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STIRLING DISTRIBUTIONS AND STIRLING NUMBERS OF THE SECOND KIND. COMPUTATIONAL PROBLEMS IN STATISTICS

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The left-truncated generalized Poisson distribution belongs to the family of the modified power series distributions. Using sufficiency and completeness of $\sum X_i$ $(\min X_i, \sum X_i))$ respectively, when the truncation point is known (resp. unknown), the minimum variance unbiased (M.V.U) estimator for certain functions of the parameter θ (resp. θ, r) involved in these distributions can be obtained (see Charalambides [3, 4], Jani [10], Voinov–Nikulin [23]).

These distributions, as well as the corresponding M.V.U estimators, was expressed in terms of the modified Stirling numbers of the second kind (S.N.S.K). In this paper we give some ways to compute these numbers: first we summarize some usual and less standard identities or relations affecting the S.N.S.K. Some basic properties are given and discussed in view of calculation.

Then, by generalizing asymptotic estimates of the usual S.N.S.K, we give and discuss alternative ways to compute the modified S.N.S.K.

1. INTRODUCTION

The problem of the M.V.U estimation for a left-truncated modified power series distribution was studied by Gupta [9], Jani [10, 11], Kumar [12], Kumar and Consul [13], Voinov [22] and many others. Tate and Goen [20] considered the problem of estimating the parameters involved in the left-truncated Poisson distribution (L.P.D).

When the truncation point is assumed known, in the case in which only the zero class is missing, the M.V.U estimator was expressed in terms of the usual S.N.S.K, while in the general case the M.V.U estimator was based on the generalized S.N.S.K. More generally, Patil [17] discussed the same problem for the generalized power series distribution (G.P.S.D), investigating the problem of the existence of the M.V.U estimator for certain functions of the parameter involved in the G.P.S.D.

The statistical situations built from these distributions are numerous and the possible applications are important (see Berg [1] and Singh [19]).

When the truncation point is known, Charalambides [3, 4] obtained the M.V.U estimator in the case of the L.P.D and, in view of the computation of the M.V.U estimators, gave recurrence relations with respect to the values of the sufficient, complete statistic $Z = \sum X_i$ and the sample size.

When the truncation point is unknown, the M.V.U estimation for certain function of the parameters was also solved by Charalambides [3,4], giving the general form of this estimator in accordance with the sufficient, complete statistic (Z, Y), where $Y = \min X_i$, in terms of the generalized S.N.S.K.

Our attention was attracted to computing in a direct way, not using recursion relations, the M.V.U estimators involved in the Left-truncated generalized Poisson distributions (L.G.P.D). Using standard properties of multiplication of series, we obtain some general well-known identities, certainly giving the value of the modified S.N.S.K, but this being too complicated for any efficient calculations. However, from some combinatorial reasons, it is possible to deduce some relations to produce interesting ways to calculate the generalized S.N.S.K. Charalambides [5] obtained recurrence identities giving the modified S.N.S.K. These relations can be used to compute recursively these numbers, and it remains in practice the only efficient way to obtain the exact values of the modified S.N.S.K.

The modified S.N.S.K can be defined through their exponential generating function. Using the Cauchy integral theorem, we can express them in terms of this generating function, and thus obtain an integral expression of the modified S.N.S.K. Then, applying analytic methods of estimating complex integrals, following respectively the approachs of Moser–Wyman [14], Bleick–Wang [2] and Temme [21], we get some different expansions of these numbers. These asymptotics are distinguished by their respective domain of convergence.

2. STIRLING DISTRIBUTIONS OF THE SECOND KIND

Consider the left-truncated generalized Poisson distribution (L.G.P.D)

$$\mathsf{P}_\theta\{X = x\} = \begin{cases} (1+\beta x)^{x-1}(e^{-\beta\theta}\theta)^x / x! B_\beta(\theta, r), & \text{if } x = r, r+1, \dots \\ 0, & \text{otherwise,} \end{cases} \tag{2.1}$$

where

$$B_\beta(\theta, r) = \sum_{\ell=r}^{\infty} \frac{(1+\beta\ell)^{\ell-1}(e^{-\beta\theta}\theta)^\ell}{\ell!} = e^\theta - \sum_{\ell=0}^{r-1} \frac{(1+\beta\ell)^{\ell-1}(e^{-\beta\theta}\theta)^\ell}{\ell!}, \tag{2.2}$$

with the conditions of convergence $0 < \theta < 1$ and $|\beta\theta| < 1$.

a) Let us suppose that $\mathbf{X} = (X_1, X_2, \cdots, X_n)$ is a sample of n random variables being distributed as the L.G.P.D with parameter θ and known r. The L.G.P.D belongs to the family of the modified power series distribution (M.P.S.D), so is an exponential distribution. Thus the sum $Z = \sum_{i=1}^n X_i$ is an exhaustive statistic for the parameter θ and is distributed as a modified Stirling distribution of the second kind (M.S.D.S.K) defined by

$$\mathsf{P}_\theta\{Z = z\} = \begin{cases} n! S(z, n, r, \beta)\,(e^{-\beta\theta}\theta)^z / z! B_\beta^n(\theta, r), & \text{if } z = nr, nr+1, \dots \\ 0, & \text{otherwise} \end{cases} \tag{2.3}$$

where the real number $S(z,n,r,\beta)$, which is none other than the coefficient of $(e^{-\beta\theta}\theta)^z/z!$ in the expansion of $B^n_\beta(\theta,r)/n!$, defines the modified S.N.S.K.

Moreover, by Jensen's inequality, we deduce from the fact that Z is sufficient and complete, that the M.V.U estimator for some function $g(\theta)$ of the parameter θ can be expressed as $\mathsf{E}_\theta(g(\theta)|Z)$. Nikulin [15] formulated a generalization of the result of Patil [17] and Joshi–Park [24], which gives a necessary and sufficient condition for the existence of the M.V.U estimator for some function $g(\theta)$, and provides a technique for constructing the M.V.U estimator in terms of the sufficient statistic; in our situation, this result can be summarized as follows:

We note $W\left[\sum \gamma_\ell \Psi^\ell(\theta)\right] = \{\ell : \gamma_\ell > 0\}$. Let $Z = \sum X_i$ be the sufficient complete minimal statistic. Then the M.V.U. estimator $T(Z)$ of $g(\theta)$ exists if and only if

$$W\left[g(\theta)B^n_\beta(\theta,r)\right] \subset W[B^n_\beta(\theta,r)]\,,$$

and

$$T(Z) = \begin{cases} c(Z,n)/b(Z,n), & \text{if } Z \subset W[g(\theta)B^n_\beta(\theta,r)] \\ 0, & \text{otherwise,} \end{cases}$$

where the numbers $b(Z,n)$ et $c(Z,n)$ designate respectively the coefficients of $\Psi^Z(\theta)$ in the series expansion of $B^n_\beta(\theta,r)$ and $g(\theta)\, B^n_\beta(\theta,r)$.

From this we obtain for $\mathsf{P}_\theta\{X = x\}$ and θ^m the following M.V.U estimators:

$$\widehat{\mathsf{P}_\theta\{X = x\}} = \binom{Z}{x}(1+\beta x)^{x-1}S(Z-x,n-1,r,\beta)/nS(Z,n,r,\beta),$$

$$\widehat{\theta^m} = \sum_{k=m}^{Z-rn} m(\beta k)^{k-m}Z!S(Z-k,n,r,\beta)/k(k-m)!(Z-k)!S(Z,n,r,\beta).$$

b) If r is unknown, the statistic (Z,Y), where Y designates the minimal order statistic $\min_{1\le i\le n} X_i$, is sufficient and complete for the parameter (θ,r). The joint distribution of (Z,Y) is

$$\mathsf{P}_{\theta,r}\{Z=z, Y=y\} = \begin{cases} \frac{n!\left(S(z,n,y,\beta)-S(z,n,y+1,\beta)\right)(e^{-\beta\theta}\theta)^z}{z!B^n_\beta(\theta,r)}, & \text{if } y \ge r \\ 0, & \text{otherwise.} \end{cases}$$

In this case, the M.V.U estimators of r and θ^m are

$$\hat{r} = Y - S(Z,n,Y+1,\beta)/\left(S(Z,n,Y,\beta) - S(Z,n,Y+1,\beta)\right),$$

$$\widehat{\theta^m} = \sum_{k=m}^{Z-nY} \frac{m(\beta k)^{k-m}Z!}{k(k-m)!(Z-k)!}\,\frac{S(Z-k,n,Y,\beta)-S(Z-k,n,Y+1,\beta)}{S(z,n,Y,\beta)-S(Z,n,Y+1,\beta)}.$$

c) Similarly, let us consider a sample $\mathbf{X} = (X_1, X_2, \cdots, X_n)$ of n random variables being distributed as a M.S.D.S.K defined in (2.3) with parameters θ, k and r. This distribution also belongs to the family of the M.P.S.D, thus the sum $Z = \sum_{i=1}^n X_i$ is exhaustive for the parameter θ when k and r are assumed known. The distribution of Z is

$$\mathsf{P}_\theta\{Z=z\} = \begin{cases} (nk)!S(z,nk,r,\beta)(e^{-\beta\theta}\theta)^z/z!B^{nk}_\beta(\theta,r), & \text{if } z \ge nkr \\ 0, & \text{otherwise.} \end{cases}$$

Let us notice now that each random variable X_i can be split up into the sum $X_i = \sum_{j=1}^{k} Y_j$ of k independent and identically distributed random variables Y_j, following a L.G.P.D with parameter θ and r.

It is also possible to compute the M.V.U estimator for some functions $g(\theta)$ of the parameter θ which can be written $\mathsf{E}_\theta(g(\theta)|Z)$. We have for instance the M.V.U estimator of $\mathsf{P}_\theta\{X = x; \theta\}$

$$\mathsf{P}_\theta\{\widehat{X = x}\} = \mathsf{P}_\theta\{X = x|Z\} = \frac{\binom{Z}{x} S(x,k,r,\beta)\, S(Z-x, nk-k, r, \beta)}{\binom{nk}{nk-k} S(Z, nk, r, \beta)}.$$

All the preceding M.V.U estimators can be used for goodness-of-fit test (see Singh [19], Nikulin [15]); they are expressed in terms of the S.N.S.K, and this shows the usefulness to compute them.

3. COMBINATORIAL IDENTITIES AND RECURRENCE RELATIONS

With the notations of the previous section, we have

$$B_\beta(\theta, r) = \sum_{x=r}^{\infty} b_x \Psi^x(\theta), \tag{3.1}$$

where $\Psi(\theta) = e^{-\beta\theta}\theta$ and $b_x = (1+\beta x)^{x-1}/x!$, $x = r, r+1, \ldots$.

We have to compute the kth power of the series $B_\beta(\theta, r)$,

$$B_\beta^k(\theta, r) = \Big(\sum_{x=r}^{\infty} b_x \Psi^x(\theta)\Big)^k = \sum_{n=kr}^{\infty} a_n(k)\, \Psi^n(\theta).$$

With the above notations, we have $a_n(k) = \frac{k!}{n!} S(n, k, r, \beta)$.

a) General expressions of the coefficient of the kth power of a series.

By Cauchy's rule of multiplication of series, we obtain

$$a_n(k) = \sum_{n_1+n_2+\cdots+n_k=n} b_{n_1} b_{n_2} \cdots b_{n_k}. \tag{3.2}$$

By multinomial theorem, we can express $a_n(k)$ as

$$a_n(k) = \sum_{\ell_1+2\ell_2+\cdots+n\ell_n=n} \frac{(k)_{\ell_1+\ell_2+\cdots+\ell_n}}{\ell_1!\ell_2!\ldots\ell_n!} b_0^{k-\ell_1-\ell_2-\cdots-\ell_n} b_1^{\ell_1} b_2^{\ell_2} \cdots b_n^{\ell_n}$$

where for k and m two positive integers, $(k)_m = k(k-1)(k-2)\cdots(k-m+1)$. This can be rewritten, regrouping the terms with the same power of b_0, as

$$a_n(k) = \sum_{m=1}^{k} (k)_m b_0^{k-m} \sum_{\substack{\ell_1+2\ell_2+\cdots+n\ell_n=n \\ \ell_1+\ell_2+\cdots+\ell_n=m}} \frac{1}{\ell_1!\ell_2!\ldots\ell_n!} b_1^{\ell_1} b_2^{\ell_2} \ldots b_n^{\ell_n}.$$

For $r \geq 1$, the r first terms in the series $B_\beta(\theta, r)$ vanish, thus we deduce the following formula for the modified S.N.S.K

$$S(n,k,r,\beta) = \sum_{\substack{r\ell_r+(r+1)\ell_{r+1}+\cdots+n\ell_n=n \\ \ell_r+\ell_{r+1}+\cdots+\ell_n=k}} \frac{n!}{\ell_r!\ell_{r+1}!\ldots\ell_n!} b_r^{\ell_r} b_{r+1}^{\ell_{r+1}} \ldots b_n^{\ell_n}. \tag{3.3}$$

b) The generalized Stirling numbers of second kind.

Now we study the special case $\beta = 0$. Let us recall that the number $S(n,k,r)$ is none other than the number of partitions of the set $\{1,2,\ldots,n\}$ of the n first nonnegative integers into k disjoint subsets each containing more than r integers of $\{1,2,\ldots,n\}$. From this, considering alternatively partitions depending on whether $\{1,2,\ldots,r\}$ forms a block or not, we can deduce the following "triangular" relation

$$S(n,k,r) = kS(n-1,k,r) + \binom{n-1}{r-1} S(n-r,k-1,r), \tag{3.4}$$

with the initial conditions $S(n,k,r) = 0$ if $n < kr$ and $S(n,k,r) = 1$ if $n = kr$.

This recurrence is very useful for computations, but unfortunately, to get the value of a certain given generalized S.N.S.K, it generates many of intermediate calculations.

Using the "including-excluding" principle, we have the following representation of the generalized S.N.S.K as

$$S(n,k,r) = \frac{1}{k!}\sum_{j=0}^{k}(-1)^j\binom{k}{j}\sum_{\ell_1,\ell_2,\ldots,\ell_j=0}^{r-1}\frac{n!(k-j)^{n-\ell_1-\ell_2-\ldots-\ell_j}}{\ell_1!\ell_2!\ldots\ell_j!(n-\ell_1-\ell_2-\ldots-\ell_j)!}.$$

If we put $r = 1$ in this formula, we obtain the well-known equality giving the standard S.N.S.K

$$S(n,k) = \frac{1}{k!}\sum_{j=0}^{k}(-1)^j\binom{k}{j}(k-j)^n. \tag{3.5}$$

These two previous formulas give the best known way to compute the S.N.S.K when we need only the value of one Stirling number. But although this method is better in terms of computing time and memory requirement than the recurrence one, it is reasonably interesting only for relative small values of k and r. On the other hand, if we don't need the exact value of $S(n,k)$, the relation (3.5) can be truncated, assuming that n is sufficiently large with regard to k, and then gives an estimation of this number.

c) We now direct our attention on the case $r = 1$. Gupta [9] gave the useful equality linking the modified S.N.S.K with $r = 1$ and the regular S.N.S.K

$$S(n,k,1,\beta) = \sum_{j=k}^{n}\binom{n-1}{j-1}(\beta n)^{n-j}S(j,k). \tag{3.6}$$

So, using either (3.4) or (3.5), it is possible and easy with (3.6) to compute the modified S.N.S.K in this special case.

d) Two recurrence relations for the modified S.N.S.K.

The modified S.N.S.K satisfy the following recursion relations (see Charalambides [4]) and Voinov–Nikulin [23]):

$$S(n+1,k,r,\beta)=\sum_{j=r}^{n-r(k-1)+1}(1+\beta j)^{j-1}\binom{n}{j-1}S(n-j+1,k-1,r,\beta),$$

$$S(n,k,r+1,\beta)=\sum_{j=0}^{k}(-1)^j(1+\beta r)^{j(r-1)}\frac{n!S(n-rj,k-j,r,\beta)}{(n-rj)!j!(r!)^j}.$$

Putting $\beta=0$ and $r=1$, we deduce a well-known relation for the regular S.N.S.K.

4. ASYMPTOTICS OF STIRLING NUMBERS

The exponential generating function of the numbers $S(n,k,r,\beta)$ is

$$\frac{1}{k!}B_\beta^k(\theta,r)=\sum_{n\geq kr}S(n,k,r,\beta)\,(e^{-\beta\theta}\theta)^n/n!. \tag{4.1}$$

Then, by the Cauchy integral theorem, we obtain the following integral expression

$$S(n,k,r,\beta)=\frac{n!}{k!}\frac{1}{2\pi i}\int_{\mathcal{C}}B_\beta^k(\Psi^{-1}(z),r)\frac{\mathrm{d}z}{z^{n+1}}, \tag{4.2}$$

where the contour $\mathcal{C}$ encloses the origin and is included in the unit disc.

Now we distinguish the two cases whether β is zero or not.

a) Suppose that $\beta=0$. The numbers $S(n,k,r,\beta)$ reduce to the generalized S.N.S.K, and $\Psi(\theta)=\theta$. Writing $B_0(z,r)=B(z,r)$, formula (4.2) gives

$$S(n,k,r)=\frac{n!}{k!}\frac{1}{2\pi i}\int_{\mathcal{C}}B^k(z,r)\frac{\mathrm{d}z}{z^{n+1}}, \tag{4.3}$$

Let us call I the integral in (4.3). Following the idea of Temme [21], we put

$$\Phi(z)=-n\ln z+k\ln B(z,r), \tag{4.4}$$

and we obtain $I=\int_{\mathcal{C}}e^{\Phi(z)}\frac{\mathrm{d}z}{z}$.

The saddle-point z_0 is defined by the positive real solution of the equation $\Phi'(z)=0$, that is

$$z_0\frac{B'(z_0,r)}{B(z_0,r)}=\frac{n}{k}. \tag{4.5}$$

We now define a local transformation of $\Phi(z)$ around the point $z=z_0$; as remarked by Temme [21] for the standard S.N.S.K, the usual saddle-point method is based on a local quadratic transformation, and does not give here approximations very accurate when $n\sim kr$. Observing that $\Phi(z)\sim(kr-n)\ln z$ when $x\to0^+$ and $\Phi(z)\sim kz$ when $x\to+\infty$, we are led to put

$$\Phi(z)=kt+(kr-n)\ln t+A, \tag{4.6}$$

where t is the new variable and A is a constant term defined by the additional conditions: $z = 0 \Leftrightarrow t = 0$, $z = +\infty \Leftrightarrow t = +\infty$ and $z = z_0 \Leftrightarrow t = t_0$, where t_0 is the point which cancels the derivative of the right hand-side of (4.6). So we have

$$t_0 = (n - kr)/k \quad \text{and} \quad A = \Phi(z_0) - kt_0 + (n - kr)\ln t_0. \tag{4.7}$$

Thus, by applying the transformation (4.6), we get

$$I = e^A \int e^{kt}\mu(t)\frac{\mathrm{d}t}{t^{n-kr+1}}, \quad \text{where} \quad \mu(t) = \frac{t}{z}\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{k(t - t_0)}{z\Phi'(z)}. \tag{4.8}$$

The function μ is analytic in a neighborhood of the origin and in a domain including the real axis, so μ is analytic at $t = 0$ and $t = t_0$. The contour C can be deformed into a contour through the point t_0.

A first approximation is now obtained by replacing $\mu(t)$ by $\mu(t_0)$. This gives in view of the relations (4.3) to (4.8) as $n \to +\infty$

$$S(n, k, r) \sim \frac{n!}{k!(n - kr)!}\left(\frac{n - kr}{e}\right)^{n-kr}\frac{B^k(z_0, r)}{z_0^{n+1}}\sqrt{\frac{kt_0}{\Phi''(z_0)}}. \tag{4.9}$$

Several computations with different values of n and r was done, and showed the uniform character of (4.9) with respect to k.

It is possible to obtain higher order approximations for the generalized S.N.S.K by following Temme's procedure. We have

$$S(n, k, r) \sim \frac{n!}{k!(n - kr)!}\left(\frac{n - kr}{e}\right)^{n-kr}\frac{B^k(z_0, r)}{z_0^n}\sum_{\ell=0}^{\infty}(-1)^{\ell}\mu_{\ell}(t_0)k^{-\ell},$$

where $$\mu_0 = \mu \quad \text{and} \quad \mu_{\ell+1}(t) = t\frac{\mathrm{d}}{\mathrm{d}t}\frac{\mu_{\ell}(t) - \mu_{\ell}(t_0)}{t - t_0}, \quad \ell \geq 0.$$

b) In the general case $\beta \neq 0$, denoting $C' = \Psi^{-1}(C)$, from (4.2) we obtain

$$S(n, k, r, \beta) = \frac{n!}{k!}\frac{1}{2\pi i}\int_{C'} B_{\beta}^k(z, r)(1 - \beta z)e^{\beta n z}\frac{\mathrm{d}z}{z^{n+1}}, \tag{4.10}$$

which can be rewritten as

$$S(n, k, r, \beta) = \frac{n!}{k!}\frac{1}{2\pi i}\left\{\int_{C'} B_{\beta}^k(z, r)e^{\beta z n}\frac{\mathrm{d}z}{z^{n+1}} - \int_{C'} B_{\beta}^k(z, r)e^{\beta z n}\frac{\mathrm{d}z}{z^n}\right\}. \tag{4.11}$$

The previous argument can be adjusted to estimate each of these two similar integrals. In this way, the first order and high order estimates follow.

c) Moser–Wyman [14] formulated two different asymptotic expansions of the standard S.N.S.K., each of them valid over two complementary domains, depending whether $n - k = o(\sqrt{n})$ or $n - k \to +\infty$ as n increases to infinity. Those can be extended easily to the generalized S.N.S.K. For obtaining similar expansions of the modified S.N.S.K, we need some modifications. According to (4.11), we have in fact to expand integrals as

$$I = \int_C B_\beta^k(z,r)e^{\beta zn}\frac{\mathrm{d}z}{z^{n+1}}. \tag{4.12}$$

Putting $z = Re^{i\theta}$ and $g(\theta, R) = k\ln B_\beta(Re^{i\theta}, r) + \beta nRe^{i\theta} - in\theta$, we get

$$I = A\int_{-\pi}^{+\pi} \exp\big(g(\theta, R)\big)\mathrm{d}\theta,$$

where A is some constant. Then we compute the MacLaurin expansion around $\theta = 0$ of $g(\theta, R)$. At this stage, we choose R such that the term in θ vanishes ($\theta = 0$ correspond to the saddle-point z_0 of the previous method). Distinguishing the contributions in I with small θ (major arcs) and with large θ (minor arcs) and extending the range of integration, we obtain

$$I \sim A'\Big\{\sum_{s=0}^{\infty}(\sqrt{kR})^{-s}\int_{-\infty}^{+\infty}\exp(-\phi^2)b_s(\phi)\mathrm{d}\phi\Big\}, \tag{4.13}$$

for some calculable constant A' and where $b_s(\phi)$ are polynomials in ϕ. The integrals in (4.13) can be easily calculated. The expansion (4.13) is valid as $n - kr$ tends to infinity.

Formula (4.13) can be completed in the following way :
Writing $f(z) = (B_\beta(z,r) - b_r\Psi^r(z))/(b_r\Psi^r(z))$ where $b_r = (1+\beta r)^{r-1}/r!$, we get from (4.12)

$$I = \frac{b_r^k}{2\pi i}\int_C \big(1 + f(z)\big)^k e^{\beta zn}\frac{\mathrm{d}z}{z^{n-kr+1}}. \tag{4.14}$$

The substitution $q = 2/k$, $z = qw$ transforms I into

$$I = q^{-(n-kr)}\frac{b_r^k}{2\pi i}\int \big(1 + f(qw)\big)^{2/q}e^{\beta qwn}\frac{\mathrm{d}w}{w^{n-kr+1}}.$$

Expanding $\big(1 + f(qw)\big)^{2/q}$ in a MacLaurin series about $q = 0$, we obtain

$$I = q^{-(n-kr)}\frac{b_r^k}{2\pi i}\int F(q, w)e^{w+\beta qwn}\frac{\mathrm{d}w}{w^{n-kr+1}},$$

where $F(q, w) = \sum_{j=0}^{\infty} P_j(w)q^j$ and $P_j(w)$ are polynoms in w. Thus we can write, putting $A_j = \left[\frac{\mathrm{d}^{n-kr}}{\mathrm{d}w^{n-kr}}\big(P_j(w)e^{w(1+\beta qn)}\big)\right]_{w=0}$

$$S(n, k, r, \beta) = \frac{n!}{k!}\left(\frac{k}{2}\right)^{n-kr} b_r^k \sum_{j=0}^{n-kr-1}\left(\frac{2}{k}\right)^j A_j. \tag{4.15}$$

If we truncate the summation in (4.15), we obtain an equivalent which is valid for $n - kr = o(\sqrt{n})$.

Using essentially the same ideas, Bleick–Wang [2] obtained another asymptotic for the standard S.N.S.K valid on a smaller domain but giving better accuracy.

In our numerical simulations, the first order approximations achieved by Temme or Moser–Wyman's approach produce approximations with relative errors less than 1/100. If we consider the first two terms of each of these expansions, the accuracy reaches 1/1000.

In the following table, we give an application for $n = 10$, $r = 1$ of the formulas (4.9), and (4.13), (4.15) truncated at the first two terms.

Table 1. Approximations of standard Stirling numbers of the second kind.

k	$S(20, k)$	(4.9)	(4.13)-(4.15)*
1	1	0.9993	0.9997
2	511	512.98	510.92
3	9330	9390.1	9332.8
4	34105	34319	34114
5	42525	42750	42531
6	22827	22922	22826
7	5880	5897.7	5877.9
8	750	751.45	749.1
9	45	45.04	45
10	1	1	1

Table 2. Approximations of generalized Stirling numbers of the second kind.

k	$S(20, k, 2)$	(4.9)	(4.13)-(4.15)*
1	1	0.9995	0.9999
2	$5.2427\ 10^5$	$5.2429\ 10^5$	$5.2409\ 10^5$
3	$5.7536\ 10^8$	$5.7766\ 10^8$	$5.7535\ 10^8$
4	$4.1388\ 10^{10}$	$4.1650\ 10^{10}$	$4.1397\ 10^{10}$
5	$5.3618\ 10^{11}$	$5.3983\ 10^{11}$	$5.3632\ 10^{11}$
6	$1.8618\ 10^{12}$	$1.8736\ 10^{12}$	$1.8622\ 10^{12}$
7	$2.0267\ 10^{12}$	$2.0377\ 10^{12}$	$2.0270\ 10^{12}$
8	$6.9474\ 10^{11}$	$6.9756\ 10^{11}$	$6.9466\ 10^{11}$
9	$6.2199\ 10^{10}$	$6.2341\ 10^{10}$	$6.2199\ 10^{10}$

(∗) best of the two formulas (4.13) and (4.15) – (†) not computed via (4.9).

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