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# Continuous Stochastic Approximation Procedure for Evaluating the Point at which the Regression Function Stops to Be Non-Positive

EL SAYED SOROUR

A continuous version of the stochastic approximation algorithm proposed in [2] is considered. A function  $r(x)$  is observed continuously with Gaussian white noise. We want to estimate the point  $\theta$  such that  $r(x) \leq 0$  for  $x \leq \theta$  while  $r(x) > 0$  for  $x > \theta$ . The proving methods developed in the book by Nevelson and Hasminskij [4] are utilized to prove the convergence with probability one and in the mean square and the asymptotic normality of the procedure.

## 1. INTRODUCTION

Consider the following stochastic differential equation

$$(1.1) \quad dX(t) = -a(t)(r(X(t)) dt + \sigma(t, X(t)) d\zeta(t)), \quad X(t_0) = x,$$

where  $\zeta(t)$  is the standard Wiener process.

Nevelson and Hasminskij [4] have proved the convergence of the procedure (1.1) with probability one to the set of roots of  $r(x)$  under general conditions.

They have also proved that the procedure can converge to a narrower set. Namely, let  $\Theta$  be the set of roots of  $r(x)$ , define  $B \subset \Theta$  such that  $\theta \in B$  if in some of its  $\eta$ -neighbourhood  $U_\eta(\theta)$  there exists a continuously differentiable function  $V(x)$  such that

$$V(\theta) = 0, \quad V(x) > 0 \quad \text{for } x \neq \theta, \quad r(x) \frac{dV}{dx} \leq 0 \quad \text{for } x \in U_\eta(\theta).$$

Then, they have proved that the procedure (1.1) cannot converge with positive probability to a point  $\theta \in B$ . Our goal is to estimate the point  $\theta$  such that  $r(x) \leq 0$  for  $x \leq \theta$  while  $r(x) > 0$  for  $x > \theta$ , when  $r(x)$  is observed continuously with Gaussian white noise. If the procedure (1.1) is used, the only thing we can deduce is that the procedure converges with probability one to the set of roots of  $r(x)$ , which does not help in our case (because  $B = \emptyset$  (empty set)).

The problem was attacked before in the discrete time case by Guttman [3] and Friedman [2], to estimate the point  $\theta$  at which the regression function stops to be a constant.

Using the continuous analogy of the procedure proposed in [2] and exploiting the proving methods in [4], we obtain results concerning the convergence of the procedure with probability one, in the mean square and the asymptotic normality of the procedure.

## 2. BASIC ASSUMPTIONS AND NOTATIONS

All random variables are supposed to be defined on a complete probability space  $(\Omega, \mathfrak{F}, P)$ . Relations between random variables are meant with probability one.  $E$  denotes the expectation. The real line is denoted by  $\mathbf{R}$  and the indicator function of a set  $A$  by  $I_A$ .

The following assumptions will be assumed to hold in the sequel.

(i) The function  $r(x)$  is real-valued and continuous;

$$(2.1) \quad r(x) \leq 0 \quad \text{for } x \leq \theta, \quad \text{while if } x > \theta \quad \text{then } r(x) > 0, \quad x \in \mathbf{R}.$$

(ii) The function  $\sigma(t, x)$  is real-valued and continuous function of its arguments for  $t \in [t_0, \infty)$ ,  $x \in \mathbf{R}$ .

(iii) For each  $N$ , there exists  $L_N$  for which

$$(2.2) \quad |r(x) - r(y)| + |\sigma(t, x) - \sigma(t, y)| \leq L_N |x - y|$$

for  $|x| \leq N$ ,  $|y| \leq N$ ;  $t_0 \leq t \leq N$ .

(iv)  $\zeta(t)$  is independent (standard) Wiener process, consistent with a non-decreasing family  $[\mathfrak{F}_t, t \geq t_0]$  of  $\sigma$ -fields of events.

(v)  $X^{s, \xi}(t)$  is the regular solution, which is continuous with probability one, of the stochastic differential equation of the form

$$(2.3) \quad dX(t) = b(t, X(t)) dt + \sigma(t, X(t)) d\zeta(t)$$

with  $X(s) = \xi$ ,  $\xi$  is  $\mathfrak{F}_s$ -measurable.

(vi) The differential operator

$$\frac{\partial}{\partial t} + b(t, x) \frac{\partial}{\partial x} + \frac{1}{2} \sigma^2(t, x) \frac{\partial^2}{\partial x^2}$$

of (2.3) is denoted by  $L$ .

The following conditions will be needed as referred to.

*Conditions on the regression function  $r(x)$*

**R1:** For some  $K_1 > 0$ ,  $r(x) \geq K_1(x - \theta)$  for  $x > \theta$ .

**R2:** For some positive constants  $K_2$  and  $K_3$ , there exist positive constants  $\varrho$  and  $h \geq \varrho$  such that

$$r(x) \leq K_2(x - \theta) \quad \text{for } x \in [\theta - \varrho, \theta],$$

$$r(x) \leq K_3(x - \theta) \quad \text{for } x \in (-\infty, -h + \theta).$$

**R3:** For some constant  $K_4 > 0$ ,  $(x - \theta)r(x) \geq K_4(x - \theta)^2$ ,  $x \in \mathbf{R}$ .

**R4:** There exists  $B > 0$ , such that

$$r(x) = B(x - \theta) + f(x, \theta), \quad x \in \mathbf{R},$$

where

$$|f(x, \theta)| = o(|x - \theta|) \quad \text{as } x \rightarrow \theta.$$

*Conditions on the functions  $a(t)$  and  $\delta(t)$*

$$\mathbf{A1:} \quad \int_{t_0}^{\infty} a(t) \delta(t) dt = \infty; \quad \int_{t_0}^{\infty} a^2(t) dt < \infty; \quad \lim_{t \rightarrow \infty} a(t) = 0^+;$$

$$\int_{t_0}^{\infty} a(t) \delta^2(t) dt < \infty \quad \text{and} \quad \lim_{t \rightarrow \infty} \delta(t) = 0.$$

$$\mathbf{A2:} \quad a(t) = \frac{a}{t^\alpha}; \quad a > 0; \quad \frac{1}{2} < \alpha < \frac{2}{3}; \quad t \geq t_0.$$

$$\mathbf{A3:} \quad \delta(t) = \frac{\delta}{t^\gamma}; \quad \delta > 0; \quad \frac{\alpha}{2} < \gamma < 1 - \alpha; \quad t \geq t_0.$$

*Conditions on  $\sigma(t, x)$*

$$\mathbf{C1:} \quad |\sigma(t, x)|^2 \leq K^2(1 + x^2) \quad \text{for all } t \geq t_0, \quad x \in \mathbf{R}.$$

$$\mathbf{C2:} \quad \lim_{\substack{t \rightarrow \infty \\ x \rightarrow \theta}} \sigma(t, x) = \sigma_0.$$

We shall also need the following theorem due to M. B. Nevelson, R. Z. Hasminskij [4].

**Theorem 3.1.** Let us have a nonnegative real-valued function  $V(t, x)$ , which is continuously differentiable with respect to  $t$ , and twice continuously differentiable with respect to  $x$ , and a set  $A$  for which

$$(3.1) \quad \inf_{t \geq t_0} V(t, x) \rightarrow \infty \quad \text{for } |x| \rightarrow \infty;$$

let us assume that

$$(3.2) \quad LV \leq -\alpha(t) \varphi(t, x) + g(t) (1 + V),$$

where

$$(3.3) \quad g(t) > 0; \quad \int_{t_0}^{\infty} g(t) dt < \infty; \quad \alpha(t) > 0; \quad \int_{t_0}^{\infty} \alpha(t) dt = \infty;$$

$$(3.4) \quad \varphi(t, x) \geq 0 \quad \text{for all } t \geq t_0; \quad x \in \mathbf{R}$$

and for all  $M > \varrho > 0$

$$\inf_{t \geq T(\varrho), x \in U_{\varrho, M}(A)} \varphi(t, x) > 0,$$

where  $U_{\varrho, M}(A) = v_{\varrho}(A) \cap \{x : |x| < M\}$ ,  $v_{\varrho}(A)$  is the complement of the  $\varrho$ -neighbourhood of the set  $A$ ;  $L$  is the differential operator of (2.3). Further let the conditions

$$(3.5) \quad \inf_{t \geq t_0} V(t, x) > 0 \quad \text{for } x \notin A; \quad V(t, x) = 0 \quad \text{for } x \in A$$

and

$$(3.6) \quad \limsup_{x \rightarrow A, t \geq t_0} V(t, x) = 0$$

be valid. Moreover let  $b(t, x)$  satisfy (2.2) ( $r(x)$  is replaced by  $b(t, x)$ ). Then the solution of (2.3) converges with probability one to the set  $A$  for all  $x \in \mathbf{R}$ .

(This is one-dimensional version of the Theorem 3.8.1 of M. B. Nevelson and R. Z. Hasminskij [4].)

#### 4. CONVERGENCE THEOREMS

Let  $a(t)$  and  $\delta(t)$  be positive real-valued continuous functions. Let  $X^x(t)$  be the regular solution of the stochastic differential equation

$$(4.1) \quad dX(t) = -a(t) [(r(X(t)) - \delta(t)) dt + \sigma(t, X(t)) d\zeta(t)],$$

with  $X(t_0) = x$ ;  $x \in \mathbf{R}$ ,  $t \geq t_0$ .

454 It is evident that the differential operator  $L$  of (4.1) is

$$(4.2) \quad L = \frac{\partial}{\partial t} - a(t) (r(x) - \delta(t)) \frac{\partial}{\partial x} + \frac{1}{2} \sigma^2(t, x) a^2(t) \frac{\partial^2}{\partial x^2}.$$

**Theorem 4.1.** If **R1**, **A1** and **C1** hold, then  $X^x(t) \rightarrow \theta$  for  $t \rightarrow \infty$  with probability one.

*Proof.* Without loss of generality we can take  $\theta = 0$ . Defining

$$(4.3) \quad V(t, x) = x^2,$$

it is evident that  $V(t, x)$  fulfils the conditions (3.1), (3.5) and (3.6) of Theorem 3.1 with the set  $A = \{0\}$ .

From (4.1), (4.2) and (4.3) we have

$$(4.4) \quad LV \leq -2x a(t) (r(x) - \delta(t)) + K^2 a^2(t) (1 + V).$$

For  $x > (2/K_1) \delta(t)$ , we have by using **R1**  $r(x) > 2 \delta(t)$ . Thus from (4.4) we get

$$(4.5) \quad LV \leq -2 a(t) \delta(t) x + K^2 a^2(t) (1 + V).$$

For  $0 \leq x \leq (2/K_1) \delta(t)$  the inequality (4.4) can be written as

$$(4.6) \quad LV \leq -2 a(t) x r(x) + \frac{4}{K_1} a(t) \delta^2(t) + K^2 a^2(t) (1 + V).$$

For  $x < 0$  the inequality (4.4) can be written as

$$LV \leq -2 a(t) x r(x) + 2 a(t) \delta(t) x + K^2 a^2(t) (1 + V).$$

Using (2.1) we get

$$(4.7) \quad LV \leq -2 a(t) \delta(t) |x| + K^2 a^2(t) (1 + V).$$

Defining

$$\varphi(t, x) = \begin{cases} x r(x) & \text{for } 0 \leq x < \frac{2}{K_1} \delta(t); \\ |x| & \text{otherwise;} \end{cases}$$

$$\alpha(t) = 2 a(t) \delta(t),$$

$$g(t) = \frac{4}{K_1} a(t) \delta^2(t) + K^2 a^2(t),$$

the inequalities (4.5), (4.6) and (4.7) can be written as

$$(4.8) \quad LV \leq -\alpha(t) \varphi(t, x) + g(t) (1 + V), \quad \text{for } t \geq T_1.$$

From (2.1) and **A1** it is easy to see that  $\varphi(t, x)$  satisfies (3.4) and from **A1** it is evident that  $g(t)$  satisfies (3.3), thus by (4.8) the condition (3.2) is also satisfied completing the proof of the theorem.

**Remark 4.2.** The condition **R1** can be somehow weakened to extend the class of the regression functions for which Theorem 4.1 is still valid. Let **A1** and **R1** in Theorem 4.1 be replaced by **A1'** and **R1'**.

$$\mathbf{A1'}: \int_{t_0}^{\infty} a(t) \delta(t) dt = \infty, \int_{t_0}^{\infty} a^2(t) dt < \infty, \lim_{t \rightarrow \infty} \delta(t) = 0 \text{ and } \lim_{t \rightarrow \infty} a(t) = 0.$$

Define  $\tau(t) > 0, \tau(t) \rightarrow 0$  for  $t \rightarrow \infty$  such that  $\inf_{x > \tau(t)} r(x) > 2 \delta(t)$  (this is possible by virtue of **A1'**).

$$\mathbf{R1'}: \int_{t_0}^{\infty} a(t) \delta(t) \tau(t) dt < \infty.$$

Still we can conclude that Theorem 4.1 is valid.

The proof can be carried out in steps as in the proof of Theorem 4.1.

In fact for  $x > \tau(t)$  the inequality (4.4) can be written as (4.5). For  $0 \leq x \leq \tau(t)$  we have

$$(4.6') \quad LV \leq -2 a(t) x r(x) + 2 a(t) \delta(t) \tau(t) + K^2 a^2(t) (1 + V).$$

For  $x < 0$  as in Theorem 4.1, we have (4.7).

Defining

$$\begin{aligned} \varphi'(t, x) &= \begin{cases} x r(x), & 0 \leq x \leq \tau(t), \\ |x| & \text{otherwise;} \end{cases} \\ \alpha(t) &= 2 a(t) \delta(t); \\ g'(t) &= 2 a(t) \delta(t) \tau(t) + K^2 a^2(t) \end{aligned}$$

then (4.5), (4.6'), (4.7) can be written as

$$(4.8') \quad LV \leq -\alpha(t) \varphi'(t, x) + g'(t) \text{ for } t \geq T_1$$

and the proof can be completed so as in Theorem 4.1.

To show that Remark 4.1 extends the class of the regression functions, the following example is used.

**Example 4.1.** Put

$$\begin{aligned} r(x) &= 0, & -\infty < x \leq 0; \\ &= x^2, & 0 < x \leq 1; \\ &= 1, & x > 1. \end{aligned}$$

456 Take  $a(t) = 1/t^{2/3}$  and  $\delta(t) = 1/t^{1/3}$ ; then from the definition of  $\tau(t)$ , we deduce that

$$\tau(t) = o\left(\frac{1}{t^{1/6}}\right).$$

It is evident that  $r(x)$  does not satisfy **R1** while **A1'** and **R1'** are satisfied.

**Theorem 4.2.** If **R1**, **R2**, **A1** and **C1** hold, then

$$\lim_{t \rightarrow \infty} \mathbf{E}[X^r(t) - \theta]^2 = 0.$$

**Proof.** As before we can take  $\theta = 0$ . Let  $V(t, x)$  be chosen as in proving Theorem 4.1. Then (4.4) can be written as

$$(4.9) \quad LV \leq -2 a(t) x r(x) + 2 a(t) \delta(t) x + K^2 a^2(t) (1 + x^2).$$

For  $x > 0$  and by using the inequality  $|x| \leq 1 + x^2$ , we can write (4.9) as

$$LV \leq -2 a(t) x r(x) + 2 a(t) \delta(t) + 2 a(t) \delta(t) x^2 + K^2 a^2(t) (1 + x^2).$$

Using **R1** we get

$$\begin{aligned} LV &\leq -2K_1 a(t) x^2 + 2 a(t) \delta(t) + K^2 a^2(t) + x^2(2 a(t) \delta(t) + K^2 a^2(t)) = \\ &= -K_1 a(t) x^2 \left(2 - \frac{2}{K_1} \delta(t) - \frac{K^2}{K_1} a(t)\right) + 2 a(t) \delta(t) + K^2 a^2(t). \end{aligned}$$

From **A1** it follows that there exists  $T_1$  such that

$$(4.10) \quad \begin{aligned} LV &\leq -K_1 a(t) x^2 + 2 a(t) \delta(t) + K^2 a^2(t) = \\ &= -K_1 a(t) V + 2 a(t) \delta(t) + K^2 a^2(t) \end{aligned}$$

for  $t \geq T_1$ .

For  $-q \leq x < 0$  the inequality (4.9), by using **R2**, can be written as

$$\begin{aligned} LV &\leq -2K_2 a(t) x^2 + 2 a(t) \delta(t) x + K^2 a^2(t) (1 + x^2) \leq \\ &\leq -2K_2 a(t) x^2 + K^2 a^2(t) (1 + x^2); \end{aligned}$$

then as before there exists  $T_2$  such that

$$(4.11) \quad LV \leq -K_2 a(t) V + K^2 a^2(t), \quad \text{for } t > T_2.$$

For  $-h \leq x \leq -q$  the inequality (4.9) can be written as

$$(4.12) \quad \begin{aligned} LV &\leq 2 a(t) \delta(t) x + K^2 a^2(t) (1 + x^2) \leq K^2 a^2(t) (1 + x^2) \leq \\ &\leq K^2 a^2(t) (1 + h^2). \end{aligned}$$



Finally for  $x < -h$  the inequality (4.9) can be written as

$$LV \leq -2 a(t) x r(x) + K^2 a^2(t) (1 + x^2).$$

Using R2 we get

$$LV \leq -2K_3 a(t) x^2 + K^2 a^2(t) (1 + x^2),$$

and as before

$$(4.13) \quad LV \leq -K_3 a(t) V + K^2 a^2(t) \quad \text{for } t > T_3.$$

Defining

$$(4.14) \quad \begin{aligned} \beta(t) &= 2 a(t) \delta(t) + K^2 a^2(t), \\ T &= \max(T_1, T_2, T_3), \\ K_5 &= \min(K_1, K_2, K_3), \\ a'(t) &= K_5 a(t), \\ W(t, x) &= V(t, x) \exp\left(\int_T^t a'(u) du\right), \end{aligned}$$

then (4.10), (4.11), (4.12) and (4.13) can be written as

$$(4.15) \quad LV \leq \begin{cases} K^2 a^2(t) (1 + h^2), & -h \leq x < -\varrho; \\ -a'(t) V + \beta(t) & \text{otherwise.} \end{cases}$$

From (4.2) and (4.14) we get

$$(4.16) \quad LW = \exp\left(\int_T^t a'(u) du\right) (LV + a'(t) V).$$

From Lemma 3.5.1 in [4], Fatou lemma and the regularity of the procedure we get

$$(4.17) \quad \mathbb{E}(W(t, X(t)) - W(s, X(s))) \leq \mathbb{E} \int_s^t L W(u, X(u)) du.$$

Let us denote

$$A = [-\varrho, \infty) \cup (-\infty, -h)$$

and

$$p(t) = \mathbb{P}(X(t) \in A).$$

Using (4.15) and (4.16), the inequality (4.17) can be written as

$$\mathbb{E}W(t, X^x(t)) - \mathbb{E}W(s, X^x(s)) \leq \mathbb{E} \left[ I_A \int_s^t \beta(u) \exp\left(\int_T^u a'(v) dv\right) du + \right.$$

$$\begin{aligned}
& + I_{A^c} \int_s^t (K^2 a^2(u)(1+h^2) + a'(u)h^2) \exp\left(\int_T^u a'(v) dv\right) du \Big] = \\
& = p(t) \int_s^t \beta(u) \exp\left(\int_T^u a'(v) dv\right) du + q(t) \int_s^t (K^2 a^2(u)(1+h^2) + \\
& \quad + a'(u)h^2) \exp\left(\int_T^u a'(v) dv\right) du, \quad \text{for } t > s \geq T \geq t_0.
\end{aligned}$$

Thus

(4.18)

$$\begin{aligned}
EV(X(t)) & \leq EW(s, X^*(s)) \exp\left(-\int_T^t a'(v) dv\right) + p(t) \int_s^t \beta(u) \exp\left(-\int_u^t a'(v) dv\right) du + \\
& \quad + K^2 q(t) \int_s^t a^2(u)(1+h^2) \exp\left(-\int_u^t a'(v) dv\right) du + \\
& \quad + q(t) h^2 \int_s^t a'(u) \exp\left(-\int_u^t a'(v) dv\right) du.
\end{aligned}$$

Consider the right-hand side of (4.18). The first term tends to zero for  $t \rightarrow \infty$  since  $EW(s, X^*(s))$  is bounded. The second and the third terms tend to zero for  $t \rightarrow \infty$  by Problem 4.4.1 in [4].

Let us denote the last term by  $g(t)$ , i.e.

$$g(t) = q(t) h^2 \int_s^t a'(u) \exp\left(-\int_u^t a'(v) dv\right) du.$$

By integration we get

$$g(t) = q(t) h^2 \left(1 - \exp\left(-\int_s^t a'(v) dv\right)\right).$$

Using Theorem 4.1,  $q(t) \rightarrow 0$  for  $t \rightarrow \infty$ , which implies with **A1** that  $g(t) \rightarrow 0$ , for  $t \rightarrow \infty$ . Thus

$$\lim_{t \rightarrow \infty} EV(X^*(t)) = \lim_{t \rightarrow \infty} E(X^*(t))^2 = 0,$$

completing the proof of the theorem.

## 5. THE ASYMPTOTIC NORMALITY OF THE PROCEDURE

To establish the asymptotic normality of the procedure, we give the following lemma. Its proof can be carried out as that of Lemma 6.2.1 in [4].

**Lemma 5.1.** If **R3**, **A2**, **A3** and **C1** hold, then

$$E|X^{s,\xi}(t) - \theta|^2 = O(t^{-\alpha}), \text{ for } t \rightarrow \infty.$$

Here  $X^{s,\xi}(t)$  is the solution of the stochastic differential equation (4.1) and  $E|\xi|^2 < \infty$ .

**Proof.** As before, we can take  $\theta = 0$ . Take  $V(t, x)$  as in proving Theorem 4.1. By using **R3**, **A2**, **A3** and **C1** in (4.4) we get

$$(5.1) \quad LV \leq -2K_4 \frac{a}{t^\alpha} V + \frac{2a\delta}{t^{\alpha+\gamma}} x + \frac{K^2 a^2}{t^{2\alpha}} (1 + V).$$

Using the inequality

$$(5.2) \quad |x| \leq \eta^{-1} t^{-\gamma} + \eta t^\gamma |x|^2, \quad \eta > 0,$$

we get

$$(5.3) \quad LV \leq -2K_4 a t^{-\alpha} V + K^2 a^2 t^{-2\alpha} V + 2a\delta \eta t^{-\alpha} V + K^2 a^2 t^{-2\alpha} + 2a\delta \eta^{-1} t^{-\alpha-2\gamma}.$$

By choosing  $\eta$  sufficiently small, since **A3** implies that  $-\alpha - 2\gamma < -2\alpha$ , we get

$$(5.4) \quad LV \leq -K_4 a t^{-\alpha} V + K^2 a^2 t^{-2\alpha}, \text{ for } t > T.$$

From Theorem 3.1, and (5.4), we can easily deduce that

$$\frac{d}{dt} EV(X^{s,\xi}(t)) = ELV(X^{s,\xi}(t)) \leq -K_4 a t^{-\alpha} EV(X^{s,\xi}(t)) + K^2 a^2 t^{-2\alpha};$$

its solution can be written as

$$EV(X^{s,\xi}(t)) \exp\left(\frac{aK_4}{1-\alpha} t^{1-\alpha}\right) \leq \int_T^t K^2 a^2 u^{-2\alpha} \exp\left(\frac{aK_4}{1-\alpha} u^{1-\alpha}\right) du + EV(X^{s,\xi}(T)) \exp\left(\frac{aK_4}{1-\alpha} T^{1-\alpha}\right).$$

Then

$$EV(X^{s,\xi}(t)) \leq m_1 \exp(-m_2 t^{1-\alpha}) \int_t^T u^{-2\alpha} \exp(m_2 u^{1-\alpha}) du + m_3 \exp(-m_2 t^{1-\alpha}).$$

(Here, as well as in the sequel,  $m$  with subscript will denote positive constants, possibly of different values in different formulas.)

$$\begin{aligned} EV(X^{s,\xi}(t)) &\leq m_1 \exp(-m_2 t^{1-\alpha}) \left[ t^{-\alpha} \exp(m_2 t^{1-\alpha}) - T^{-\alpha} \exp(m_2 T^{1-\alpha}) + \right. \\ &\quad \left. + m_2 \int_T^t u^{-\alpha-1} \exp(m_2 u^{1-\alpha}) du \right] + m_3 \exp(-m_2 t^{1-\alpha}) = \\ &= m_1 t^{-\alpha} + m_1 \exp(-m_2 t^{1-\alpha}) + m_1 I(t), \end{aligned}$$

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$$I(t) = \int_T^t u^{-\alpha-1} \exp(-m_2(t^{1-\alpha} - u^{1-\alpha})) du.$$

It is sufficient to prove that  $I(t) = O(t^{-\alpha})$ , for  $t \rightarrow \infty$ . Substituting  $z = t^{1-\alpha} - u^{1-\alpha}$ , we obtain

$$t^\alpha I(t) = \frac{1}{1-\alpha} \int_0^{t^{1-\alpha}-T^{1-\alpha}} \frac{t^{\alpha-1}}{\left(1 - \frac{z}{t^{1-\alpha}}\right)^{1/(1-\alpha)}} \exp(-m_2 z) dz.$$

Then  $\lim_{t \rightarrow \infty} t^\alpha I(t) = 0$ , which completes the proof of the lemma.

**Theorem 5.1.** If **R1**, **R4**, **A2**, **A3**, **C1** and **C2** hold, then the asymptotic distribution of  $t^{\alpha/2}(X^\alpha(t) - \theta)$  is normal with

$$(5.5) \quad \text{mean} = 0$$

and

$$(5.6) \quad \text{variance} = \frac{a\sigma_0^2}{2B}.$$

*Proof.* From **R4**, it follows that there exists  $\eta > 0$  such that

$$(5.7) \quad |f(x, \theta)| \leq \frac{B}{2} |x - \theta| \quad \text{for } |x - \theta| < \eta.$$

From **C1** it follows that

$$(5.8) \quad |\sigma(t, x)| \leq K' \quad \text{for } |x - \theta| < \eta.$$

Let us define

$$(5.9) \quad \begin{aligned} f^*(x, \theta) &= f(x, \theta) && \text{for } |x - \theta| \leq \eta, \\ &= f\left(\eta + \theta \frac{(x - \theta)}{|x - \theta|}, \theta\right) \frac{|x|}{\eta} && \text{for } |x - \theta| > \eta \end{aligned}$$

and

$$(5.10) \quad \begin{aligned} \sigma^*(t, x) &= \sigma(t, x) && \text{for } |x - \theta| \leq \eta, \\ &= \sigma\left(t, \left(\eta + \theta \frac{(x - \theta)}{|x - \theta|}\right)\right) && \text{for } |x - \theta| > \eta. \end{aligned}$$

Without loss of generality, we can take  $\theta = 0$ . Let us consider the following auxiliary stochastic differential equation

$$(5.11) \quad dX^*(t) = -at^{-\alpha}[(BX^*(t) + f^*(X^*(t))) - \delta t^{-\gamma}] dt + \sigma^*(t, X^*(t)) d\zeta(t)]$$

with  $X^*(s) = \xi$  where  $\xi$  is  $\mathfrak{F}_s$ -measurable and  $E[\xi]^2 < \infty$ .

From (5.9) it follows that **R3** holds; then from Lemma 5.1

$$(5.12) \quad \mathbb{E}|X^{**,\xi}(t)|^2 = O(t^{-\alpha}) \quad \text{for } t \rightarrow \infty.$$

Denoting  $Y^*(t) = t^{\alpha/2} X^{**,\xi}(t)$ , from (5.11) we get

$$(5.13) \quad dY^*(t) = \left[ \left( \frac{\alpha}{2} t^{-1} - aBt^{-\alpha} \right) Y^*(t) - at^{-\alpha/2} f^*(X(t)) + a\delta t^{-\alpha/2-\gamma} \right] dt - at^{-\alpha/2} \sigma^*(t, X^*(t)) d\zeta(t).$$

Then

$$(5.14) \quad \begin{aligned} Y^*(t) = & a\delta \int_s^t \left( \frac{t}{u} \right)^{\alpha/2} \exp \left( \frac{-aB}{1-\alpha} (t^{1-\alpha} - u^{1-\alpha}) \right) u^{-\alpha/2-\gamma} du - \\ & - a \int_s^t \left( \frac{t}{u} \right)^{\alpha/2} \exp \left( \frac{-aB}{1-\alpha} (t^{1-\alpha} - u^{1-\alpha}) \right) u^{-\alpha/2} f^*(X^*(u)) du - \\ & - a \int_s^t \left( \frac{t}{u} \right)^{\alpha/2} \exp \left( \frac{-aB}{1-\alpha} (t^{1-\alpha} - u^{1-\alpha}) \right) u^{-\alpha/2} \sigma^*(u, X(u)) d\zeta + \\ & + t^{\alpha/2} \exp \left( \frac{-aB}{1-\alpha} (t^{1-\alpha} - s^{1-\alpha}) \right) \zeta. \end{aligned}$$

Let us consider the right-hand side of (5.14). Denote the 2-nd integral by  $I_2(t)$ . From (5.9) it follows that given  $\eta > 0$ , there exists  $\varrho_1 > 0$ , such that

$$|f^*(x)| \leq \eta^2 |x| \quad \text{for } |x| < \varrho_1.$$

From Theorem 4.1 it follows

$$\mathbb{P} \left[ \sup_{u \geq T} |X^\xi(t)| < \varrho_1 \right] > 1 - \eta;$$

by this we can write

$$(5.15) \quad \begin{aligned} \mathbb{P}[|I_2(t)| > \eta] & \leq \eta + \mathbb{P} \left[ \left| \int_T^t \left( \frac{t}{u} \right)^{\alpha/2} \exp \left( \frac{-aB}{1-\alpha} (t^{1-\alpha} - u^{1-\alpha}) \right) f^* u^{-\alpha/2} du \right| > \eta \right] \\ & \leq \eta + \mathbb{P} \left[ \sup_{|X^\xi(t)| > \varrho_1} \left| \int_T^t \left( \frac{t}{u} \right)^{\alpha/2} \exp \left( \frac{-aB}{1-\alpha} (t^{1-\alpha} - u^{1-\alpha}) \right) |f^*| u^{-\alpha/2} du > \eta \right] \leq \\ & \leq \eta + \eta \left[ \int_T^t \left( \frac{t}{u} \right)^{\alpha/2} \exp \left( \frac{-aB}{1-\alpha} (t^{1-\alpha} - u^{1-\alpha}) \right) \mathbb{E}|X^*(u)| u^{-\alpha/2} du \right]. \end{aligned}$$

462 By using the inequality  $|X(t)| \leq t^{-\alpha/2} + t^{\alpha/2}|X(t)|^2$  and (5.12), the right-hand side of (5.15) is less than or equal to

$$\eta + 2\eta \int_T^t \left(\frac{t}{u}\right)^{\alpha/2} u^{-\alpha} \exp\left(\frac{-aB}{1-\alpha}(t^{1-\alpha} - u^{1-\alpha})\right) du.$$

Denoting

$$G(t) = \eta \int_T^t \left(\frac{t}{u}\right)^{\alpha/2} u^{-\alpha} \exp\left(\frac{-aB}{1-\alpha}(t^{1-\alpha} - u^{1-\alpha})\right) du$$

and using the substitution (5.3)  $z = t^{1-\alpha} - u^{1-\alpha}$ , we have

$$(5.16) \quad G(t) = m' \eta \int_0^{t^{1-\alpha} - T^{1-\alpha}} \frac{1}{\left(1 - \frac{z}{t^{1-\alpha}}\right)^{\alpha/2(1-\alpha)}} \exp\left(\frac{-aB}{1-\alpha} z\right) dz.$$

Thus

$$\lim_{t \rightarrow \infty} G(t) = \eta m' \int_0^{\infty} \exp\left(\frac{-aB}{1-\alpha} z\right) dz = \eta m'',$$

and then

$$\lim_{t \rightarrow \infty} P[|I_2(t)| > \eta] = 0.$$

Considering the 3-rd integral and defining

$$\begin{aligned} \eta_1(t) &= -a \int_s^t \left(\frac{t}{u}\right)^{\alpha/2} u^{-\alpha/2} \exp\left(\frac{-aB}{1-\alpha}(t^{1-\alpha} - u^{1-\alpha})\right) (\sigma^* - \sigma_0) d\zeta(u), \\ \eta_2(t) &= -a \int_s^t \left(\frac{t}{u}\right)^{\alpha/2} u^{-\alpha/2} \exp\left(\frac{-aB}{1-\alpha}(t^{1-\alpha} - u^{1-\alpha})\right) \sigma_0 d\zeta(u) \end{aligned}$$

then

$$E(\eta_1(t))^2 = a^2 \int_s^t \left(\frac{t}{u}\right)^{\alpha} u^{-\alpha} E|\sigma^* - \sigma_0|^2 \exp\left(\frac{-2aB}{1-\alpha}(t^{1-\alpha} - u^{1-\alpha})\right) du.$$

From Theorem 4.1, (5.8), (5.10) and C2

$$\lim_{t \rightarrow \infty} E|\sigma^*(t, X^*(t)) - \sigma_0|^2 = 0.$$

Thus given  $\eta > 0$ , there exists  $T(\eta)$ , such that

$$(5.17) \quad E(\eta_1(t))^2 \leq a^2 \int_s^{T(\eta)} \left(\frac{t}{u}\right)^{\alpha} u^{-\alpha} E|\sigma^* - \sigma_0|^2 \exp\left(\frac{-2aB}{1-\alpha}(t^{1-\alpha} - u^{1-\alpha})\right) du + \\ + a^2 \eta \int_{T(\eta)}^t \left(\frac{t}{u}\right)^{\alpha} u^{-\alpha} \exp\left(\frac{-2aB}{1-\alpha}(t^{1-\alpha} - u^{1-\alpha})\right) du.$$

Using (5.8), the first integral in the right hand side of (5.17) tends to zero for  $t \rightarrow \infty$ , while the 2-nd integral tends to 0 for  $t \rightarrow \infty$ . 463

Then  $E(\eta_1(t))^2 \rightarrow 0$ , for  $t \rightarrow \infty$  which implies that

$\eta_1(t) \rightarrow 0$  in probability;

$\eta_2(t)$  is normal random variable with mean zero, and with

$$\text{variance} = \lim_{t \rightarrow \infty} a^2 \sigma_0^2 \int_s^t \left(\frac{t}{u}\right)^\alpha u^{-\alpha} \exp\left(\frac{-2aB}{1-\alpha}(t^{1-\alpha} - u^{1-\alpha})\right) du.$$

Using the substitution (5.3)  $z = t^{1-\alpha} - u^{1-\alpha}$ , the

$$\begin{aligned} \text{variance} &= -\frac{a^2 \sigma_0^2}{2B'} \int_0^\infty \exp\left(-\frac{2aB}{1-\alpha} z\right) dz, \quad B' = \frac{\alpha-1}{2}. \\ &= a\sigma_0^2/2B. \end{aligned}$$

It is evident that the 4-th term tends to zero for  $t \rightarrow \infty$ . Then  $Y^*(t)$  is normal with variance  $= a\sigma_0^2/2B$ , and with

$$\text{mean} = \lim_{t \rightarrow \infty} a\delta \int_s^t \left(\frac{t}{u}\right)^{\alpha/2} \exp\left(\frac{-aB}{1-\alpha}(t^{1-\alpha} - u^{1-\alpha})\right) u^{-\alpha/2-\gamma} du.$$

From A3 it follows that  $-\alpha/2 - \gamma < -\alpha$ ; then  $u^{-\alpha/2-\gamma} = \varrho(u) u^{-\alpha}$ ,  $\varrho(u) \rightarrow 0$  as  $u \rightarrow \infty$ , thus

$$\text{mean} = \lim_{t \rightarrow \infty} a\delta \int_s^t \left(\frac{t}{u}\right)^{\alpha/2} u^{-\alpha} \exp\left(\frac{-aB}{1-\alpha}(t^{1-\alpha} - u^{1-\alpha})\right) \varrho(u) du.$$

This integral tends to 0 for  $t \rightarrow \infty$  as (5.16). Thus  $t^{2/2} X^*(t)$  is normal with

$$\begin{aligned} \text{mean} &= 0, \\ \text{variance} &= \frac{a\sigma_0^2}{2B}. \end{aligned}$$

The proof can be completed as in Theorem 6.5.1 in [4].

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- [1] V. Dupač: The continuous dynamic Robbins-Monro procedure. *Kybernetika* 12 (1976), 6, 414—420.
- [2] S. Friedman: On stochastic approximations. *The Annals of Mathematical Statistics* 34 (1963), 343—346.
- [3] L. Guttman, R. Guttman: An illustration of the use of stochastic approximation. *Biometrics* 15 (1959), 551—559.
- [4] М. Б. Невельсон, Р. З. Хасьминский: Стохастическая аппроксимация и рекуррентное оценивание. Наука, Москва 1972.
- [5] H. Robbins, S. Monro: A stochastic approximation method. *The Annals of Mathematical Statistics* 22 (1951), 400—407.
- [6] E. Sorour: Stochastic approximation methods. Candidate Dissertation Work. Institute of Information Theory and Automation of the Czechoslovak Academy of Sciences. Prague 1977.

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