Jiří Zlatuška

Normal forms in the typed λ-calculus with tuple types

Kybernetika, Vol. 21 (1985), No. 5, 366--381

Persistent URL: http://dml.cz/dmlcz/124994

KYBERNETIKA — VOLUME 21 (1985), NUMBER 5

NORMAL FORMS IN THE TYPED λ-CALCULUS WITH TUPLE TYPES

JIŘÍ ZLATUŠKA

A modified typed λ-calculus with types containing, in addition to function types, also product types is studied. A notion of reduction, including bijective tuple and projection operations, is introduced and it is shown that it is both strongly normalizing and Church-Rosser.

INTRODUCTION

It is well-known that the λ-calculus has the Church-Rosser property [3] and that, if we restrict the terms to those of the typed λ-calculus, a unique normal form exists to every couple of convertible terms [4]. This makes the typed λ-calculus a good notational formalism. In order to achieve better means of expression of the typed λ-calculus (e.g. for [10]), it is useful to generalize the types corresponding to functions [2] to their cartesian products as can be found e.g. in [7]. Such a generalized calculus has been shown to be in very close correspondence with the cartesian closed categories [8]. In the present paper we shall present a notion of reduction in the calculus with tuple types ($\lambda^{\times}$-calculus) that is Church-Rosser with unique normal forms and contains bijective pairing.

TYPES AND TERMS

First, let us introduce our type structure. By a base we shall mean a set of pairwise different symbols.

Let $\mathbf{B}$ be a base. The set $\mathsf{Typ}(\mathbf{B})$ of the *types over the base* $\mathbf{B}$ is inductively defined as follows:

(1) $\mathbf{B} \subset \mathsf{Typ}(\mathbf{B})$

(2) if $\xi, \eta \in \mathsf{Typ}(\mathbf{B})$, then $(\eta\xi) \in \mathsf{Typ}(\mathbf{B})$

(3) if $\xi_1, \ldots, \xi_n \in \mathsf{Typ}(\mathbf{B})$, then $(\xi_1, \ldots, \xi_n) \in \mathsf{Typ}(\mathbf{B})$

The semantics of the type symbols from $\mathsf{Typ}(\mathbf{B})$ is given by the values (some sets or, according to [7], some domains) of an interpretation $\mathscr{I}$ at the types from the base $\mathbf{B}$ (the so-called base types). We inductively define the interpretation $\mathscr{I}(\eta\xi)$ of the function types of the form $(\eta\xi)$ as a set of functions from $\mathscr{I}\xi$ to $\mathscr{I}\eta$ (e.g.

$\mathscr{I}(\eta\xi) = \mathscr{I}(\eta)^{\mathscr{I}(\xi)}$) and the interpretation $\mathscr{I}(\xi_1, \ldots, \xi_n)$ of the tuple types as the cartesian product of $\mathscr{I}\xi_1, \ldots, \mathscr{I}\xi_n$ (i.e. $\mathscr{I}(\xi_1, \ldots, \xi_n) = \mathscr{I}\xi_1 \times \ldots \times \mathscr{I}\xi_n$). To avoid tiresome conversions in the notation, with respect to the defined semantics of the tuple types we shall assume the associativity of the cartesian product and, therefore, identify, at the level of type symbols, any type $\xi \in \mathsf{Typ}(\mathbf{B})$ with the corresponding one-tuple (ξ). The inner parentheses in the tuple types are of no importance, e.g. if we take $\xi_i = (\zeta_1, \ldots, \zeta_k)$, then $(\xi_1, \ldots, \xi_i, \ldots, \xi_n)$ is considered to be identical with $(\xi_1, \ldots, \xi_{i-1}, \zeta_1, \ldots, \zeta_k, \xi_{i+1}, \ldots, \xi_n)$.

The identification of certain tuple types can be made formally correct in the following way: We can define a reduction $\nrightarrow^*$ of type symbols as a transitive closure of relation $\nrightarrow$ defined by:

if $\xi = (\xi_1, \ldots, \xi_n)$ and $\xi_i = (\zeta_1, \ldots, \zeta_k)$, then $\xi \nrightarrow \eta$,
  where $\eta = (\xi_1, \ldots, \xi_{i-1}, \zeta_1, \ldots, \zeta_k, \xi_{i+1}, \ldots, \xi_n)$;

if $\xi = (\xi_1, \ldots, \xi_n)$ and $\xi_i \nrightarrow \zeta$, then $\xi \nrightarrow \eta$,
  where $\eta = (\xi_1, \ldots, \xi_{i-1}, \zeta, \xi_{i+1}, \ldots, \xi_n)$.

Clearly, $\nrightarrow^*$ is Church-Rosser and every reduction sequence has only a finite number of steps; consequently, there exists a unique normal form for every (tuple) type symbol. Factorizing the type symbols using the equivalence relation induced by $\nrightarrow^*$, we can use unique representations for every type symbol – we call them *normal type symbols* (or *normal types* for short).

The $\lambda^\times$-terms of the calculus are defined as words over the alphabet of variables (where we assume infinite number of variables for every type) and auxiliary symbols $(\,)$, λ $_{(1)}$ $_{(2)}$ $_{(3)}$ $\ldots$ $_{(n)}$ $\ldots$ (subscripts for every integer number) as follows:

(i) The set of $\lambda^\times$*-terms* (*over the base* $\mathbf{B}$) *of the type* ξ is inductively defined as the least set $\Lambda^{\mathbf{B}}_\xi$ satisfying:

(a) if v^ξ is a variable of a type $\xi \in \mathsf{Typ}(\mathbf{B})$, then $v^\xi \in \Lambda^{\mathbf{B}}_\xi$;
(b) if $X \in \Lambda^{\mathbf{B}}_\xi$, $Y \in \Lambda^{\mathbf{B}}_{(\eta\xi)}$ then $(YX) \in \Lambda^{\mathbf{B}}_\eta$;
(c) if $x_1^{\xi_1}, \ldots, x_n^{\xi_n}$ are mutually different variables of respective types $\xi_1, \ldots, \xi_n$ from $\mathsf{Typ}(\mathbf{B})$ and $Y \in \Lambda^{\mathbf{B}}_\eta$, then $\lambda x_1 \ldots x_n(Y) \in \Lambda^{\mathbf{B}}_{(\eta(\xi_1, \ldots, \xi_n))}$;
(d) if $X_1 \in \Lambda^{\mathbf{B}}_{\xi_1, \ldots, \xi_n} \in \Lambda^{\mathbf{B}}_{\xi_n}$, then $(X_1, \ldots, X_n) \in \Lambda^{\mathbf{B}}_{(\xi_1, \ldots, \xi_n)}$;
(e) if $X \in \Lambda^{\mathbf{B}}_{(\xi_1, \ldots, \xi_n)}$, where $(\xi_1, \ldots, \xi_n)$ is a normal type, then $X_{(i)} \in \Lambda^{\mathbf{B}}_{\xi_i}$, for every i, $1 \leqq i \leqq n$.

(ii) The set $\Lambda^{\mathbf{B}}$ of all typed terms (over the base $\mathbf{B}$) is $\Lambda^{\mathbf{B}} = \bigcup_{\xi \in \mathsf{Typ}(\mathbf{B})} \Lambda^{\mathbf{B}}_\xi$.

In the definition above, (a)–(c) correspond to the common notation of typed λ-terms, (d) represents the "tuple-forming" operation which is a generalization of the sometimes used pairing symbol (note that we do *not* introduce special symbols for the operation; instead we use it in the same way as it is usual to use the abstraction (c)), and (e) represents the "projections" from tuples (the condition $(\xi_1, \ldots, \xi_n)$ being normal makes the projection unambiguous because no ξ_i is a tuple type) into their components.

Remark. The pairing symbol, $\boldsymbol{P}$, of type $(\alpha_1, \alpha_2)\,\alpha_2\alpha_1$ can be expressed by

$$\boldsymbol{P} = \lambda x^{\alpha_1} y^{\alpha_2} \,.\, (x, y)$$

The corresponding projection symbols, ${}^k\boldsymbol{\Pi}$, of type $\alpha_k(\alpha_1, \alpha_2)$, $1 \leqq k \leqq 2$, can be expressed by

$${}^k\boldsymbol{\Pi} = \lambda x^{(\alpha_1,\alpha_2)} x_{(k)} \,,$$

provided that (α_1, α_2) is normal. If (α_1, α_2) is not normal, say $\alpha_1 = (\beta_1, \beta_2)$, then similarly, without great difficulties,

$${}^1\boldsymbol{\Pi} = \lambda x^{(\beta_1,\beta_2,\beta_3)} \,.\, (x_{(1)}, x_{(2)}) \,; \quad {}^2\boldsymbol{\Pi} = \lambda x^{(\beta_1,\beta_2,\beta_3)} \,.\, x_{(3)} \,,$$

etc.

Note, moreover, that the subscripts for the projections contain only *fixed* numbers they are, in fact, improper symbols) and, therefore, it is clearly impossible to "compute" their values in $\lambda^\times$-terms — they must be given when one writes $\lambda^\times$-term.

For the sake of notational convenience, we shall omit tiresome parenthesing whenever it is possible to do so without confusion. Particularly, we shall omit outermost parentheses in applications and we shall, moreover, assume implicit parenthesing with association to the left if no other parenthesing is implied from the type context.

The notions of free and bound variables as well as the standard conventions avoiding confusions of free and bound variables (especially in abstractions) are supposed. To avoid unnecessary troubles with renaming of variables, we assume $\lambda^\times$-terms modulo α-conversion (renaming of bound variables) — cf. [1], App. C.

Convention. Whenever it is possible, without causing any confusion or misunderstanding, we shall omit the corresponding type symbols (then we assume any admissible typing), or we shall indicate types only in the defining occurrences of $\lambda^\times$-terms or variables.

REDUCTIONS

We define the following notions of reduction expressing natural transformations of $\lambda^\times$-terms:

β: $(\lambda x_1^{\xi_1} \ldots x_n^{\xi_n} \,.\, A)\, B^{(\xi_1,\ldots,\xi_n)} \rightarrow A[x_1/B_{(1)}, \ldots, x_n/B_{(n)}]$,
provided that $(\xi_1, \ldots, \xi_n)$ is normal type;

η: $\lambda x_1 \ldots x_n \,.\, A(x_1, \ldots, x_n) \rightarrow A$,
provided that x_i, $1 \leqq i \leqq n$, do not occur free in A;

π: $(A_1^{\xi_1}, \ldots, A_n^{\xi_n})_{(i)} \rightarrow A_i$,
provided that $(\xi_1, \ldots, \xi_n)$ is normal type;

τ: $\lambda x_1 \ldots x_i^{(\xi_1,\ldots,\xi_k)} \ldots x_n \,.\, A \rightarrow \lambda x_1 \ldots x_{i-1} y_{i_1}^{\xi_1} \ldots y_{i_k}^{\xi_k} x_{i+1} \ldots x_n \,.$
$.\, A[x_i/(y_{i_1}, \ldots, y_{i_k})]$,
provided that y_{i_j}, $1 \leqq j \leqq k$, do not occur in A;

σ: $(A_1, \ldots, A_i^{(\xi_1,\ldots,\xi_k)}, \ldots, A_n) \to (A_1, \ldots, A_{i-1}, A_{i(i)}, \ldots, A_{i(k)}, A_{i+1}, \ldots, A_n)$,
provided that $(\xi_1, \ldots, \xi_k)$ is normal type;
μ: $(A_{(1)}, \ldots, A_{(n)}) \to A^{(\xi_1,\ldots,\xi_n)}$,
provided that $(\xi_1, \ldots, \xi_n)$ is normal type.

Remark. The notation $N[x_1/L_1, \ldots, x_k/L_k]$ represents simultaneuous substitution of $L_1, \ldots, L_k$ for the (pairwise different) variables $x_1, \ldots, x_k$, respectively. To be exact:

$y[x/L] \equiv y$ iff y is a variable, $y \not\equiv x$
$y[x/L] \equiv L$ iff $y \equiv x$
$(MN)[x/L] \equiv (M[x/L])(N[x/L])$
$(\lambda y_1 \ldots y_n . M)[x/L] \equiv \lambda y_1 \ldots y_n . M[x/L]$ (here we assume, according to the standard conventions, the variable x being different from each of $y_1, \ldots, y_n$)
$(N_1, \ldots, N_n)[x/L] \equiv (N_1[x/L], \ldots, N_n[x/L])$
$(N_{(i)})[x/L] \equiv (N[x/L])_{(i)}$.

By an easy induction the *substitution lemma* holds:

$M[x/N][y/L] \equiv M[y/L][x/N[y/L]]$, provided that $x \not\equiv y$ and x does not occur free in L.

According to the substitution lemma it follows:

$M[x/N][y/L] \equiv M[y/L][x/N]$, provided that $x \not\equiv y$ and neither x occurs free in L nor y occurs free in N;

therefore we can write in such a case $M[x/N, y/L]$ instead of $M[x/N][y/L]$ and, straight away, we have the notation $M[x_1/N_1, \ldots, x_k/N_k]$, or, for short, only $M[\bar{x}/\bar{N}]$ (if the number of the components is clear), provided that $x_i \not\equiv x_j$, $i \neq j$, and x_i does not occur free in any N_j, $i \neq j$.

One step reductions are formed from $\beta \cup \eta \cup \pi \cup \tau \cup \sigma \cup \mu$ (we shall abbreviate this notion of reduction by $\boldsymbol{\Gamma}$) as the *compatible closure*, i.e.

$M \to M' = ZM \to ZM'$
$M \to M' = MZ \to M'Z$
$M \to M' = \lambda\bar{x} . M \to \lambda\bar{x} . M'$
$M_i \to M_i' = (M_1, \ldots, M_i, \ldots, M_n) \to (M_1, \ldots, M_i', \ldots, M_n)$
$M \to M' = M_{(i)} \to M_{(i)}'$

General reductions, $\to^*$, are generated as the reflexive and transitive closure of $\to$ [1].

For the study of the equality of $\lambda^\times$-terms constructed as the equivalence relation generated by $\to^*$, the reduction relation $\to^*$ should have certain good properties, especially to be Church-Rosser and (as we have our calculus typed) strongly normalising:

A notion of reduction is said to be *Church-Rosser* (***CR***) iff whenever $A \to^* B$ and $A \to^* C$ then there exists D such that $B \to^* D$ and $C \to^* D$.

A notion of reduction is *strongly normalizing* (***SN***) iff for any $\lambda^\times$-term A there is no infinite reduction sequence $A \to A_1 \to \ldots$.

An important property is the existence of normal forms with respect to a notion of reduction which is both ***CR*** and ***SN***: Term A is said to be in *normal form* iff $B \equiv A$ follows from $A \rightarrow^* B$. Moreover, if two terms A, B are equivalent in the equivalence relation generated by $\rightarrow^*$ and $\rightarrow^*$ is ***CR*** and ***SN***, then there exists a unique term C in the normal form such that $A \rightarrow^* C$ and $B \rightarrow^* C$. Any reduction sequence starting with A or B terminates in this term C.

Remark. The following lemma is useful for work with reductions.

$M \rightarrow^* N \Rightarrow M[x_1/L_1, \ldots, x_k/L_k] \rightarrow^* N[x_1/L_1, \ldots, x_k/L_k]$ (i.e. $\boldsymbol{\Gamma}$ is *substitutive*).

It is quite sufficient to check the substitutivity for the relation $\rightarrow$; the substitutivity of $\rightarrow^*$ follows by simple induction on the definition of $\rightarrow^*$ (Cf. [1], Prop. 3.1.15). The proof of the substitutivity of $\rightarrow$ follows by simple induction on the definition of $\rightarrow$.

In the following we shall prove the Church-Rosser property and the strong normalization of $\boldsymbol{\Gamma}$ in our calculus.

First, we shall prove the weak Church-Rosser property (***WCR***) (i.e. whenever $A \rightarrow B$, $A \rightarrow C$, then there exists D such that $B \rightarrow^* D$ and $C \rightarrow^* D$). Then, the strong normalization will be proved using the method shown in [5] (in which a strongly normalizing notion of reduction is studied; however, it does not contain the bijective pairing rule μ – it contains only the rules analogous to our rules β and π). ***SN*** together with ***WCR*** suffices for the validation of ***CR*** (Cf. [6]).

(WEAK) CHURCH-ROSSER PROPERTY

Lemma. $\boldsymbol{\Gamma}$ is ***WCR***.

P r o o f. In order to prove that $\boldsymbol{\Gamma}$ is ***WCR*** it suffices to chase the diagram

$$\begin{tikzcd} M \arrow[rr] \arrow[d] & & M_2 \arrow[d, dashed, "*"] \\ M_1 \arrow[rr, dashed, "*"] & & M_3 \end{tikzcd}$$

Suppose $M \rightarrow M_1$ and $M \rightarrow M_2$ to be direct consequences (through the compatible closure) of $\Delta_1 \rightarrow \Delta_1'$ and $\Delta_2 \rightarrow \Delta_2'$ (Δ_i are redex occurrences and Δ_i' contractum occurrences in the considered $\lambda^\times$-terms). The possible relationships between the occurrences Δ_1 and Δ_2 are listed in the following table:

(1) $\Delta_1 \cap \Delta_2 = \emptyset$
(2) $\Delta_1 = \Delta_2$
(3) $\Delta_1 \subset \Delta_2$
(4) $\Delta_1 \supset \Delta_2$

In the case (1), the diagram is clearly satisfied for every couple of reductions β, η, π, τ, σ, μ, because if we have

$$M \equiv \ldots \Delta_1 \ldots \Delta_2 \ldots$$
$$M_1 \equiv \ldots \Delta_1' \ldots \Delta_2 \ldots$$
$$M_2 \equiv \ldots \Delta_1 \ldots \Delta_2' \ldots,$$

we can take

$$M_3 \equiv \ldots \Delta_1' \ldots \Delta_2' \ldots.$$

In the remaining situations (2)–(4) we must tediously try the possible cases for the of reductions $\Delta_1 \to \Delta_1'$ and $\Delta_2 \to \Delta_2'$:

Take β-reduction: $\Delta_1 \equiv (\lambda x_1 \ldots x_n \,.\, P^1)\, Q^1$,
$\Delta_1' \equiv P^1[x_1/Q^1_{(1)}, \ldots, x_n/Q^1_{(n)}]$.

Case $\beta\beta$: $\Delta_2 \equiv (\lambda y_1 \ldots y_m \,.\, P^2)\, Q^2$,
$\Delta_2' \equiv P^2[y_1/Q^2_{(1)}, \ldots, y_m/Q^2_{(m)}]$.

Subcase (2): Then $M_1 \equiv M_2$ and therefore we can take $M_3 \equiv M_1$.

Subcase (3): (a) $\Delta_1 \subset P^2$, then

$$M \equiv \ldots (\lambda y_1 \ldots y_m (\ldots \Delta_1 \ldots))\, Q^2 \ldots, \text{ where } \ldots \Delta_1 \ldots = P^2,$$
$$M_1 \equiv \ldots (\lambda y_1 \ldots y_m (\ldots \Delta_1' \ldots))\, Q^2 \ldots,$$
$$M_2 \equiv \ldots (\ldots \Delta_1 \ldots)\, [y_1/Q^2_{(1)}, \ldots, y_m/Q^2_{(m)}] \ldots;$$

take

$$M_3 \equiv \ldots (\ldots \Delta_1' \ldots)\, [y_1/Q^2_{(1)}, \ldots, y_m/Q^2_{(m)}] \ldots$$

and the result follows from the substitutivity of Γ.

(b) $\Delta_1 \subset Q^2$, then

$$M = \ldots (\lambda y_1 \ldots y_m \,.\, P^2)\, (\ldots \Delta_1 \ldots) \ldots, \text{ where } \ldots \Delta_1 \ldots = Q^2,$$
$$M_1 \equiv \ldots (\lambda y_1 \ldots y_m \,.\, P^2)\, (\ldots \Delta_1' \ldots) \ldots,$$
$$M_2 \equiv \ldots P^2[y_1/(\ldots \Delta_1 \ldots)_{(1)}, \ldots, y_m/(\ldots \Delta_1 \ldots)_{(m)}] \ldots;$$

take

$$M_3 \equiv \ldots P^2[y_1/(\ldots \Delta_1' \ldots)_{(1)}, \ldots, y_m/(\ldots \Delta_1' \ldots)_{m)}] \ldots$$

and the result follows from the compatibility of Γ.

Subcase (4) is analogous to (3).

Case $\beta\eta$: $\Delta_2 \equiv \lambda y_1 \ldots y_m \,.\, P^2(y_1, \ldots, y_m)$; $\Delta_2' \equiv P^2$.

Subcase (2): impossible

Subcase (3): then $\Delta_1 \subset P^2$:

$$M \equiv \ldots \lambda y_1 \ldots y_m \,.\, (\ldots \Delta_1 \ldots)\, (y_1, \ldots, y_m)) \ldots, \text{ where } \ldots \Delta_1 \ldots = P^2,$$
$$M_1 \equiv \ldots (\lambda y_1 \ldots y_m \,.\, (\ldots \Delta_1' \ldots)\, (y_1, \ldots, y_m)) \ldots,$$
$$M_2 \equiv \ldots (\ldots \Delta_1 \ldots) \ldots;$$

take

$$M_3 \equiv \ldots (\ldots \Delta_1' \ldots) \ldots$$

and the result is clear.

Subcase (4): (*a*) $\Delta_2 \subset P^1$:

$$M \equiv \ldots \lambda x_1 \ldots x_n \,.\, (\ldots \Delta_2 \ldots)) \, Q^1 \ldots, \text{ where } \ldots \Delta_2 \ldots \equiv P^1,$$
$$M_1 \equiv \ldots (\ldots \Delta_2 \ldots) \, [x_1/Q^1_{(1)}, \ldots, x_n/Q^1_{(n)}] \ldots,$$
$$M_2 \equiv \ldots (\lambda x_1 \ldots x_n \,.\, (\ldots \Delta_2' \ldots)) \, Q^1 \ldots;$$

take

$$M_3 \equiv \ldots (\ldots \Delta_2' \ldots) \, [x_1/Q^1_{(1)}, \ldots, x_n/Q^1_{(n)}] \ldots$$

and the result follows from the substitutivity of Γ.

(b) $\Delta_2 \equiv \lambda x_1 \ldots x_n \,.\, P^1(x_1, \ldots, x_n)$, then

$$M \equiv \ldots (\lambda x_1 \ldots x_n \,.\, P^1(x_1, \ldots, x_n)) \, Q^1 \ldots,$$
$$M_1 \equiv \ldots (P^1(x_1, \ldots, x_n)) \, [x_1/Q^1_{(1)}, \ldots, x_n/Q^1_{(n)}] \ldots,$$
$$M_2 \equiv \ldots P^1 Q^1 \ldots;$$

take

$$M_3 \equiv M_2$$

and the result follows from $M_1 = \ldots P^1(Q^1_{(1)}, \ldots, Q^1_{(n)}) \ldots$ (because P^1 does not contain any free occurrence of $x_1, \ldots, x_n$) using μ-reduction.

(c) $\Delta_2 \subset Q^1$, then

$$M \equiv \ldots (\lambda x_1 \ldots x_n \,.\, P^1) \, (\ldots \Delta_2 \ldots) \ldots, \text{ where } \ldots \Delta_2 \ldots \equiv Q^1,$$
$$M_1 \equiv \ldots P^1[x_1/(\ldots \Delta_2 \ldots)_{(1)}, \ldots, x_n/(\ldots \Delta_2 \ldots)_{(n)}] \ldots,$$
$$M_2 \equiv (\lambda x_1 \ldots x_n \,.\, P^1) \, (\ldots \Delta_2' \ldots) \ldots;$$

take

$$M_3 \equiv P^1[x_1/(\ldots \Delta_2' \ldots)_{(1)}, \ldots, x_n/(\ldots \Delta_2' \ldots)_{(n)}] \ldots$$

and the result is clear from the compatibility of Γ.

Case $\beta\pi$: $\Delta_2 \equiv (P_1^2, \ldots, P_m^2)_{(i)}$, $\Delta_2' \equiv P_i^2$.

Subcase (2): impossible.

Subcase (3): (a) $\Delta_1 \subset P_j^2$, $j \neq i$, then

$$M \equiv \ldots (P_1^2, \ldots, (\ldots \Delta_1 \ldots), \ldots, P_m^2)_{(i)} \ldots, \text{ where } \ldots \Delta_1 \ldots \equiv P_j^2$$
$$M_1 \equiv \ldots (P_1^2, \ldots, (\ldots \Delta_1' \ldots), \ldots, P_m^2)_{(i)} \ldots,$$
$$M_2 = \equiv \ldots P_i^2 \ldots;$$

take

$$M_3 \equiv M_2$$

and the result is clear.

(b) $\Delta_1 \subset P_i^2$, then

$$M \equiv \ldots (P_1^2, \ldots, (\ldots \Delta_1 \ldots), \ldots, P_m^2)_{(i)} \ldots, \text{ where } \ldots \Delta_1 \ldots \equiv P_i^2,$$
$$M_1 \equiv \ldots (P_1^2, \ldots, (\ldots \Delta_1' \ldots), \ldots, P_m^2)_{(i)} \ldots,$$
$$M_2 \equiv \ldots (\ldots \Delta_1 \ldots) \ldots;$$

take

$$M_3 \equiv \ldots (\ldots \Delta_1' \ldots) \ldots$$

and the result is clear (from the compatibility of Γ).

Subcase (4): similarly to $\beta\beta$ (4).

Case $\beta\tau$: $\Delta_2 \equiv \lambda y_1 \ldots y_i^{(\xi_1, \ldots, \xi_k)} \ldots y_m . P^2$,

$\Delta_2' \equiv y_1 \ldots y_{i_1}^{\xi_1} \ldots y_{i_k}^{\xi_k} \ldots y_m . P^2[y_i/(y_{i_1}, \ldots, y_{i_k})]$.

Subcase (2): impossible.

Subcase (3): $M \equiv \ldots \lambda y_1 \ldots y_i \ldots y_m(\ldots \Delta_1 \ldots) \ldots$, where $\ldots \Delta_1 \ldots \equiv P^2$.

$$M_1 \equiv \ldots \lambda y_1 \ldots y_i \ldots y_m(\ldots \Delta_1' \ldots) \ldots,$$
$$M_2 \equiv \ldots \lambda y_1 \ldots y_{i_1} \ldots y_{i_k} \ldots y_m(\ldots \Delta_1 \ldots) [y_i/(y_{i_1}, \ldots, y_{i_k})] \ldots;$$

take

$$M_3 \equiv \ldots \lambda y_1 \ldots y_{i_1} \ldots y_{i_k} \ldots y_m(\ldots \Delta_1' \ldots) [y_i/(y_{i_1}, \ldots, y_{i_k}) \ldots$$

and the result is clear (from compatibility of Γ).

Subcase (4): similarly to $\beta\beta(4)$.

Case $\beta\sigma$: $\Delta_2 \equiv (P_1^2, \ldots, P_i^2, \ldots, P_m^2)$,

$\Delta_2' \equiv (P_1^2, \ldots, P_{i(1)}^2, \ldots, P_{i(k)}^2, \ldots, P_m^2)$.

Subcase (2): impossible.

Subcase (3): $\Delta_1 \subset P_j^2$, $j \neq i$ ($j = i$ is clearly impossible!),

then

$$M \equiv \ldots (P_1^2, \ldots, (\ldots \Delta_1 \ldots), \ldots, P_i^2, \ldots, P_m^2) \ldots, \text{ where } \ldots \Delta_1 \ldots \equiv P_j^2,$$
$$M_1 \equiv \ldots (P_1^2, \ldots, (\ldots \Delta_1' \ldots), \ldots, P_i^2, \ldots, P_m^2) \ldots,$$
$$M_2 \equiv \ldots (P_1^2, \ldots, (\ldots \Delta_1 \ldots), \ldots, P_{i(1)}^2, \ldots, P_{i(k)}^2, \ldots, P_m^2) \ldots;$$

take

$$M_3 \equiv \ldots (P_1^2, \ldots, (\ldots \Delta_1' \ldots), \ldots, P_{i(1)}^2, \ldots, P_{i(k)}^2, \ldots, P_m^2) \ldots$$

and the result is clear from the compatibility.

Subcase (4): similarly to $\beta\beta(4)$.

Case $\beta\mu$: $\Delta_2 \equiv (P_{(1)}^2, \ldots, P_{(k)}^2$, $\Delta_2' \equiv P^2$.

Subcase (2): impossible.

Subcase (3): $\Delta_1 \subset P_{(i)}^2$ (remember that Δ's are particular occurrences of redexes!), then

$$M \equiv \ldots (P_{(1)}^2, \ldots, (\ldots \Delta_1 \ldots)_{(i)}, \ldots, P_{(k)}^2) \ldots, \text{ where } \ldots \Delta_1 \ldots \equiv P^2,$$
$$M_1 \equiv \ldots (P_{(1)}^2, \ldots, (\ldots \Delta_1' \ldots)_{(i)}, \ldots, P_{(k)}^2) \ldots,$$
$$M_2 \equiv \ldots P^2 \ldots;$$

take

$$M_3 = \ldots (\ldots \Delta_1' \ldots) \ldots$$

and the result is clear from $\ldots \Delta_1 \ldots \equiv P^2$ using compatibility of Γ.

Subcase (4): similarly to $\beta\beta(4)$.

Take η-reduction: $\Delta_1 \equiv \lambda x_1 \ldots x_n . P^1(x_1, \ldots, x_n)$,

$\Delta_1' \equiv P^1$.

Case $\eta\eta$: $\Delta_2 \equiv \lambda y_1 \dots y_m . P^2(y_1, \dots, y_m)$, $\Delta'_2 \equiv P^2$.

Subcase (2): $\Delta_1 \equiv \Delta_2$, i.e.

$$M \equiv \dots \lambda x_1 \dots x_n(P^1(x_1, \dots, x_n)) \dots ,$$
$$M_1 \equiv \dots P^1 \dots ,$$
$$M_2 \equiv \dots P^1 \dots ;$$

trivially $M_3 \equiv M_1 \equiv M_2$.

Subcase (3): $\Delta_1 \subset P^2$, then

$$M \equiv \dots \lambda y_1 \dots y_m((\dots \Delta_1 \dots)(y_1, \dots, y_m)) \dots , \quad \text{where} \quad \dots \Delta_1 \dots \equiv P^2,$$
$$M_1 \equiv \dots \lambda y_1 \dots y_m((\dots \Delta'_1 \dots)(y_1, \dots, y_m)) \dots ,$$
$$M_2 \equiv \dots (\dots \Delta_1 \dots) \dots ,$$

take

$$M_3 \equiv \dots (\dots \Delta'_1 \dots) \dots$$

and the result is clear.

Subcase (4): similarly to (3).

Case $\eta\pi$: $\Delta_2 \equiv (P^2_1, \dots, P^2_m)_{(i)}$, $\Delta'_2 \equiv P^2_i$.

Subcase (2): impossible.

Subcase (4): similarly to $\beta\pi(3)$.

Subcase (4): similarly to $\eta\eta(4)$.

Case $\eta\tau$: $\Delta_2 \equiv \lambda y_1 \dots y_i \dots y_m . P^2$,

$$\Delta'_2 \equiv \lambda y_1 \dots y_{i_1} \dots y_{i_k} \dots y_m . P^2[y_i/(y_{i_1}, \dots, y_{i_k})].$$

Subcase (2): $M \equiv \dots \lambda y_1 \dots y_i \dots y_m(P^1(y_1, \dots, y_i, \dots, y_m)) \dots ,$

$$M_1 \equiv \dots P^1 \dots ,$$
$$M_2 \equiv \dots \lambda y_1 \dots y_{i_1} \dots y_{i_k} \dots y_m(P^1(y_1, \dots (y_{i_1}, \dots, y_{i_k}), \dots, y_m)) \dots ;$$

take

$$M_3 \equiv \dots P^1 \dots$$

and the result follows using σ, π and η reductions to the $\lambda^\times$-term M_2.

Subcase (3): similarly to $\beta\tau(3)$.

Subcase (4): similarly to $\eta\eta(4)$.

Case $\eta\sigma$: $\Delta_2 \equiv (P^2_1, \dots, P^2_i, \dots, P^2_m)$,

$$\Delta'_2 \equiv (P^2_1, \dots, P^2_{i(1)}, \dots, P^2_{i(k)}, \dots, P^2_m).$$

Subcase (2): impossible.

Subcase (3): similarly to $\beta\sigma(3)$.

Subcase (4): similarly to $\eta\eta(4)$.

Case $\eta\mu$: $\Delta_2 \equiv (P^2_{(1)}, \dots, P^2_{(k)})$, $\Delta'_2 \equiv P^2$.

Subcase (2): impossible.

Subcase (3): similarly to $\beta\mu(3)$.

Subcase (4): similarly to $\eta\eta(4)$.

Take π-reduction: $\Delta_1 \equiv (P^1_1, \dots, P^1_n)_{(j)}$, $\Delta'_1 \equiv P^1_j$.

Case $\pi\pi$: $\Delta_2 \equiv (P_1^2, \ldots, P_m^2)_{(k)}$, $\Delta_2' \equiv P_k^2$.

Subcase (2): then $M_1 \equiv M_2$ and so we can take $M_3 \equiv M_1$.

Subcase (3): (a) $\Delta_1 \subset P_i^2$, $i \neq k$, then

$$M \equiv \ldots (P_1^2, \ldots, (\ldots \Delta_1 \ldots), \ldots, P_m^2)_{(k)} \ldots, \text{ where } \ldots \Delta_1 \ldots \equiv P_i^2,$$
$$M_1 \equiv \ldots (P_1^2, \ldots, (\ldots \Delta_1' \ldots), \ldots, P_m^2)_{(k)} \ldots,$$
$$M_2 \equiv \ldots P_k^2 \ldots;$$

take

$$M_3 \equiv M_2$$

and the result is clear.

(b) $\Delta_1 \subset P_k^2$, then

$$M \equiv \ldots (P_1^2, \ldots, (\ldots \Delta_1 \ldots), \ldots, P_m^2)_{(k)} \ldots, \quad \text{where} \quad \ldots \Delta_1 \ldots \equiv P_k^2,$$
$$M_1 \equiv \ldots (P_1^2, \ldots, (\ldots \Delta_1' \ldots), \ldots, P_m^2)_{(k)} \ldots,$$
$$M_2 \equiv \ldots (\ldots \Delta_1 \ldots) \ldots;$$

take

$$M_3 \equiv \ldots (\ldots \Delta_1' \ldots) \ldots$$

and the result follows using one reduction.

Subcase (4): similarly to (3).

Case $\pi\tau$: $\Delta_2 \equiv \lambda y_1 \ldots y_i \ldots y_m . P^2$,

$$\Delta_2' \equiv \lambda y_1 \ldots y_{i_1} \ldots y_{i_k} \ldots y_m . P^2[y_i/(y_{i_1}, \ldots, y_{i_k})].$$

Subcase (3): impossible.

Subcase (3): similarly to $\beta\tau(3)$.

Subcase (4): similarly to $\pi\pi(4)$.

Case $\pi\sigma$: $\Delta_2 \equiv (P_1^2, \ldots, P_i^2, \ldots, P_m^2)$,

$$\Delta_2' \equiv (P_1^2, \ldots, P_{i(1)}^2, \ldots, P_{i(k)}^2, \ldots, P_m^2).$$

Subcase (2): impossible.

Subcase (3): similarly to $\beta\sigma(3)$.

Subcase (4): similarly to $\pi\pi(4)$.

Case $\pi\mu$: $\Delta_2 \equiv (P_{(1)}^2, \ldots, P_{(k)}^2)$, $\Delta_2' \equiv P^2$.

Subcase (2): impossible.

Subcase (3): similarly to $\beta\mu(3)$.

Subcase (4): (a) $\Delta_2 \subset P_k^1$ similarly to $\pi\pi(4)$.

(b) $\Delta_1 \equiv \Delta_{2(j)}$, then

$$M \equiv \ldots (P_{(1)}^2, \ldots, P_{(n)}^2)_{(j)} \ldots,$$
$$M_1 \equiv \ldots P_{(j)}^2 \ldots,$$
$$M_2 \equiv \ldots P_{(j)}^2 \ldots;$$

take $M_3 \equiv M_2 \equiv M_1$ and the result is clear.

Take τ-reduction: $\Delta_1 = x_1 \ldots x_j \ldots x_n . P_1$,

$$\Delta_1' = x_1 \ldots x_{j_1} \ldots x_{j_k} \ldots x_n . P^1[x_j/(x_{j_1}, \ldots, x_{j_k})].$$

Case $\tau\tau$: $\Delta_2 \equiv \lambda y_1 \ldots y_i \ldots y_m . P^2$,

$$\Delta_2' \equiv \lambda y_1 \ldots y_{i_1} \ldots y_{i_l} \ldots y_m . P^2[y_i/(y_{i_1}, \ldots, y_{i_l})].$$

Subcase (2): $\Delta_1 \equiv \Delta_2$:

$$M \equiv \ldots \lambda x_1 \ldots x_i \ldots x_j \ldots x_n(P^1) \ldots,$$
$$M_1 \equiv \ldots \lambda x_1 \ldots x_i \ldots x_{j_1} \ldots x_{jk} \ldots x_n(P^1[x_j/(x_{j_1}, \ldots, x_{j_k})] \ldots,$$
$$M_2 \equiv \ldots \lambda x_1 \ldots x_{i_1} \ldots x_{i_l} \ldots x_j \ldots x_n(P^1[x_i/(x_{i_1}, \ldots, x_{i_l})]) \ldots;$$

take

$$M_3 \equiv \ldots \lambda x_1 \ldots x_{i_1} \ldots x_{i_l} \ldots x_{j_1} \ldots x_{j_k} \ldots x_n(P^1[x_i/(x_{i_1}, \ldots, x_{i_l}), x_j/(x_{j_1}, \ldots, x_{j_k})]) \ldots$$

and the result follows using substitution lemma (or, in case of $i = j$, as a trivial $M_3 \equiv M_1 \equiv M_2$).

Subcase (3): similarly to $\eta\eta(3)$ using substitutivity of $\boldsymbol{\Gamma}$.

Subcase (4): similarly to (3).

Case $\tau\sigma$: $\Delta_2 \equiv (P_1^2, \ldots, P_i^2, \ldots, P_m^2)$,
$\Delta_2' \equiv (P_1^2, \ldots, P_{i(1)}^2, \ldots, P_{i(k)}^2, \ldots, P_m^2)$.

Subcase (2): impossible.

Subcase (3): similarly to $\beta\sigma(3)$.

Subcase (4): similarly to $\tau\tau(4)$.

Case $\tau\mu$: $\Delta_2 \equiv (P_{(1)}^2, \ldots, P_{(m)}^2)$, $\Delta_2' \equiv P^2$.

Subcase (2): impossible.

Subcase (3): similarly to $\beta\mu(3)$.

Subcase (4): similarly to $\tau\tau(4)$.

Take σ-reduction: $\Delta_1 \equiv (P_1^1, \ldots, P_j^1, \ldots, P_n^1)$,
$\Delta_1' \equiv (P_1^1, \ldots, P_{j(1)}^1, \ldots, P_{j(k)}^1, \ldots, P_n^1)$.

Case $\sigma\sigma$: $\Delta_2 \equiv (P_1^2, \ldots, P_i^2, \ldots, P_m^2)$,
$\Delta_2' \equiv (P_1^2, \ldots, P_{i(1)}^2, \ldots, P_{i(l)}^2, \ldots, P_m^2)$.

Subcase (2): clear, abalogously to $\tau\tau(2)$.

Subcase (3): similarly to $\beta\sigma(3)$.

Subcase (4): similarly to (3).

Case $\beta\mu$: $\Delta_2 \equiv (P_{(1)}^2, \ldots, P_{(m)}^2)$, $\Delta_2' \equiv P^2$.

Subcase (2): impossible (type restrictions!).

Subcase (3): similarly to $\beta\mu(3)$.

Subcase (4): similarly to $\sigma\sigma(4)$.

Take μ-reduction: $\Delta_1 \equiv (P_{(1)}^1, \ldots, P_{(n)}^1)$, $\Delta_1' \equiv P^1$.

We have the last $\mu\mu$: $\Delta_2 \equiv (P_{(1)}^2, \ldots, P_{(m)}^2)$, $\Delta_2' \equiv P^2$.

Subcase (2): then $M_1 \equiv M_2$ and we can take $M_3 \equiv M_1 \equiv M_2$.

Subcase (3): similarly to $\beta\mu(3)$.

Subcase (4): similarly to (3). □

Lemma. $\boldsymbol{\Gamma}$ is *SN*.

Proof. To prove the lemma, we shall use a generalization of the elegant method shown in [5]. The method uses monotonocity of $\lambda^\times I$-terms.

Remark. $\lambda^\times I$*-terms* are defined as subset of $\Lambda^{\mathbf{B}}$ satisfying:

(i) variables are $\lambda^\times I$-terms;

(ii) if X, Y are $\lambda^\times I$-terms, then (XY) is $\lambda^\times I$-term;

(iii) if X is $\lambda^\times I$-term with free occurrences of variables $x_1, \ldots, x_n$, then $\lambda x_1 \ldots x_n . X$ is $\lambda^\times I$-term;

(iv) if $X_1, \ldots, X_n$ are $\lambda^\times I$-terms with the same sets of free variables, then $(X_1, \ldots, X_n)$ is $\lambda^\times I$-term;

(v) if X is $\lambda^\times I$-term, then $X_{(i)}$ is $\lambda^\times I$-term;

(vi) if $X_1, \ldots, X_n$ are $\lambda^\times I$-terms such that $\lambda^\times$-term $(X_1, \ldots, X_n)$ is of normal type $(\xi_1, \ldots, \xi_m)$ and Y is $\lambda^\times I$-term [of type $((\ldots(\eta\xi_m)\ldots)\,\xi_1)$], then $Y(X_1, \ldots \ldots, X_n)_{(1)} \ldots (X_1, \ldots, X_n)_{(m)}$ is $\lambda^\times I$-term.

Suppose the base $\mathbf{B} = \{o_1, \ldots, o_n\}$. Let us interpret every member of the base assigning to o_i a set $\mathbf{T}_{o_i} = \mathscr{I}(o_i)$ ordered by orderings $<_{o_i}$, $1 \leqq i \leqq n$. Let us define collections $\mathbf{H}_\alpha$ (for normal types α) of hereditarily monotonic members of the type structure in the following way:

$$(\mathbf{H}_{o_i}, <_{o_i}) = (\mathbf{T}_{o_i}, <_{o_i})$$

$$\mathbf{H}_{\beta\alpha} = \{\mathrm{f} \in \mathbf{T}_{\beta\alpha}: \forall \mathrm{a}, \mathrm{a}' \in \mathbf{H}_\alpha . \mathrm{fa} \in \mathbf{H}_\beta \wedge (\mathrm{a} <_\alpha \mathrm{a}' \Rightarrow \mathrm{fa} <_\beta \mathrm{fa}')\}$$

$$\text{for} \quad \mathrm{f}, \mathrm{g} \in \mathbf{H}_{\beta\alpha}: \mathrm{f} <_{\beta\alpha} \mathrm{g} \quad \text{iff} \quad \forall \mathrm{a} \in \mathbf{H}_\alpha . \mathrm{fa} <_\beta \mathrm{ga}$$

$$\mathbf{H}_{(\alpha_1, \ldots, \alpha_n)} = \mathbf{H}_{\alpha_1} \times \ldots \times \mathbf{H}_{\alpha_n}$$

$$(\mathrm{a}_1, \ldots, \mathrm{a}_n) <_{(\alpha_1, \ldots, \alpha_n)} (\mathrm{a}'_1, \ldots, \mathrm{a}'_n) \quad \text{iff} \quad \mathrm{a}_i <_{\alpha_i} \mathrm{a}'_i \quad \text{for all } i, \quad 1 \leqq i \leqq n .$$

[5] contains the proof that under any interpretation of free variables of $\lambda^\times I$-term N by values in $\mathbf{H}$'s, the value of N in the interpretation is monotonic and is contained in $\mathbf{H}_\eta$, where η is the type of N (one only needs to extend the proof also to the case (vi) of our definition of $\lambda^\times I$-terms; however, it is sufficient to note that if the value of $(X_1, \ldots, X_n)$ is some $(\mathrm{a}_1, \ldots, \mathrm{a}_m)$, then the $\lambda^\times$-terms $(X_1, \ldots, X_n)_{(1)}, \ldots \ldots, (X_1, \ldots, X_n)_{(m)}$ will have their denotations $\mathrm{a}_1, \ldots, \mathrm{a}_m$, respectively). Therefore, we can use the orderings $<_\alpha$ for the $\lambda^\times I$-terms if we assume their symbols to be interpreted only by values from $\mathbf{H}$'s. Using $\lambda^\times I$-terms, we shall not consider μ-reduction in order to make standard arguments about the $\lambda^\times I$-calculus possible. Assuming that every numerical term has a numerical value (the assumption we can take from the ordinary typed λ-calculus, cf. e.g. [9], 2.2), it is possible to introduce such a mapping of $\lambda^\times$-terms into $\lambda^\times I$-terms in which the image of a redex will be greater than that of the corresponding contractum. Let us remark that the exclusion of the μ-reduction from the $\lambda^\times I$-calculus makes no problems because the norms of the $\lambda^\times$-terms will be constructed in such a way that they will not contain μ-redexes.

Choose numerical type o from the base $\boldsymbol{B}$. Let us have symbols 0^o, S_o, $+_o$ for zero (of the type o), successor (of the type o) and addition (of the type $(oo)\,o$).

Define $+I$-*calculus* by extending our $\lambda^\times I$-calculus by these $\lambda^\times I$-terms 0^o, S_o, $+_o$ and the $\lambda^\times I$-terms defined by

$S_{\beta\alpha} = \lambda f^{\beta\alpha} . \lambda x^\alpha . S_\beta(fx)$

$S_{(\alpha_1,\dots,\alpha_n)} = \lambda z^{(\alpha_1,\dots,\alpha_n)} . (S_{\alpha_1}(z_{(1)}), \dots, S_{\alpha_n}(z_{(n)})$, where $(\alpha_1, \dots, \alpha_n)$ is normal type

$+_{\beta\alpha} = \lambda f^{\beta\alpha} \lambda g^{\beta\alpha} . \lambda x^\alpha . (fx) +_\beta (gx)$

$+_{(\alpha_1,\dots,\alpha_n)} = \lambda x^{(\alpha_1,\dots,\alpha_n)} \lambda y^{(\alpha_1,\dots,\alpha_n)} . (x_{(1)} +_{\alpha_1} y_{(n)}, \dots, x_{(n)} +_{\alpha_n} y_{(n)})$ with $(\alpha_1, \dots, \alpha_n)$ being normal type.

Moreover, define $\lambda^\times I$-terms L (of normal types only) by

(i) $L^o = 0^o$

(ii) $L^{oo} = \lambda x^o . x$

(iii) $L^{o(\beta\alpha)} = \lambda f^{\beta\alpha} . L^{o\beta} . fL^\alpha$

(iv) $L^{(\gamma\beta)\alpha} = \lambda x^\alpha . \lambda y^\beta . (L^{\gamma\alpha} x) +_\gamma (L^{\gamma\beta} y)$

(v) $L^{(\alpha_1,\dots,\alpha_n)} = \lambda z^{(\alpha_1,\dots,\alpha_n)} . (L^{o\alpha_1} z_{(1)} +_o \dots +_o (L^{o\alpha_1} z_{(n)})$

(vii) $L^{(\alpha_1,\dots,\alpha_n)\beta} = \lambda y^\beta . (L^{\alpha_1\beta} y, \dots, L^{\alpha_n\beta} y)$.

It is shown in [5] that all these terms belong to $\boldsymbol{H}$'s.

Now, we shall define a transformation embedding $\lambda^\times$-terms into $+I$-terms in such a way that redexes have their images greater than those of contracta. Extending the concepts from [5], we define

$$\lambda^* x_1^{\alpha_1} \dots x_n^{\alpha_n} . M^\beta = \lambda x_1 \dots x_n .$$
$$. S_\beta(M +_\beta \underbrace{S_\beta \dots S_\beta}_{k\text{-times}}(L^{((\dots(\beta\gamma_m)\dots)\gamma_1)}(x_1, \dots, x_n)_{(1)} \dots (x_1, \dots, x_n)_{(m)})),$$

where k is the total number of the occurrences of embedded tuple types contained in $\alpha_1, \dots, \alpha_n$ and $(\alpha_1, \dots, \alpha_n) = (\gamma_1, \dots, \gamma_m)$ (where the right hand side is a normal type). First, define the transformation $*$ of the type symbols as follows:

$$\xi^* = o \text{ iff } \xi \text{ is a member of the base}$$
$$(\eta\xi)^* = (\eta^*\xi^*)$$
$$(\xi_1, \dots, \xi_n)^* = (\xi_1^*, \dots, \xi_n^*)$$

and further assume that every type symbol has been transformed using such a transformation $*$.

Now define transformation $*$ from $\lambda^\times$-terms into $+I$-terms as

$x^* = x$ iff x is a variable

$(MN)^* = M^*N^*$

$(\lambda x_1 \dots x_n . M)^* = \lambda^* x_1 \dots x_n . M^*$

$(M_{(i)})^* = (M^*)_{(i)}$

$$(M_1^*, \dots, M_n^*) = (\underbrace{S_{\beta_1} \dots S_{\beta_1}}_{(k+1)\text{-times}}((M_1^* +_{\alpha_1} (L^{((\dots(\alpha_1\beta_m)\dots)\beta_1)}(M_1^*, \dots, M_n^*)_{(1)} \dots$$
$$\dots (M_1^*, \dots, M_n^*)_{(m)}), \dots, M_n^* +_{\alpha_n} (L^{((\dots(\alpha_n\beta_m)\dots)\beta_1)}(M_1^*, \dots, M_n^*)_{(1)} \dots$$
$$\dots (M_1^*, \dots, M_n^*)_{(m)}))_{(1)}), \dots$$

$$\ldots, \underbrace{S_{\beta_m} \ldots S_{\beta_m}}_{(k+1)\text{-times}}((M_1^* +_{\alpha_1} (L^{((\ldots(\alpha_1\beta_m)\ldots)\beta_1)}(M_1^*, \ldots, M_n^*)_{(1)} \ldots$$

$$\ldots (M_1^*, \ldots, M_n^*)_{(m)}), \ldots, M_n^* +_{\alpha_n} (L^{((\ldots(\alpha_n\beta_m)\ldots)\beta_1)}(M_1^*, \ldots, M_n^*)_{(1)} \ldots$$

$\ldots (M_1^*, \ldots, M_n^*)_{(m)}))_{(m)}))$ where $(\alpha_1, \ldots, \alpha_n) = (\beta_1, \ldots, \beta_m)$ (where the right hand side is a normal type) and k is the total number of embedded tuple types contained in $\alpha_1, \ldots, \alpha_n$.

Using 1.4 of [5], to prove strong normalization of $\boldsymbol{\Gamma}$ it is sufficient to show that whenever

$$M_1 \to M_2,$$

then

$$M_1^* > M_2^*$$

(because then $L_{o\alpha}(M_1^\alpha) > L_{o\alpha}(M_2^\alpha)$ by monotonicity of L's) for each reduction of $\beta, \eta, \pi, \tau, \sigma, \pi$:

β-case: $M_1 \equiv (\lambda x_1 \ldots x_n . P)\, Q$, $M_2 \equiv P[x_1/Q_{(1)}, \ldots, x_n/Q_{(n)}]$;

$$\begin{aligned} M_1^* &= (\lambda^* x_1 \ldots x_n . P^*)\, Q^* = \\ &= (\lambda x_1 \ldots x_n . S(P^* + L x_1 \ldots x_n))\, Q^* = \\ &= S(P^*[x_1/Q^*_{(1)}, \ldots, x_n/Q^*_{(n)}] + L Q^*_{(1)} \ldots Q^*_{(n)} \\ M_2^* &= P^*[x_1/Q^*_{(1)}, \ldots, x_n/Q^*_{(n)}] \end{aligned}$$

and $M_1^* > M_2^*$ follows from the monotonicity of $+I$-terms.

η-case: $M_1 \equiv \lambda x_1 \ldots x_n . P(x_1, \ldots, x_n)$, $M_2 \equiv P$;

$$\begin{aligned} M_1^* &= \lambda^* x_1 \ldots x_n . P^*(S^{k+1}((x_1 + L(x_1 \ldots x_n)_{(1)} \ldots (x_1, \ldots, x_n)_{(m)}, \ldots, x_n + \\ &\quad + L(x_1, \ldots, x_n)_{(1)} \ldots (x_1, \ldots, x_n)_{(m)})_{(1)}, \ldots, S^{k+1}((x_1 + L(x_1, \ldots, x_n)_{(1)} \ldots \\ &\quad \ldots (x_1, \ldots, x_n)_{(m)}, \ldots, x_n + L(x_1, \ldots, x_n)_{(1)} \ldots (x_1, \ldots, x_n)_{(m)})_{(m)})) = \\ &= \lambda x_1 \ldots x_n . S(P^*(S^{k+1}((x_1 + L(x_1, \ldots, x_n)_{(1)} \ldots (x_1, \ldots, x_n)_{(m)}, \ldots, x_n + \\ &\quad + L(x_1, \ldots, x_n)_{(1)} \ldots (x_1, \ldots, x_n)_{(m)})_{(1)}, \ldots, S^{k+1}((x_1 + L(x_1, \ldots, x_n)_{(1)} \ldots \\ &\quad \ldots (x_1, \ldots, x_n)_{(m)}, \ldots, x_n + L(x_1, \ldots, x_n)_{(1)} \ldots (x_1, \ldots, x_n)_{(m)})_{(m)})) + \\ &\quad + S^k L(x_1, \ldots, x_n)_{(1)} \ldots (x_1, \ldots, x_n)_{(m)})) \\ M_2^* &= P^* \end{aligned}$$

and $M_1^* > M_2^*$ because for any $a_1, \ldots, a_m \in \boldsymbol{H}_{\beta_1} \times \ldots \times \boldsymbol{H}_{\beta_m}$ we have $M_1^*(a_1, \ldots, a_m) = S(P^*(S^{k+1}(a_1 + L(\ldots)), \ldots, S^{k+1}(a_m + L(\ldots)) + S^k(L(\ldots))) > P^*(a_1, \ldots, a_m) = M_2^*(a_1, \ldots, a_m)$ from the monotonicity of P^*.

π-case: $M_1 \equiv ((P_1, \ldots, P_n))_{(i)})$, $M_2 \equiv P_i$;

$$\begin{aligned} M_1^* &= (P_1, \ldots, P_n)^*_{(i)} = \\ &= (S(P_1^* + L P_1^* \ldots P_n^*), \ldots, S(P_n^* + L P_1^* \ldots P_n^*))_{(i)} = \\ &= S(P_i^* + L P_1^* \ldots P_n^*) \\ M_2^* &= P_i^*, \end{aligned}$$

and $M_1^* > M_2^*$ follows.

τ-case: $M_1 \equiv \lambda x_1 \dots x_i \dots x_n . P,$

$$M_2 \equiv \lambda x_1 \dots y_{i_1} \dots y_{i_j} \dots x_n . P[x_i/(y_{i_1}, \dots, y_{i_j})];$$

$$M_1^* = \lambda^* x_1 \dots x_i \dots x_n . P^* = \\ = \lambda x_1 \dots x_i \dots x_n . S(P^* + S^k(L(x_1, \dots, x_i, \dots, x_n)_{(1)} \dots \\ \dots (x_1, \dots, x_i, \dots, x_n)_{(m)}))$$

$$M_2^* = \lambda^* x_1 \dots y_{i_1} \dots y_{i_j} \dots x_n . (P[x_i/y_{i_1}, \dots, y_{i_j})])^* = \\ = \lambda x_1 \dots y_{i_1} \dots y_{i_j} \dots x_n . S(P[x_i/(y_{i_1}, \dots, y_{i_j})]^* + \\ + S^l(L(x_1, \dots, y_{i_1}, \dots, y_{i_j}, \dots, x_n)_{(1)} \dots (x_1, \dots, y_{i_1}, \dots, y_{i_j}, \dots, x_n)_{(m)})),$$

and $M_1^* > M_2^*$ follows from $k > l$ and $x_i^* > (y_{i_1}, \dots, y_{i_j})^*$ according to the type restrictions in the rule τ.

σ-case: $M_1 \equiv (P_1, \dots, P_i, \dots, P_n),$

$$M_2 \equiv (P_1, \dots, P_{i(1)}, \dots, P_{i(j)}, \dots, P_n);$$

$$M_1^* = (S^k((P_1^* + LM_{1(1)}^* \dots M_{1(m)}^*, \dots, P_n^* + LM_{1(1)}^* \dots M_{1(m)}^*)_{(1)}), \dots \\ \dots, S^k((P_1^* + LM_{1(1)}^* \dots M_{1(m)}^*, \dots, P_n^* + LM_{1(1)}^* \dots M_{1(m)}^*)_{(m)}))$$

$$M_2^* = (S^k((P_1^* + LM_{2(1)}^* \dots M_{2(m)}^*, \dots, P_{i(1)}^* + LM_{2(1)}^* \dots M_{2(m)}^*, \dots, P_{i(j)}^* + \\ + LM_{2(1)}^* \dots M_{2(m)}^*, \dots, P_n^* + LM_{2(1)}^* \dots M_{2(m)}^*)_{(1)}, \dots \\ \dots, S^k((P_1^* + LM_{2(1)}^* \dots M_{2(m)}^*, \dots, P_{i(1)}^* + LM_{2(1)}^* \dots M_{2(m)}^*, \dots, P_{i(j)}^* + \\ + LM_{2(1)}^* \dots M_{2(m)}^*, \dots, P_n^* + LM_{2(1)}^* \dots M_{2(m)}^*)_{(m)}))$$

and $M_1^* > M_2^*$ follows from $k > l$ (type restrictions!) and the comparison of the corresponding projections $M_{1(1)}^*, \dots, M_{1(m)}^*$ and $M_{2(1)}^*, \dots, M_{2(m)}^*$.

μ-case: $M_1 \equiv (P_{(1)}, \dots, P_{(n)}),\ M_2 \equiv P;$

$$M_1^* = (S(P_{(1)}^* + LP_{(1)}^* \dots P_{(n)}^*), \dots, S(P_{(n)}^* + LP_{(1)}^* \dots P_{(n)}^*))$$

$$M_2^* = P^* = (P_{(1)}^*, \dots, P_{(n)}^*),$$

and $M_1^* > M_2^*$ follows.

Therefore, according to the properties of $*$, reductions from Γ cannot create an infinite sequence. □

CONCLUSION

Theorem. The notion of reduction $\boldsymbol{\Gamma}$ is Church-Rosser and strongly normalizing. The proof follows from the lemmata above and from $\boldsymbol{WCR} \wedge \boldsymbol{SN} \Rightarrow \boldsymbol{CR}$ (cf., e.g., [1], 3.1.25).

Corollary. The $\lambda^\times$-calculus has the well-known pleasant properties implied by ***CR*** and ***SN***, e.g.:

(i) to every $\lambda^\times$-term of the calculus with tuple types there exists a uniquely determined normal form;

(ii) the normal form of any $\lambda^\times$-term is reached after a finite number of reductions;

(iii) any reduction strategy leads to the normal form;
(iv) two $\lambda^\times$-terms are equal in the theory induced by Γ iff they have identical normal forms;
(v) the equality of $\lambda^\times$-terms in the theory induced by Γ is decidable.

Example. Suppose U, V are terms of respective types $(\alpha_1, \alpha_2, \alpha_3)$ and $\beta(\alpha_1, \alpha_2, \alpha_3)$, with $(\alpha_1, \alpha_2, \alpha_3)$ being normal. Let us show how $\lambda^\times$-term

$$T = (\lambda x^{\alpha_3} y^{(\alpha_1, \alpha_2)} \, . \, V(y, x)) ((U_{(3)}, U_{(1)}), U_{(2)}) ,$$

(where, as it is easy to see, ordinary β-reduction cannot be performed because immeadiate substitutions are not defined) will be transformed using our reductions:

$$\begin{aligned} T &\to_\sigma (\lambda xy \, . \, V(y, x)) ((U_{(3)}, U_{(1)})_{(1)}, (U_{(3)}, U_{(1)})_{(2)}, U_{(2)}) \to^* \\ &\to^*_{\tau\pi\pi} (\lambda x y_1^{\alpha_1} y_2^{\alpha_2} \, . \, V((y_1, y_2), x)) (U_{(3)}, U_{(1)}, U_{(2)}) \to^* \\ &\to^*_{\sigma\pi\pi} (\lambda x y_1 y_2 \, . \, V(y_1, y_2, x)) (U_{(3)}, U_{(1)}, U_{(2)}) \to \\ &\to_\beta V((U_{(3)}, U_{(1)}, U_{(2)})_{(2)}, (U_{(3)}, U_{(1)}, U_{(2)})_{(3)}, (U_{(3)}, U_{(1)}, U_{(2)})_{(1)}) \to^* \\ &\to^*_{\pi\pi\pi} V(U_{(1)}, U_{(2)}, U_{(3)}) \to_\mu VU. \end{aligned}$$

Clearly, our reductions enable to transform the $\lambda^\times$-terms in the naive way, just as one expects.

ACKNOWLEDGMENTS

This work has been supported by the Research Project I-5-7/02.

(Received August 24, 1984.)

REFERENCES

[1] H. P. Barendregt: The Lambda Calculus. Its syntax and semantics. (Studies in Logic and the Foundations of Mathematics 103.), North-Holland, Amsterdam 1981.
[2] A. Church: A formulation of the simple theory of types. J. Symb. Logic *5* (1948), 1, 56–68.
[3] A. Church: The Calculi of λ-conversion. (Annals of Mathematics Studies No. 6.), Princeton University Press, Princeton 1941 (1951).
[4] R. O. Gandy: An early proof of normalization by A. M. Turing. In: To H. B. Curry: Essays on Combinatory Logic, Lambda-calculus and Formalism (J. R. Hindley, J. P. Seldin, eds.), Academic Press, London 1980, pp. 453–456.
[5] R. O. Gandy: Proofs of strong normalization. In: [4], pp. 457–477.
[6] M. H. A. Newman: On theories with a combinatorial definition of "equivalence". Ann. of Math. (2), *43* (1942), 223–243.
[7] D. S. Scott: Lectures on a Mathematical Theory of Computation. Oxford University Computing Laboratory, Technical Monograph PRG-19, 1981.
[8] D. S. Scott: Relating theories of the λ-calculus. In: [4], pp. 403–450.
[9] A. S. Troelstra (ed.): Metamathematical Investigations of Intuitionistic Arithmetic and Analysis (Lecture Notes in Mathematics 344). Springer-Verlag, Berlin 1973.
[10] J. Zlatuška: HIT data model. Data bases from the functional point of view. In: Proc. 11th VLDB (A. Pirotte, Y. Vassiliou, eds.), Stockholm 1985, pp. 470–477.

RNDr. Jiří Zlatuška, Ústav výpočetní techniky UJEP (Computer Science Department — University of Brno), Kotlářská 2, 611 37 Brno. Czechoslovakia.