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ON HODGES–LEHMANN OPTIMALITY OF LR TESTS<sup>1</sup>

FRANTIŠEK RUBLÍK

It is shown that the likelihood ratio test statistics are Hodges–Lehmann optimal for testing the null hypothesis against the whole parameter space, provided that certain regularity conditions are fulfilled. These conditions are verified for the non-singular normal, multinomial and Poisson distribution.

Let  $\{\bar{P}_\gamma; \gamma \in \Xi\}$  be a family of probability measures, defined on  $(X, \mathcal{F})$  by means of the densities  $\{f(x, \gamma); \gamma \in \Xi\}$  with respect to a measure  $\nu$ . If we denote the  $q$ -fold products

$$S = X^\infty \times \dots \times X^\infty, \quad \mathcal{S} = \mathcal{F}^\infty \times \dots \times \mathcal{F}^\infty, \quad \Theta = \Xi^q \quad (1)$$

then for  $\theta = (\theta_1, \dots, \theta_q) \in \Theta$  the corresponding product measure

$$P_\theta = \bar{P}_{\theta_1}^\infty \times \dots \times \bar{P}_{\theta_q}^\infty, \quad (2)$$

defined on the  $\sigma$ -algebra  $\mathcal{S}$ , describes independent sampling from the  $q$  populations  $(X, \mathcal{F}, \bar{P}_{\theta_j}), j = 1, \dots, q$ .

Throughout the paper we shall assume that

$$\emptyset \neq \Omega_0 \subset \Theta. \quad (3)$$

In describing asymptotic properties of tests of the null hypothesis we shall use the notation

$$\mathcal{P} = \left\{ p \in R^q; \sum_{j=1}^q p_j = 1 \text{ and } p_j > 0 \text{ for all } j \right\} \quad (4)$$

and for  $\theta, \theta^* \in \Theta, p \in \mathcal{P}$  we denote

$$K(\theta^*, \theta, p) = \sum_{j=1}^q p_j K(\theta_j^*, \theta_j), \quad (5)$$

$$K(\Omega_0, \theta, p) = \inf \{ K(\theta^*, \theta, p); \theta^* \in \Omega_0 \}, \quad (6)$$

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where  $K(\theta_j^*, \theta_j) = K(\overline{P}_{\theta_j^*}, \overline{P}_{\theta_j})$  is the Kullback–Leibler information number.

We shall suppose that a test  $\varphi_u$  of  $\Omega_0$  against  $\Theta - \Omega_0$  depends on

$$s = \left( \{x_j^{(1)}\}_{j=1}^\infty, \dots, \{x_j^{(q)}\}_{j=1}^\infty \right) \in S$$

through

$$x^{(u)} = \left( y(1, n_u^{(1)}), \dots, y(q, n_u^{(q)}) \right) \quad (7)$$

only, where

$$y \left( j, n_u^{(j)} \right) = \left( x_1^{(j)}, \dots, x_{n_u^{(j)}}^{(j)} \right) \quad (8)$$

is a sample from the  $j$ th population. The sample sizes will be subjected to the following assumption, which in the one sample case  $q = 1$  simply means that the sample size  $n$  tends to infinity.

(C1) *In the notation*

$$n_u = \sum_{j=1}^q n_u^{(j)}, \quad p_u^{(j)} = n_u^{(j)} / n_u \quad (9)$$

the relations

$$\lim_{u \rightarrow \infty} n_u = +\infty, \quad \lim_{u \rightarrow \infty} p_u^{(j)} = p_j \in (0, 1), \quad j = 1, \dots, q \quad (10)$$

hold.

If a test  $\varphi_u$  of  $\Omega_0$  against  $\Theta - \Omega_0$  is based on (7) and for  $\beta_u(\theta) = E_\theta(\varphi_u)$  the relation

$$\sup \left\{ \lim_{u \rightarrow \infty} \beta_u(\theta^*); \theta^* \in \Omega_0 \right\} = \alpha \in (0, 1) \quad (11)$$

holds, then according to Lemma 6.1 in [1] and Theorem 2.1 in [8] under validity of (C1) for each parameter  $\theta \in \Theta - \Omega_0$

$$\liminf_{u \rightarrow \infty} \frac{1}{n_u} \log[1 - \beta_u(\theta)] \geq -K(\Omega_0, \theta, p). \quad (12)$$

We remark that an extension of this inequality to the case of stochastic processes and random fields can be found in [11].

In accordance with [4], [7] and [8] we shall say that the tests  $\{\varphi_u\}$  are *Hodges–Lehmann optimal (H–L optimal) for testing  $\Omega_0$  against  $\Theta - \Omega_0$* , if (C1) and (11) imply that

$$\lim_{u \rightarrow \infty} \frac{1}{n_u} \log[1 - \beta_u(\theta)] = -K(\Omega_0, \theta, p) \quad (13)$$

for each  $\theta \in \Theta - \Omega_0$ .

The H–L optimality was investigated by Brown in [2], where testing  $\Omega_0$  against  $\Omega_1$  is replaced with testing  $\Omega_0^*$  against  $\Omega_1^*$ , and  $\Omega_i^*$  is the closure of  $\Omega_i$  in a set  $\Theta^*$ , into which the original parameter set  $\Theta$  is embedded. As pointed out in [2], p. 1208, the

likelihood ratio statistics  $T(x_1, \dots, x_n, \Omega_0, \Omega_1) = \log[L(x_1, \dots, x_n, \Omega_1)/L(x_1, \dots, x_n, \Omega_0)]$  can be essentially different from  $T(x_1, \dots, x_n, \Omega_0^*, \Omega_1^*)$ , which is in [2] proved to be optimal for the extended problem  $\Omega_0^*$  against  $\Omega_1^*$ .

In some particular cases was the H-L optimality of the LR tests proved in [8]. In the case of exponential families was this H-L optimality in the multisample case  $q > 1$  proved in [7] under the conditions, including the following assumptions.

(D 1) *The effective set*

$$B = \left\{ x; \sup_{\gamma} \log f(x, \gamma) < +\infty \right\} \tag{14}$$

is open.

(D 2) *The arithmetic mean  $\bar{x} = \frac{1}{n} \sum_{i=1}^n x_i$  of the i.i.d. observations  $x_1, \dots, x_n$  belongs to B with probability 1 for all  $n \geq N$  and  $\gamma \in \Xi$ .*

These assumptions are in [7] imposed to assure existence of the MLE with probability 1 for all  $n \geq N$ . They are fulfilled by the family generated by the exponential reparametrization of the non-singular normal distributions, but for important exponential families—generated by the exponential reparametrization of the Poisson or the multinomial distributions—the set (14) is closed, and the MLE does not exist with positive probability for all  $n$ . Moreover, the null hypothesis  $\Omega_0$  is in [7] assumed to have the property that the function (25) of the parameter  $\theta^*$  is continuous. The aim of this paper is to present regularity assumptions, fulfilled by the non-singular normal, the Poisson and the multinomial distributions. As it can be seen from Theorem 1, the presented assumptions in difference from [7] facilitate a unified approach to the H-L optimality of the LR tests for all integers  $q \geq 1$ , without imposing any restrictions on value of the quantity  $K(\theta)$  (defined in [7] on p. 7), or on continuity of  $K(\cdot, \Omega_0, p)$ .

(A I) *There exists a  $\sigma$ -compact metric space  $\Xi_1$  such that  $\Xi$  is dense in  $\Xi_1$  (i. e.  $\bar{\Xi} = \Xi_1$ ), and the original system  $\{f(x, \gamma); \gamma \in \Xi\}$  can be extended to a system  $\{f(x, \gamma); \gamma \in \Xi_1\}$  of densities with respect to the same measure  $\nu$  (by the extension we understand that  $f(x, \gamma)$  coincides with the original density if  $\gamma \in \Xi$ ). Moreover, the extended system  $\{\bar{P}_{\gamma}; \gamma \in \Xi_1\}$  consists of mutually different probabilities, the Kullback-Leibler information quantity  $K(\theta^*, \cdot)$  is continuous on  $\Xi_1$  for each  $\theta^* \in \Xi_1$  (i. e.  $\gamma_n^* \rightarrow \gamma^*$  implies  $K(\theta^*, \gamma_n^*) \rightarrow K(\theta^*, \gamma^*)$ ), and  $f(x, \gamma)$  is continuous on  $\Xi_1$  for each  $x \in X$ .*

(A II) *The function  $K(\cdot, \theta)$  is finite and continuous on  $\Xi_1$  for each  $\theta \in \Xi$ .*

(A III) *Let  $\{\theta_n^*\}_{n=1}^{\infty}$  be parameters from  $\Xi_1$ ,  $\gamma \in \Xi$  and*

$$\limsup_{n \rightarrow \infty} K(\theta_n^*, \gamma) < +\infty.$$

(a) *If  $\theta \in \Xi$ , then  $\limsup_{n \rightarrow \infty} K(\theta_n^*, \theta) < +\infty$ .*

(b) The sequence  $\{\theta_n^*\}_{n=1}^\infty$  possesses a limit point in  $\Xi_1$  and if the parameters  $\{\theta_n\}_{n=1}^\infty$  belonging to  $\Xi$  are such that  $\limsup K(\theta_n^*, \theta_n) < +\infty$ , then also the sequence  $\{\theta_n\}_{n=1}^\infty$  possesses a limit point in  $\Xi_1$ .

(c) Let  $\lim_{n \rightarrow \infty} \theta_n^* = \theta^*$ . If  $\{\theta_n\}_{n=1}^\infty$  is a sequence of parameters from  $\Xi$  and  $\lim_{n \rightarrow \infty} \theta_n = \theta$ , then  $\liminf_{n \rightarrow \infty} K(\theta_n^*, \theta_n) \geq K(\theta^*, \theta)$ .

(A IV) There exist measurable sets  $A_n \subset X^n$ , an integer  $N$  and measurable mappings  $\hat{\theta}_n : A_n \rightarrow \Xi_1$ ,  $g_n : A_n \rightarrow R$  such that for each  $\gamma \in \Xi$  and  $n \geq N$

$$\bar{P}_\gamma(A_n) = 1 \quad (15)$$

and on the set  $A_n$  in the notation  $L(x_1, \dots, x_n, \gamma) = \prod_{i=1}^n f(x_i, \gamma)$  the equality

$$\log L(x_1, \dots, x_n, \gamma) = g_n(x_1, \dots, x_n) - nK(\hat{\theta}_n, \gamma) \quad (16)$$

holds.

(A V) In the notation from (A IV) and

$$p_u = (p_u^{(1)}, \dots, p_u^{(g)}), \quad (17)$$

for  $\theta = (\theta_1, \dots, \theta_g) \in \Xi^g$  let (cf. (8))

$$K(\hat{\theta}, \theta, p_u) = \sum_{j=1}^g p_u^{(j)} K(\hat{\theta}_{n_u^{(j)}}(y(j, n_u^{(j)})), \theta_j). \quad (18)$$

If (10) holds and  $\theta \in \Xi^g$ , then for every real  $t$

$$P_\theta [n_u K(\hat{\theta}, \theta, p_u) \geq t] \leq \exp[-t + h_u(t)] \quad (19)$$

where

$$\limsup_{u \rightarrow \infty} \frac{h_u(t_u)}{n_u} = 0 \quad (20)$$

whenever the inequality

$$\limsup_{u \rightarrow \infty} \frac{t_u}{n_u} < +\infty \quad (21)$$

holds.

We recall that a metric space is said to be  $\sigma$ -compact, if it can be expressed as a countable union of its compact subsets. The reader can easily verify that both this property of  $\Xi_1$  and the continuity of  $f(x, \cdot)$ , which are postulated in (A I), imply measurability of (22) for any non-empty set  $\Omega \subset \Theta = \Xi^g$ .

The regularity assumptions imposed for the multisample case in [7] include only the exponential families, for which the MLE exists with probability 1 for all  $n \geq N$ . Instead of the exponential reparametrization we use an enlargement of the class of

probabilities. Since the Kullback–Leibler information number is non-negative, the axiom (A IV) guarantees that for all  $n \geq N$  there exists with probability 1 a MLE  $\hat{\theta}_n$  of the unknown parameter  $\gamma \in \Xi$ , but taking values in the enlarged parameter set  $\Xi_1$ , and having the property that the likelihood functions computed for the original parameters  $\gamma \in \Xi$  can be expressed in the specific postulated way. We remark that for the exponential families, satisfying the assumptions (D 1) and (D 2), imposed in [7], the assumption (A IV) is fulfilled with  $\Xi_1 = \Xi$ . Validity of (A IV) makes possible to use the inequality (2.8) from [3] in the proof of H–L optimality of LR tests.

If we put for  $\Omega \subset \Theta$

$$L(x^{(u)}, \Omega) = \sup \left\{ \prod_{j=1}^q \prod_{i=1}^{n^{(j)}} f(x_i^{(j)}, \theta_j); \theta = (\theta_1, \dots, \theta_q) \in \Omega \right\}, \quad (22)$$

then by the likelihood ratio test statistics  $T_u$  for testing  $\Omega_0$  against  $\Theta - \Omega_0$  we shall understand the statistics

$$T_u = 2 \log \frac{L(x^{(u)}, \Theta)}{L(x^{(u)}, \Omega_0)}. \quad (23)$$

In the proof of the theorem on H–L optimality of (23) the following lemma will be used.

**Lemma 1.** Let the assumptions (A I)–(A III) be fulfilled, (3) hold and for  $\delta \geq 0$  let

$$D(\delta, p) = \{ \theta^* \in \Xi_1^q; K(\theta^*, \Omega_0, p) \leq \delta \}, \quad (24)$$

where

$$K(\theta^*, \Omega_0, p) = \inf \{ K(\theta^*, \tilde{\theta}, p); \tilde{\theta} \in \Omega_0 \}. \quad (25)$$

If  $\lim_{u \rightarrow \infty} \delta_u = 0, \lim_{u \rightarrow \infty} p_u = p \in \mathcal{P}$ , then for each  $\theta \in \Theta$  in the notation (6)

$$d_u = K[D(\delta_u, p_u), \theta, p_u] \quad (26)$$

is a real number and

$$\liminf_{u \rightarrow \infty} d_u \geq K(\Omega_0, \theta, p). \quad (27)$$

**Proof.** Since  $d_u \leq K(\theta^*, \theta, p_u)$  whenever  $\theta^* \in \Omega_0$ , taking into account (A II) we see that

$$\limsup_{u \rightarrow \infty} d_u < +\infty. \quad (28)$$

Since the sets (24) are non-empty, there exist  $\theta_u^* \in D(\delta_u, p_u)$  such that

$$d_u \leq K(\theta_u^*, \theta, p_u) \leq d_u + u^{-1}. \quad (29)$$

Let  $\theta_u^{(0)} \in \Omega_0$  be such that

$$K(\theta_u^*, \Omega_0, p_u) \leq K(\theta_u^*, \theta_u^{(0)}, p_u) \leq K(\theta_u^*, \Omega_0, p_u) + u^{-1}. \quad (30)$$

Let us choose an increasing sequence  $\{u_v\}_{v=1}^\infty$  of positive integers such that

$$\liminf_{u \rightarrow \infty} d_u = \lim_{v \rightarrow \infty} d_{u_v}. \tag{31}$$

Fixing  $\gamma_0 \in \Omega_0$  and utilizing (28)–(30),  $p_u \rightarrow p$  and (A III)(a) we get

$$\limsup_{u \rightarrow \infty} K(\theta_u^*, \theta_u^{(0)}, p_u) \leq \limsup_{u \rightarrow \infty} K(\theta_u^*, \gamma_0, p_u) < +\infty$$

which together with (A III)(b) means that there exists a subsequence  $\{u_{v_t}\}_{t=1}^\infty$  and points  $\theta^* \in \Xi_1$ ,  $\theta^{(0)} \in \bar{\Omega}_0$  such that in the notation

$$z_t = u_{v_t}, \quad \gamma_t^* = \theta_{z_t}^*, \quad \gamma_t^{(0)} = \theta_{z_t}^{(0)}$$

for  $t \rightarrow \infty$

$$\gamma_t^* \rightarrow \theta^*, \quad \gamma_t^{(0)} \rightarrow \theta^{(0)}.$$

If  $\tilde{\theta} \in \Omega_0$ , then (A II), (30) and (A III)(c) yield

$$\begin{aligned} K(\theta^*, \tilde{\theta}, p) &= \lim_{t \rightarrow \infty} K(\gamma_t^*, \tilde{\theta}, p_{z_t}) \geq \limsup_{t \rightarrow \infty} K(\gamma_t^*, \gamma_t^{(0)}, p_{z_t}) \geq \\ &\geq \liminf_{t \rightarrow \infty} K(\gamma_t^*, \gamma_t^{(0)}, p_{z_t}) \geq K(\theta^*, \theta^{(0)}, p) \geq K(\theta^*, \bar{\Omega}_0, p). \end{aligned}$$

Since according to (A I)

$$K(\theta^*, \bar{\Omega}_0, p) = K(\theta^*, \Omega_0, p), \tag{32}$$

obviously  $K(\theta^*, \Omega_0, p) = \lim_{t \rightarrow \infty} K(\gamma_t^*, \gamma_t^{(0)}, p_{z_t})$ . Combining this with (30) and  $\theta_u^* \in D(\delta_u, p_u)$  we see that  $K(\theta^*, \Omega_0, p) = 0$ . Utilizing (A III)(b), (A I) we obtain that  $\theta^* \in \bar{\Omega}_0$  and by virtue of (31), (29), (A II)

$$\liminf_{u \rightarrow \infty} d_u = \lim_{t \rightarrow \infty} K(\gamma_t^*, \theta, p_{z_t}) = K(\theta^*, \theta, p) \geq K(\bar{\Omega}_0, \theta, p).$$

The inequality (27) now follows from (A II). □

**Theorem 1.** Let a family of probabilities  $\{\bar{P}_\gamma; \gamma \in \Xi\}$ , determined by densities  $\{f(x, \gamma); \gamma \in \Xi\}$  with respect to a measure  $\nu$  be such that the assumptions (A I)–(A V) are fulfilled. Let

$$\emptyset \neq \Omega_0 \subset \Theta = \Xi^q \tag{33}$$

and in the notation (23)

$$\varphi_u(x^{(u)}) = \begin{cases} 1 & T_u > t_u \\ 0 & T_u \leq t_u \end{cases} \tag{34}$$

If validity of (CI) and (11) implies that the critical constants  $t_u$  in (34) can be chosen in such a way that

$$\limsup_{u \rightarrow \infty} t_u < +\infty \tag{35}$$

then the tests (34) are H-L optimal.

PROOF. We shall proceed analogously as in the proof of Theorem 4.5 in [7]. Let us assume that both (CI) and (11) hold. In accordance with (AIV) let

$$B_u = A_{n_u^{(1)}} \times \dots \times A_{n_u^{(g)}}$$

and for  $x^{(u)} \in B_u$  put

$$\hat{\theta}_{(u)}(x^{(u)}) = \left( \hat{\theta}_{n_u^{(1)}}(y(1, n_u^{(1)})), \dots, \hat{\theta}_{n_u^{(g)}}(y(g, n_u^{(g)})) \right). \tag{36}$$

Making use of (AIV), (AII) we get for  $u \geq u_0$  in the notation (24) that

$$\begin{aligned} P_\theta \left[ 2 \log \frac{L(x^{(u)}, \Theta)}{L(x^{(u)}, \Omega_0)} \leq t_u \right] &= P_\theta [2n_u K(\hat{\theta}_{(u)}, \Omega_0, p_u) \leq t_u] = \\ &= P_\theta [\hat{\theta}_{(u)} \in D(\delta_u, p_u)] \end{aligned} \tag{37}$$

where  $\delta_u = t_u/2n_u$ . According to the assumptions  $\lim_{u \rightarrow \infty} \delta_u = 0$ . Utilizing the first inequality in (2.8) in [3] and applying the assumption (AV) we see that for  $u \geq u_1$

$$P_\theta [\hat{\theta}_{(u)} \in D(\delta_u, p_u)] \leq P_\theta [K(\hat{\theta}_{(u)}, \theta, p_u) \geq d_u] \leq \exp[-n_u d_u + o_u] \tag{38}$$

where  $\lim_{u \rightarrow \infty} o_u/n_u = 0$ . Combining (37) and (38) with (27) we get

$$\limsup_{u \rightarrow \infty} \frac{1}{n_u} \log P_\theta [\varphi_u = 0] \leq -K(\Omega_0, \theta, p)$$

and (13) follows from (12). □

According to Theorem 1 the LR tests (34) based on (23) are H-L optimal for arbitrary non-empty set  $\Omega_0 \subset \Xi^g$ , having the property (35). However, if  $\theta^* \in \Omega_0$ , then

$$2 \log \frac{L(x^{(u)}, \Theta)}{L(x^{(u)}, \Omega_0)} \leq \tilde{T}_u(x^{(u)}, \theta^*), \quad \tilde{T}_u(x^{(u)}, \theta^*) = 2 \log \frac{L(x^{(u)}, \Theta)}{L(x^{(u)}, \theta^*)}. \tag{39}$$

If  $\Xi$  is an open subset of  $R^m$  and the densities  $\{f(x, \gamma); \gamma \in \Xi\}$  satisfy regularity conditions on partial derivatives, then the MLE  $\hat{\theta}_n$  of the unknown parameter from  $\Xi$  has the property that  $\mathcal{L}(\sqrt{n}(\hat{\theta}_n - \gamma)|P_\gamma) \rightarrow N(0, J^{-1}(\gamma))$ , where  $J(\gamma)$  is the Fisher information matrix, and according to the well-known classical results under validity of (CI) for  $u \rightarrow \infty$  the weak convergence of distributions

$$\mathcal{L}(\tilde{T}_u(x^{(u)}, \theta^*)|P_{\theta^*}) \rightarrow \chi_{m,g}^2 \tag{40}$$



takes place, where  $\chi_{mq}^2$  denotes the chi-square distribution with  $mq$  degrees of freedom. From (39) and (40) one can easily find out, that validity of (CI) and (11) implies (35). This fact will be used in the proofs of theorems in the further text.

Let  $k > 1$  be an integer and  $a = (k - 1)k/2$ . Let us put  $m = 2k + a$  and denote

$$\Xi = \{ \gamma = (\mu', \sigma_1, \dots, \sigma_k, \rho')' \in R^m; \mu \in R^k, \min \sigma_i > 0, \rho \in R^a, R(\rho) \gg 0 \} \quad (41)$$

the set of parameters of the non-singular  $k$ -dimensional normal distributions, i. e.  $\mu$  is the vector of means,  $\sigma_i^2$  are the variances,  $\rho = (\rho_{12}, \dots, \rho_{k-1k})'$  are the correlation coefficients and  $R(\rho)$  is the symmetric matrix with  $R(\rho)_{ij} = \rho_{ij}$  if  $i < j$ , and  $R(\rho)_{ii} = 1$ , the notation  $R(\rho) \gg 0$  means that the matrix  $R(\rho)$  is positive definite. For  $\gamma \in \Xi$  we shall denote by  $V(\gamma)$  the covariance matrix of the corresponding normal distribution and  $f(x, \gamma)$  its density. In this notation the following theorem holds.

**Theorem 2.** If  $\Omega_0$  is a non-empty subset of  $\Theta = \Xi^g$  and  $T_u$  are the statistics (23), then the tests (34) are H-L optimal for testing  $\Omega_0$  against  $\Theta - \Omega_0$ .

**Proof.** Since (40) holds with  $m = \frac{k(k+3)}{2}$ , the tests (34) have the property (35). We shall prove that also the assumptions (A I)–(A V) are fulfilled.

Let us denote  $\Xi_1 = \Xi$ . For  $\gamma, \gamma^* \in \Xi$

$$K(\gamma, \gamma^*) = \frac{1}{2}(\mu^* - \mu)' V(\gamma^*)^{-1}(\mu^* - \mu) + \frac{1}{2} \text{tr}[V(\gamma)V(\gamma^*)^{-1}] + \frac{1}{2} \log \frac{|V(\gamma^*)|}{|V(\gamma)|} - \frac{k}{2}$$

and (A I), (A II) obviously hold.

According to the inequality (3.31) in [6]

$$K(\gamma^*, \gamma) \geq \frac{1}{2}(\mu^* - \mu)' V(\gamma)^{-1}(\mu^* - \mu) + \frac{1}{2} \sum_{j=1}^k g\left(\frac{\lambda_j(\gamma)}{\lambda_j(\gamma^*)}\right) - \frac{k}{2}$$

where  $\lambda_1(\gamma) \geq \dots \geq \lambda_k(\gamma)$  are characteristic roots of  $V(\gamma)$  and  $g(z) = z^{-1} + \log z$ . Since  $g(z)$  attains its minimum in  $z = 1$  and  $g(z) \rightarrow +\infty$  if either  $z \rightarrow +\infty$  or  $z \rightarrow 0^+$ , the assumption (A III) holds.

The assumption (A V) can be proved either similarly as Theorem 2.1 in [3], or after the exponential reparametrization by means of Lemma 4.4 in [7]. Since (A IV) holds with the usual MLE  $\hat{\theta}_n$  and with the set  $A_n$  of the  $n$ -tuples  $(x_1, \dots, x_n) \in (R^k)^n$  for which the matrix  $\sum_{j=1}^n (x_j - \bar{x})(x_j - \bar{x})'$  is positive definite, the assumptions of Theorem 1 are fulfilled.  $\square$

As we have already noted, this Theorem 2 can be applied to testing the null hypothesis (3) on parameters of the non-singular normal distribution by means of the test statistics (23). Since in some situations the likelihood ratio test statistics (23) are expressible as monotone transformations  $Z_u(T_u^*)$  of some usually used test statistics  $T_u^*$ , in such a case also H-L optimality of  $T_u^*$  is established. In the following example the statistics  $T_u^*$  have not this property.

**Example. Testing equality of covariances.** Let us denote in accordance with (1) and (41)

$$\Omega_0 = \{ \theta = (\theta_1, \dots, \theta_q) \in \Theta; V(\theta_1) = \dots = V(\theta_q) \} \tag{42}$$

the hypothesis that the covariance matrices of the  $q$  normal populations are equal. Let

$$\bar{x}_j, \hat{\Sigma}_j$$

denote the sample mean and the sample covariance matrix constructed from the sample drawn from the  $j$ th population. If we put  $S_j = n_u^{(j)} \hat{\sigma}_j$ ,  $S = \sum_{j=1}^q S_j$ , then

$$T_u = 2 \log \frac{L(x^{(u)}, \Theta)}{L(x^{(u)}, \Omega_0)} = \log \tilde{T}_u, \quad \tilde{T}_u = \left| \frac{1}{n_u} S \right|^{n_u} / \prod_{j=1}^q \left| \frac{1}{n_u^{(j)}} S_j \right|^{n_u^{(j)}}.$$

As pointed out in [10], p. 225, to obtain an unbiased test, instead of  $\tilde{T}_u$  the modified test statistic

$$T_u^* = \left| \frac{1}{n_u - q} S \right|^{n_u - q} / \prod_{j=1}^q \left| \frac{1}{n_u^{(j)} - 1} S_j \right|^{n_u^{(j)} - 1} \tag{43}$$

is used. We shall prove the H-L optimality of the statistic  $T_u^*$ .

As it is shown in the proof of Theorem 3.1 in [6], if  $\gamma \in \Xi$  and  $a$  is a real number, then there exist real numbers  $0 < \varepsilon < \beta$  such that in the notation

$$B_n = \{ (x_1, \dots, x_n); \lambda_k(\hat{\Sigma}) \geq \varepsilon, \lambda_1(\hat{\Sigma}) \leq \beta \}, \tag{44}$$

where  $\lambda_k(\hat{\Sigma})$  is the smallest and  $\lambda_1(\hat{\Sigma})$  is the largest characteristic root of  $\hat{\Sigma}$ , the inequality

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log [1 - \bar{P}_\gamma(B_n)] \leq -a \tag{45}$$

holds. Let (10) hold. If  $a = K(\Omega_0, \theta, p)$  and  $B_n^{(j)}$  are the sets (44) satisfying (45) with  $\gamma_j = \theta_j$  and  $a_j^* = a/p_j$ , then in the notation  $X = R^k$ ,  $C_u = B_{n_u^{(1)}}^{(1)} \times \dots \times B_{n_u^{(q)}}^{(q)}$

$$\limsup_{u \rightarrow \infty} \frac{1}{n_u} \log P_\theta [X^{n_u} - C_u] \leq -a. \tag{46}$$

Under validity of the null hypothesis  $\tilde{T}_u/T_u^* \rightarrow 1$  a. e., and according to [9], p. 404 the distributions  $\mathcal{L}(\log \tilde{T}_u) \rightarrow \chi_u^2$ . Thus if constants  $\{t_u\}$  are such that the tests (34) based on  $T_u = \log T_u^*$  satisfy (11), then (35) is fulfilled. Further, it is obvious that there exist an index  $u_1$  and a positive real number  $M$  such that for all  $u \geq u_1$  on the set  $C_u$  the inequality  $\tilde{T}_u/T_u^* \leq M$  holds. Hence according to Theorem 2

$$\begin{aligned} & \limsup_{u \rightarrow \infty} \frac{1}{n_u} \log P_\theta [C_u \cap \{ \log T_u^* \leq t_u \}] \leq \\ & \leq \limsup_{u \rightarrow \infty} \frac{1}{n_u} \log P_\theta [ \log \tilde{T}_u \leq t_u + \log M ] = -K(\Omega_0, \theta, p). \end{aligned}$$

Combining this with (46) we get

$$\limsup_{u \rightarrow \infty} \frac{1}{n_u} \log P_\theta [\log T_u^* \leq t_u] \leq -K(\Omega_0, \theta, p),$$

which together with (12) means that the tests based on  $T_u^*$  are H-L optimal for testing the hypothesis (42) under the normality assumptions.

Let  $X = \{1, \dots, k\}$  be a finite set,

$$\Xi = \left\{ (p_1, \dots, p_{k-1})' \in R^{k-1}; \min_i p_i > 0, \sum_{i=1}^{k-1} p_i < 1 \right\} \tag{47}$$

and

$$f(x, p) = p_x, \quad p_k = 1 - \sum_{j=1}^{k-1} p_j \tag{48}$$

denotes a density with respect to the counting measure  $\mu$  on  $(X, 2^X)$ . In this notation the following theorem holds.

**Theorem 3.** If  $\Omega_0$  is a non-empty subset of  $\Theta = \Xi^g$  and  $T_u$  are the statistics (23), then the tests (34) are H-L optimal for testing  $\Omega_0$  against  $\Theta - \Omega_0$ .

*Proof.* Since (40) holds with  $m = k - 1$ , the tests (34) have the property (35). To prove (A I)–(A V), let us denote

$$\Xi_1 = \bar{\Xi} \tag{49}$$

the closure of (47), and let  $f(x, p)$  be the densities (48). Since

$$K(p, p^*) = \sum_{j=1}^k p_j \log \frac{p_j}{p_j^*}$$

where  $0 \log \frac{0}{x} = 0$ , the assumption (A III)(c) can be proved similarly as Lemma 4.4(a) in [5], and all the other statements in (A I)–(A III) can be easily proved by means of the compactness of  $\Xi_1$ .

The assumption (A IV) is fulfilled with  $A_n = X^n$  and

$$\hat{\theta}_n = (\hat{p}_1, \dots, \hat{p}_{k-1})', \quad \hat{p}_x = \frac{n_x}{n}$$

where  $n_x$  denotes the number of occurrences of  $x$  in  $x_1, \dots, x_n$ .

Making use of the first equality in (48), the relation (2.4) in [5] and proceeding as in the proof of the inequality (2.10) in [5], we obtain, that (A V) holds.

Thus the assumptions of Theorem 1 are fulfilled and the proof is completed.  $\square$

Let  $X = \{0, 1, 2, \dots\}$ ,

$$\Xi = (0, +\infty) \tag{50}$$

and

$$f(x, \lambda) = \frac{e^{-\lambda} \lambda^x}{x!} \tag{51}$$

be density of the Poisson distribution  $\bar{P}_\lambda$  with respect to the counting measure  $\mu$  on  $(X, 2^X)$ . In this notation the following theorem holds.

**Theorem 4.** If  $\Omega_0$  is a non-empty subset of  $\Theta = \Xi^{\mathcal{V}}$  and  $T_u$  are the statistics (23), then the tests (34) are H-L optimal for testing  $\Omega_0$  against  $\Theta - \Omega_0$ .

*Proof.* Since (40) holds with  $m = 1$ , the tests (34) have the property (35). To prove (A I) - (A V), we denote

$$\Xi_1 = (0, +\infty).$$

Since for  $\lambda, \lambda^* \in \Xi_1$

$$K(\lambda, \lambda^*) = \begin{cases} \lambda^* - \lambda + \lambda \log \frac{\lambda}{\lambda^*} & \lambda^* > 0 \\ +\infty & \lambda^* = 0, \lambda > 0 \end{cases}$$

where  $0 \log \frac{0}{x} = 0$ , the assumptions (A I) - (A III) hold. The assumption (A IV) holds with  $A_n = X^n$  and  $\hat{\theta}_n = n^{-1} \sum_{j=1}^n x_j$ . Since the axiom (A V) can be proved either by means of the relation (6.22) in [6] or by means of Lemma 4.3 in [7], the assumptions of Theorem 1 are fulfilled. □

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