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BIFURCATIONS AND CHAOS IN A PERIODICALLY FORCED PROTOTYPE ADAPTIVE CONTROL SYSTEM[1]

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An adaptive control system with a first-order plant and the so-called σ-modification adaptation law is analyzed in the case of periodic disturbance or reference input. The local bifurcations of the low-period solutions are numerically detected by means of a continuation method, and the different modes of behavior are classified as well as the transitions among them. As predicted by the theory, the control system is robust in the sense that all trajectories are bounded regardless to the action of the disturbance. However, the periodicity of the input can give rise to chaotic behavior. The result of the analysis will aid the designer in selecting the controller parameters in order to achieve an acceptable behavior.

1. INTRODUCTION

It has been pointed out in the last few years that even simple continuous-time autonomous adaptive control systems can have a rather complex behavior, displaying several types of bifurcations with respect to the design or to the plant parameters (Cyr et al. [1]; Rubio et al. [2]; Salam and Bai [3,4]; Bai and Salam [5]; Mareels and Bitmead [6]). In this work, a model reference adaptive control system with a first-order plant and an adaptation law generally referred to as *σ-modification* (Ioannou and Kokotovic [7]; Riedle et al. [8]) is considered, and attention is paid to a situation which is of definite interest in control systems analysis, namely the superposition of a sinusoidal component over the constant disturbance or the reference input.

As is known, the σ-modification scheme guarantees robustness, in the sense that the output error and the adapting parameter remain bounded for all initial conditions and parameter values, regardless to the action of the (bounded) disturbance. In the autonomous case (i.e. with constant disturbance and reference input) Salam and Bai [3, 4] showed that the system experiences several bifurcations (saddle-node, Hopf, and homoclinic) with respect to the disturbance, and presented a detailed picture of the possible system behaviors.

In the periodically forced case considered in this work, the aim of the analysis is to classify the modes of behavior of the system with respect to the system parameters, and to evidence the mechanisms of transition among them. As is known,

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the study of the bifurcations of the periodic solutions is equivalent to the study of the bifurcations of the fixed-points of an autonomous discrete-time system (the *Poincaré map*) associated to the original periodic continuous-time system (Arnold [9]; Guckenheimer and Holmes [10]). This discrete-time system is not available in analytical form, so that a numerical analysis must be carried out. In particular, the bifurcations of the fixed-points of the Poincaré map will be numerically analyzed in this work by means of a *continuation method* (Seydel [11]; Khibnik et al. [12]). The final result, as already pointed out, is a classification of the modes of behavior of the system and of the transitions among them, that will aid the designer in selecting the controller parameters in order to achieve an acceptable system behavior.

2. SYSTEM DESCRIPTION

The object of the control is the first-order plant

$$\dot{y}_p = \rho y_p + u + d$$

where $u = u(t)$ and $d = d(t)$ are, respectively, the control input and the disturbance. It is assumed that d is uniformly bounded, i.e. there exists a positive constant D such that $|d(t)| \leq D \; \forall t$. The plant output y_p is required to track the output of the reference model

$$\dot{y}_m = -y_m + r$$

where $r = r(t)$ is the reference input. With this goal, the control input u is selected according to the feedback law

$$u = -ky_p + r.$$

In the ideal case, i.e. when ρ is known and no disturbance affects the system ($d(t) = 0 \; \forall t$), the nominal feedback gain $k^* = \rho + 1$ is such that the transfer function of the plant matches that of the model. In the real case, k is varied in order to counteract uncertainty, according to the following adaptation law (Ioannou and Kokotovic [7]; Riedle et al. [8]):

$$\dot{k} = -\sigma k + \alpha y_p (y_p - y_m) \qquad \sigma, \alpha > 0.$$

By introducing the output error $e = y_p - y_m$ and the displacement $\phi = k - k^* = k - (\rho + 1)$ of the feedback gain k from its nominal value, the above system is reduced to

$$\dot{e} = -e - \phi(e + y_m) + d \tag{1a}$$

$$\dot{\phi} = -\sigma\phi + \alpha e(e + y_m) - \sigma(\rho + 1) \tag{1b}$$

$$\dot{y}_m = -y_m + r. \tag{1c}$$

Consider the function

$$V(e, \phi) = \frac{1}{2}\alpha e^2 + \frac{1}{2}\phi^2 \tag{2}$$

whose time derivative along the system trajectories is

$$\dot{V} = \frac{\partial V}{\partial e}\dot{e} + \frac{\partial V}{\partial \phi}\dot{\phi} = -\alpha e^2 - \sigma\phi^2 + \alpha e d - \sigma(\rho + 1)\,\phi \tag{3}$$

Since d is bounded, $\dot{V}$ is negative for $||(e,\phi)||$ sufficiently large, so that there exists a positive constant M such that all the system trajectories are attracted into the region delimited by the closed line $V(e,\phi) = M$ (an approximation of the lowest M having the above property can easily be derived from (2), (3)). Therefore system (1) has the property that all its solutions are uniformly bounded for all possible disturbances, reference signal, and system parameters (Ioannou and Kokotovic [7]).

We are interested in analyzing the behavior of system (1) in the following two cases:

a) *periodic disturbance:* we let

$$d(t) = d_0 + \delta\cos(\omega t)$$

and $r(t) = 0\ \forall t$. After transient we can assume $y_m(t) = 0$, so that system (1) reduces to

$$\dot{e} = -e - \phi e + d_0 + \delta\cos(\omega t) \tag{4a}$$

$$\dot{\phi} = -\sigma\phi + \alpha e^2 - \sigma(\rho + 1) \tag{4b}$$

b) *periodic reference:* we let

$$r(t) = r_0 + \delta\cos(\omega t)$$

and $d(t) = 0\ \forall t$. After transient we can assume

$$y_m(t) = r_0 + \delta'\cos(\omega t + \psi)$$

where $\delta' = \delta/(1+\omega^2)^{1/2}$ and $\psi = -\text{arctg}(\omega)$. By introducing the time shifting $\tau = t + \psi/\omega$, system (1) reduces to

$$\dot{e} = -e - \phi(e + r_0 + \delta'\cos(\omega\tau)) \tag{5a}$$

$$\dot{\phi} = -\sigma\phi + \alpha e(e + r_0 + \delta'\cos(\omega\tau)) - \sigma(\rho + 1). \tag{5b}$$

The numerical analysis of systems (4) and (5) revealed that they possess qualitatively the same bifurcation picture. Therefore, the following presentation and discussion will be limited to the case of periodic disturbance.

3. BIFURCATION ANALYSIS

The bifurcations of the periodic solutions of system (4) have been analyzed with respect to δ and d_0, namely the two parameters characterizing the disturbance. The other parameters have been fixed at the values suggested by Salam and Bai [3], namely $\rho = 1.5$, $\sigma = 0.5$, and $\alpha = 1$. The frequency of the periodic disturbance has been fixed at $\omega = 1.5$.

We briefly recall the bifurcations taking place in system (4) in the autonomous case ($\delta = 0$) with respect to the (constant) disturbance d_0 (see Salam and Bai [3] for a detailed description). At $d_0 = 0$ system (4) has three equilibria, two stable foci ($\mathcal{E}_1 : (e,\phi) = (-0.866,-1)$; $\mathcal{E}_2 : (e,\phi) = (0.866,-1)$) and a saddle ($\mathcal{E}_3 : (e,\phi) = (0,-2.5)$). At $d_0 = d_0' \cong 0.26336$ the stable and unstable manifolds of the saddle $\mathcal{E}_3$ form a homoclinic orbit around the focus $\mathcal{E}_1$. By increasing d_0, the homoclinic orbit disappears giving rise to an unstable limit cycle. This cycle has decreasing amplitude as d_0 increases, and disappears at $d_0 = d_0'' \cong 0.35355$ through a subcritical Hopf bifurcation by colliding with $\mathcal{E}_1$, giving rise to an unstable focus. Finally, at $d_0 \cong 0.49905$ the unstable focus becomes an unstable node which disappears at $d_0 = 0.5$ through a saddle- node bifurcation by colliding with $\mathcal{E}_3$, so that for $d_0 > 0.5$ the stable focus $\mathcal{E}_2$ is the only existing equilibrium. The same sequence of bifurcations takes place when d_0 is decreased from 0, with the role of the two foci interchanged.

In the periodically forced case ($\delta > 0$) system (4) defines a Poincaré map $P : \mathbf{R}^2 \to \mathbf{R}^2$, namely a discrete-time autonomous system of the form

$$(e(T), \phi(T)) = P(e(0), \phi(0))$$

where $T = 2\pi/\omega$. The fixed points of the m-th iterate $P^{(m)}$ ($m = 1, 2, \ldots$) of the Poincaré map clearly correspond to period-mT solutions of system (4). The eigenvalues of the Jacobian matrix of $P^{(m)}$ evaluated at a fixed point are called *multipliers* of the period-mT solution. A fixed point is *structurally stable* if and only if it has no multipliers with module equal to 1. Having fixed all the parameters except δ and d_0, the values (δ, d_0) at which a fixed point is structurally unstable define a *bifurcation curve* in the (δ, d_0)-plane. We will use the following notation to denote codimension-one bifurcation curves:[1]

t^m-*saddle-node (tangent) bifurcation curve* (on this curve $P^{(m)}$ has a fixed point with a multiplier equal to 1).

f^m-*flip (period-doubling) bifurcation curve* (on this curve $P^{(m)}$ has a fixed point with a multiplier equal to -1).

h^m-*Naimark-Sacker (secondary Hopf) bifurcation curve* (on this curve $P^{(m)}$ has a fixed point with a pair of complex conjugate multipliers with module equal to 1).

The bifurcations taking place across these curves are very well known and described, for example, in Arnold [9] or in Guckenheimer and Holmes [10]. At certain points on these curves some non-degeneracy condition for the corresponding codimension-one bifurcation can be violated, thus giving rise to a codimension-two bifurcation. We will use the following notation to denote codimension-two points:[2]

A^m-*strong resonance 1:2* (at this point $P^{(m)}$ has a fixed point with a double multiplier equal to -1).

B^m-*strong resonance 1:1* (at this point $P^{(m)}$ has a fixed point with a double multiplier equal to 1).

C^m-*cusp* (at this point $P^{(m)}$ has a fixed point with a multiplier equal to 1 and an extra nonlinear degeneration [9, 10]).

D^m-*degenerate flip* (at this point $P^{(m)}$ has a fixed point with a multiplier equal to -1 and an extra nonlinear degeneration [9, 10]).

[1] Curves of the same class will be distinguished by a subscript number.
[2] Points of the same class will be distinguished by a subscript number.

The parametric portrait of system (4) for the above specified values of ρ, σ, α, and ω is presented in Figure 1. It has been numerically obtained by means of a continuation method implemented on PC by the interactive program LOCBIF (see Khibnik et al. [12] for a detailed description).

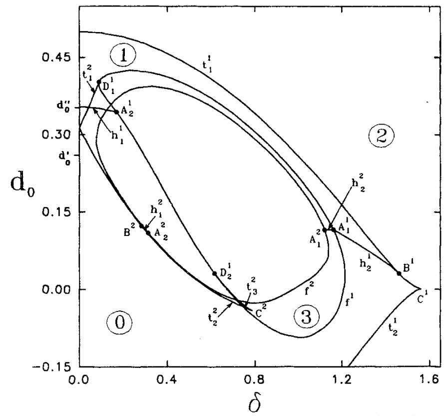

Fig. 1. Parametric portrait of system (4) in the (δ, d_0)-plane.

In region 0 system (4) has three period$-T$ solutions: two of them are stable and correspond to the perturbation of the foci $\mathcal{E}_1$ and $\mathcal{E}_2$ of the autonomous system (we denote these solutions by $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively), and one is unstable (denoted by $\mathcal{C}_3$) corresponding to the saddle $\mathcal{E}_3$. (We only describe the bifurcations related to $\mathcal{C}_1$ and $\mathcal{C}_3$, since $\mathcal{C}_2$ experiences the same bifurcations as $\mathcal{C}_1$ once the sign of d_0 is reversed). The curves t_1^1 and t_2^1 reveal a typical (symmetric) fold structure for these period$-T$ cycles. More in detail, on segment $B^1 C^1$ of curve t_1^1, the cycles $\mathcal{C}_1$ and $\mathcal{C}_3$ collide and disappear, while $\mathcal{C}_1$ becomes unstable when crossing h_1^1 or h_2^1 toward region 1, and then collides with $\mathcal{C}_3$ and disappears on t_1^1. At the cusp point C^1 the three cycles $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ coincide, and outside the curves t_1^1 and t_2^1 (region 2) only one period$-T$ solution exists.

In the autonomous system ($\delta = 0$), by decreasing d_0 from d_0'' (Hopf) to d_0' (homoclinic) the period ϑ of the unstable cycle generated through Hopf bifurcation increases from $\vartheta \cong 1.15T$ to infinity. Thus, in the periodically forced system ($\delta > 0$) we expect resonance tongues at the values of d_0 where ϑ is an integer multiple of the period T of the forcing term, i.e. $\vartheta = mT$ ($m = 2, 3, \ldots$). These tongues are formed by pairs of saddle-node bifurcation curves which delimit the region where a pair of period-mT orbits exist: these orbits are unstable at least for δ close to zero, since the limit cycle in the autonomous system is unstable. For $m = 2$ these curves are depicted in Figure 1 (t_1^2 and t_2^2). They terminate respectively on a flip bifurcation curve f^1 (point D_1^1), and at a cusp point C^2. At point C^2, another branch of period$-2T$ saddle-node bifurcation curve (t_3^2), which originates on f^1 at point D_2^1, terminates. Moreover, an infinite sequence $f^2, f^4, \ldots$ of flip bifurcation curves intersects with f^1 (f^2 and f^4 have been computed, and only f^2 is depicted in Figure 1). This sequence accumulates on a curve f^∞ that delimits the region of chaotic behavior. The curve f^2 is connected to t_2^2 and to f^1 by two Naimark-Sacker bifurcation curves h_1^2 and h_2^2, while the curves f^m and f^{2m} ($m = 2, 4, \ldots$) are connected by a pair of Naimark-Sacker bifurcation curves h_1^{2m} and h_2^{2m} (not depicted in the figure).

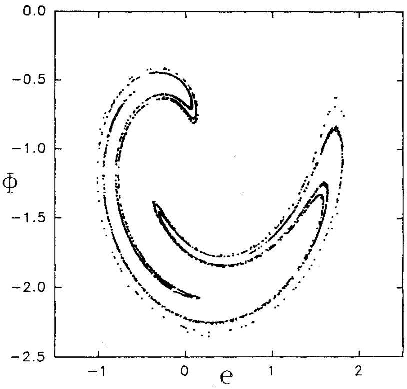

Fig. 2. Poincaré section of a chaotic attractor at $(\delta, d_0) = (0.9, 0)$.

This complex structure of bifurcations gives rise to a partition of the (δ, d_0)-plane into a number of regions characterized by the existence of solutions of different period, which may appear/disappear or change their stability by passing from one region to another. For example, if we cross f^1 from region 0 to 3, the period$-T$ cycle C_1 becomes unstable and a stable period$-2T$ cycle appears. Then, if $f^2, f^4, \ldots$ are crossed, a cascade of period-doublings takes place leading to chaotic behavior (the Poincaré section of a chaotic attractor is depicted in Figure 2). If, on the contrary, we cross t_2^2 from the left to the right below point B^2, *a* stable period$-2T$ cycle appears (together with an unstable one) through saddle-node bifurcation. Then, this cycle may disappear again through saddle-node bifurcation if t_3^2 is crossed, or undergo a cascade of period-doublings if the curves $f^2, f^4, \ldots$ are crossed.

4. DISCUSSION OF THE RESULTS

The bifurcation analysis presented in the previous section shows that the adaptive control system (4) subject to periodic disturbance displays an extremely complex set of solutions when the parameters characterizing the disturbance are varied. The careful examination of the bifurcation picture in Figure 1 reveals that, in any region of the (δ, d_0)-plane, there exists at least one attractor, which can be periodic or chaotic. As predicted by the theory, all system trajectories converge to one of these attractors, so that a bounded behavior is guaranteed regardless to the action of the disturbance.

Since for some parameter values the attractors are multiple, unmodeled perturbations might move the system state from one basin of attraction to another. On the other hand, slow variations of some parameters might also yield a transition from one attractor to another via bifurcation. This latter transition can be either *non-catastrophic*, as it happens, for instance, when crossing the period-doubling curve f^1 from region 0 to 3 (see Figure 1), or *catastrophic* (i.e. with a sudden jump), as, for instance, when crossing the Naimark-Sacker bifurcation curves h_1^1 or h_2^1 from region 0 to 1. Therefore, the combined action of slow parameter variations and of fast state perturbations can switch the system from one attractor to another, yet giving rise, in any case, to bounded trajectories.

It can be observed that the system is characterized by the most regular behavior when it is strongly excited, namely when δ and/or d_0 are sufficiently high (region 2 in Figure 1). Indeed, in this case there is only one attractor (whose basin is the whole state space) having the same period as the input signal.

For synthesis purposes, one could also be interested in analyzing how the design parameters σ and α affect the dimension and location of the region of chaotic behavior. For example, if we fix $d_0 = 0$ and assume that the chaotic region is practically delimited by the second-flip curve f^2, we can plot the chaotic region in the (σ, δ)-plane for several values of α (Figure 3). It emerges that chaos does not take place for very high and very low σ, while α merely shifts, for fixed σ, the interval of δ where chaos takes place.

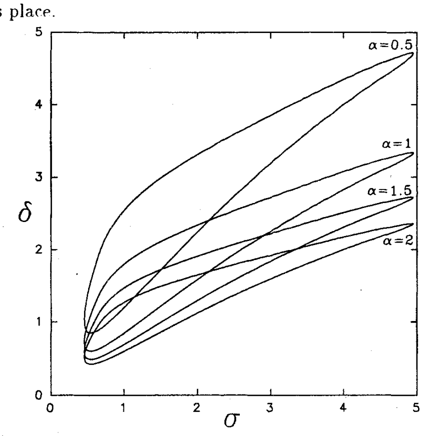

Fig. 3. The region of chaotic behavior in the (σ, δ)-plane for several values of α ($d_0 = 0$).

Finally, it must be reported that, for different values of the parameters ρ, σ, α, and

ω, the picture of Figure 1 is partially altered (e.g., at $\omega = 1$ curve f^2 is completely contained inside f^1), but all the above discussed properties remain valid.

5. CONCLUSIONS

In this work a model reference adaptive control system involving a first-order plant and the σ-modification adaptation law (Ioannou and Kokotovic [7]) has been analyzed in a case of particular interest in control systems analysis, namely when the disturbance or the reference input are of the form constant-plus-sinusoid. A bifurcation analysis of the periodic solutions has been carried out via numerical techniques, allowing to classify the different modes of behavior and the transitions among them. The study has revealed that, as predicted by the theory, the adaptive control system is robust (the output error and the adapting parameter remain bounded), but the periodicity of the input can give rise to chaotic behavior.

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