = I_l\,. \tag{14}$$

Writing equations (14) and (13) in the compact form

$$\begin{bmatrix} S \\ H \end{bmatrix} F + \begin{bmatrix} B_2 \\ -A_2 \end{bmatrix} R = \begin{bmatrix} I_l \\ 0 \end{bmatrix} \tag{15}$$

we have got equation (6). Now we can write e and u in the compact form

$$\begin{bmatrix} e \\ u \end{bmatrix} = -\begin{bmatrix} P_2 \\ Q_2 \end{bmatrix} C + \begin{bmatrix} S \\ H \end{bmatrix} G \tag{16}$$

Since $\begin{bmatrix} e \\ u \end{bmatrix}$ is to be of least possible degree independently of C and G, we must take the solutions $\begin{bmatrix} P_2 \\ Q_2 \end{bmatrix}$ and $\begin{bmatrix} S \\ H \end{bmatrix}$ of equations (5) and (6) respectively which minimize the degree of every column of $\begin{bmatrix} P_2 \\ Q_2 \end{bmatrix}$ and $\begin{bmatrix} S \\ H \end{bmatrix}$.

To carry out the second part of the proof, observe that equation (5) has a solution if and only if the condition (i) holds, that is, A and B are left coprime polynomial matrices.

As shown in [1], equation (6) has a solution if and only if the matrices

$$\begin{bmatrix} B_2 & 0 \\ -A_2 & 0 \\ 0 & F \end{bmatrix} \text{ and } \begin{bmatrix} B_2 & I_l \\ -A_2 & 0 \\ 0 & F \end{bmatrix} \tag{17}$$

are equivalent. We are left to prove that this is the case if and only if the condition (ii) is satisfied. The B_2 and $-A_2$ are right coprime matrices and so there is an $(m + l) \times (l + m)$ unimodular matrix

$$J = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}$$

such that

$$\begin{aligned} J_{11}B_2 + J_{12}(-A_2) &= I_m \\ J_{21}B_2 + J_{22}(-A_2) &= 0 \end{aligned} \tag{18}$$

Here J_{21} and J_{22} are left coprime matrices. Hence

$$\begin{bmatrix} I_m & 0 \\ 0 & 0 \\ 0 & F \end{bmatrix} = \begin{bmatrix} J_{11} & J_{12} & 0 \\ J_{21} & J_{22} & 0 \\ 0 & 0 & I_l \end{bmatrix} \begin{bmatrix} B_2 & 0 \\ -A_2 & 0 \\ 0 & F \end{bmatrix}$$

and so

$$\begin{bmatrix} B_2 & 0 \\ -A_2 & 0 \\ 0 & F \end{bmatrix} \sim \begin{bmatrix} I_m & 0 \\ 0 & 0 \\ 0 & F \end{bmatrix} \sim \begin{bmatrix} I_m & 0 \\ 0 & F \\ 0 & 0 \end{bmatrix} \tag{19}$$

where $\sim$ stands for the equivalence of polynomial matrices. Analogically

$$\begin{bmatrix} B_2 & I_l \\ -A_2 & 0 \\ 0 & F \end{bmatrix} \sim \begin{bmatrix} I_m & J_{11} \\ 0 & J_{21} \\ 0 & F \end{bmatrix} \sim \begin{bmatrix} I_m & 0 \\ 0 & J_{21} \\ 0 & F \end{bmatrix} \tag{20}$$

Now there is an $(l + l) \times (l + l)$ unimodular matrix

$$L = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix}$$

such that

$$\begin{aligned} L_{11}J_{21} + L_{12}F &= D \\ L_{21}J_{21} + L_{22}F &= 0 \end{aligned} \tag{21}$$

where the $l \times l$ polynomial matrix D is a greatest common right divisor of matrices J_{21} and F. So due to (20) and

$$\begin{bmatrix} B_2 & I_l \\ -A_2 & 0 \\ 0 & F \end{bmatrix} \sim \begin{bmatrix} I_m & 0 \\ 0 & D \\ 0 & 0 \end{bmatrix} \tag{22}$$

Hence (19) and (22) imply that the matrices from (17) are equivalent if and only if F is a right divisor of J_{21}. Now from (7)

$$AB_2 = BA_2$$

where A and B are left coprime matrices and, at the same time, from (18)

$$J_{21}B_2 = J_{22}A_2$$

with J_{21} and J_{22} left coprime.
So

$$A = KJ_{21}$$

where K is some $l \times l$ unimodular matrix. Hence F is a right divisor of J_{21} if and only if F is a right divisor of A and so condition (ii) is proven. This completes the proof of the theorem. □

The solvability conditions in the Theorem can be given a nice physical interpretation. The former means complete controllability of the given system while the latter says that all infinite dynamical modes to be followed must be present in the plant. By an infinite mode we mean here any separately excitable response which does not vanish in a finite time.

5. DESIGN PROCEDURE

The design procedure is seen to be very simple. It consists only in the solution of three linear equations in polynomial matrices. All needed algorithms contain only elementary row and column operations and can be found, for example, in [1].

Given A, B and F, we must solve the equation (5) at first, i.e.,

$$AP_2 + BQ_2 = I_l \tag{23}$$

Using elementary column operations, carry out the reduction

$$\begin{bmatrix} A & B \\ I_l & 0 \\ 0 & I_m \end{bmatrix} \to \begin{bmatrix} I_l & 0 \\ X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} \tag{24}$$

where X_{11}, X_{12}, X_{21}, X_{22} are respectively $l \times l$, $l \times m$, $m \times l$, $m \times m$ polynomial matrices.

Then a general solution of the equation (23) has the form (see [1])

$$\begin{aligned} P_2 &= X_{11} + X_{12}T \\ Q_2 &= X_{21} + X_{22}T \end{aligned} \tag{25}$$

where T is an arbitrary polynomial $m \times l$ matrix. From (25) we must take the solution which minimizes the degree of every column of $\begin{bmatrix} P_2 \\ Q_2 \end{bmatrix}$. A simple algorithm for choosing such a solution is described in [2].

To find the feedback part of the optimal controller, let us use elementary row operations for carrying out the reduction

$$\begin{bmatrix} P_2 & X_{12} & I_l & 0 \\ Q_2 & X_{22} & 0 & I_m \end{bmatrix} \rightarrow \begin{bmatrix} I_l & 0 & Y_{11} & Y_{12} \\ 0 & I_m & Y_{21} & Y_{22} \end{bmatrix} \tag{26}$$

where Y_{11}, Y_{12}, Y_{21} and Y_{22} are respectively $l \times l$, $l \times m$, $m \times l$ and $m \times m$ matrices Now

$$\begin{aligned} P &= -Y_{22} \\ Q &= Y_{21} \end{aligned} \tag{27}$$

Finally, we get the feedforward part R as a solution of equation (6), i.e.,

$$\begin{bmatrix} S \\ H \end{bmatrix} F + \begin{bmatrix} X_{12} \\ X_{22} \end{bmatrix} R = \begin{bmatrix} I_l \\ 0 \end{bmatrix} \tag{28}$$

Form a general solution, which is of the form

$$\begin{aligned} R &= U + VF \\ \begin{bmatrix} S \\ H \end{bmatrix} &= Z - \begin{bmatrix} X_{12} \\ X_{22} \end{bmatrix} V \end{aligned} \tag{29}$$

where U, Z is some particular solution and V is an arbitrary $m \times l$ polynomial matrix, we must again take the solution minimizing the degree of every column of $\begin{bmatrix} S \\ H \end{bmatrix}$.

6. EXAMPLE

To illustrate the design procedure, consider the deadbeat servo problem for the plant (1) given by

$$A = \begin{bmatrix} 1-d & -d \\ 1-d & 1 \end{bmatrix} \quad B = \begin{bmatrix} d \\ 0 \end{bmatrix}$$

and for the class of reference sequences (2) described by

$$F = \begin{bmatrix} 1-d & 0 \\ 0 & 1 \end{bmatrix}$$

The C and G are arbitrary polynomial 2-vectors of degree less than or equal to 1 which represent arbitrary initial state of the plant and of the reference generator, respectively.

The equation (23) has now the form

$$\begin{bmatrix} 1-d & -d \\ 1-d & 1 \end{bmatrix} P_2 + \begin{bmatrix} d \\ 0 \end{bmatrix} Q_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Carrying out the reduction (24)

$$\left[\begin{array}{cc|c} 1-d & -d & d \\ 1-d & 1 & 0 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 1 & 0 & d \\ -1+d & 1 & -d+d^2 \\ \hline d & 1 & -1+d^2 \end{array}\right]$$

we get the general solution

$$P_2 = \begin{bmatrix} 1 & 0 \\ -1+d & 1 \end{bmatrix} + \begin{bmatrix} d \\ -d+d^2 \end{bmatrix} [t_1 \ t_2]$$

$$Q_2 = [\mathrm{d} \ 1] + (-1+d^2)[t_1 \ t_2]$$

and the minimal solution

$$P_2 = \begin{bmatrix} 1 & 0 \\ -1+d & 1 \end{bmatrix} \quad Q_2 = [d \ 1]$$

Then we carry out the reduction (26)

$$\left[\begin{array}{cc|c|ccc} 1 & 0 & d & 1 & 0 & 0 \\ -1+d & 1 & -d+d^2 & 0 & 1 & 0 \\ \hline d & 1 & -1+d^2 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|c|cc|c} 1 & 0 & 0 & 1-d & -d & d \\ 0 & 1 & 0 & 1-d & 1 & 0 \\ \hline 0 & 0 & 1 & 1 & 1 & -1 \end{array}\right]$$

and so

$$P = 1$$
$$Q = [1 \ 1]$$

Finally, the equation (28) is

$$\begin{bmatrix} S \\ H \end{bmatrix} \begin{bmatrix} 1-d & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} d \\ -d+d^2 \\ -1+d^2 \end{bmatrix} R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

and has a general solution

(30) $$R = [1, 0] + [v_1 \ v_2] \begin{bmatrix} 1-d & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} S \\ H \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ d & 1 \\ 1+d & 0 \end{bmatrix} - \begin{bmatrix} d \\ -d+d^2 \\ -1+d^2 \end{bmatrix} [v_1 \ v_2]$$

The minimal solution is calculated to be

$$R = [1 \ 0] \qquad \begin{bmatrix} S \\ H \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ d & 1 \\ 1 + d & 0 \end{bmatrix}$$

So the optimal deadbeat controller is

$$\begin{aligned} P &= 1 \\ Q &= [1 \ 1] \\ R &= [1 \ 0] \end{aligned} \tag{31}$$

and the resulting deadbeat response is given by

$$\begin{bmatrix} e \\ u \end{bmatrix} = - \begin{bmatrix} 1 & 0 \\ -1 + d & 1 \\ d & 1 \end{bmatrix} C + \begin{bmatrix} 1 & 0 \\ d & 1 \\ 1 + d & 0 \end{bmatrix} G \tag{32}$$

The exact tracking is accomplished in no more then three steps.

7. DISCUSSION

It is worthwhile to notice that in (31) $R \neq Q$. Hence the classical, error-actuated compensator can never achieve the performance obtained in the preceding example. Forcing the control law to operate on the tracking error means putting $R = Q$. So in order to obtain the classical controller we need

$$R = [1 \ 1]$$

for which we must take

$$[v_1 \ v_2] = [0 \ 1]$$

in (30). But this entails that

$$\begin{bmatrix} S \\ H \end{bmatrix} = \begin{bmatrix} 1 & -d \\ d & d - d^2 \\ 1 + d & 1 - d^2 \end{bmatrix}$$

thereby the degree of

$$\begin{bmatrix} e \\ u \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ -1 + d & 1 \\ d & 1 \end{bmatrix} C + \begin{bmatrix} 1 & -d \\ d & d - d^2 \\ 1 + d & 1 - d^2 \end{bmatrix} G$$

has increased and so, generally, the transients are one step longer. That is why the classical, error-actuated controller is only suboptimal.

Further, the pseudocharacteristic polynomial of the designed servo system is equal to 1. So the deadbeat performance is not lost even if any additional exogenous finite input enters the servo system.

Last but not least, if the discrete-time plant and reference originated from continuous-time ones by the process of sampling, the condition that u vanish in a finite time implies exact tracking not only at the sampling points but also in between.

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