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CONJUGATED AND SYMMETRIC POLYNOMIAL EQUATIONS
II: Discrete-Time Systems

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This paper investigates the properties of conjugated and symmetric polynomial equations which occur in the synthesis of discrete quadratically optimal controllers.

INTRODUCTION

This paper is devoted to the equation

$$a(\zeta^{-1})\,x(\zeta) + a(\zeta)\,x(\zeta^{-1}) = b(\zeta) + b(\zeta^{-1}) \tag{1}$$

where $a(\zeta)$, $b(\zeta)$ are given real polynomials of indeterminate ζ or ξ^{-1}, $x(\zeta)$ is an unknown polynomial. The equation plays the same role in the discrete control theory [1] as the corresponding equation [2] does in the continuous one, see Part I. A related equation

$$a(\zeta)\,x(\zeta^{-1}) + b(\zeta^{-1})\,y(\zeta) = c(\zeta) + d(\zeta^{-1}) \tag{2}$$

with two unknown polynomials is also studied.

Unlike the continuous case, the equation above cannot be made equivalent to a polynomial equation

$$\alpha(\lambda)\,\xi(\lambda) + \beta(\lambda)\,\eta(\lambda) = \gamma(\lambda) \tag{3}$$

whose theory is well developed [3]. General solution of (3) contains polynomials of arbitrary high degree. As will be seen, general solution of (1) or (2) is always of finite dimension. Mathematical tools for (1), (2) must be built separately from those for (3) but they are very similar to them. This is done in the sequel.

PRELIMINARIES

Instead of the ring of polynomials, it is better to work here with the ring of "two-sided polynomials" (tsp). It can be defined as a ring of expressions $f(\zeta) = \sum_{i=\eta f}^{\partial f} f_i \zeta^i$

with obvious rules for addition and multiplication. Here ηf, ∂f are integers (positive, zero or negative) which can be named "lower degree" resp. "upper degree"; $\partial f \geqq \eta f$ otherwise $f(\zeta)$ is the zero tsp. The following rules are evident:

$$\partial(f+g) \leqq \max(\partial f, \partial g) \quad \eta(f+g) \geqq \min(\eta f, \eta g)$$
$$\partial(fg) = \partial f + \partial g \qquad \eta(fg) = \eta f + \eta g$$

A given f is a polynomial iff $\eta f \geqq 0$, polynomials form a subring. The ring of tsp's has many properties common with that of polynomials: it is an integral domain, Euclidean domain, principal ideal domain, unique factorization domain. But units (elements which have an inverse) are different: every $K\zeta^r$ is a unit, K being a nonzero number. Elements f, $f' = K\zeta^r f$ are associated, their divisibility properties are the same. Every tsp is associated to a polynomial. In prime factor decomposition, only factors $(\zeta - \zeta_i)^k$ where $\zeta_i \neq 0$ are considered primes.

An operation of "conjugation" is defined $f^*(\zeta) = f(\zeta^{-1})$. It is evident that

$$(f+g)^* = f^* + g^*, \quad (fg)^* = f^*g^*, \quad f^{**} = f,$$
$$\partial f^* = -\eta f, \quad \eta f^* = -\partial f$$

To every tsp, two polynomials are defined

$$\{f\}_+ = \frac{f_0}{2} + \sum_{i=1}^{\partial f} f_i\zeta^i, \quad \{f\}_- = \frac{f_0}{2} + \sum_{i=\eta f}^{-1} f_i\zeta^{-i} \tag{4}$$

(right part, left part). It is evident that $\{f^*\}_+ = \{f\}_-$. The f can be decomposed as $f = \{f\}_+ + \{f\}_-^*$, $f^* = \{f\}_- + \{f\}_+^*$.

In certain circumstances, another definition is more convenient:

$$[f]_+ = \sum_{i=1}^{\partial f} f_i\zeta^i \quad [f]_- = \sum_{i=\eta f}^{-1} f_i\zeta^{-i} \tag{5}$$

(strictly right part, strictly left part). It is $[f^*]_+ = [f]_-$, the decomposition is $f = [f]_+ + f_0 + [f]_-^*$, $f^* = [f]_- + f_0 + [f]_*^+$.

In the next chapters, we shall need three following lemmas. The first of them is a variant of the Euclidean algorithm.

Lemma 1. Let b be a tsp, a a polynomial. Then $b = aq^* + r$ where q, r are polynomials.

Proof. For $\eta b \geqq 0$ the claim holds, it is $q = 0$, $r = b$. For $\eta b < 0$ we shall prove it by induction. Let the lemma holds for all tsp's whose lower degree is greater than ηb. We construct

$$b' = b - \frac{b_{\eta b}}{a_{\eta a}} \zeta^{\eta b - \eta a} a, \quad \eta b < \eta b' \leqq 0$$

From the induction assumption $b' = aq'^* + r$,

$$b = aq'^* + r + \frac{b_{\eta b}}{a_{\eta a}}\zeta^{\eta b - \eta a}a = \left(q'^* + \frac{b_{\eta b}}{a_{\eta a}}\zeta^{\eta b - \eta a}\right)a + r \qquad \square$$

The next two lemmas concern greatest common divisors in the ring of tsp's.

Lemma 2. For every polynomial a, $g = \gcd(a, a^*)$ can be selected so that it satisfies one of the four following conditions:

$$\text{(a) } g^* = g \quad \text{(b) } g^* = -g \quad \text{(c) } g^* = \zeta^{-1}g \quad \text{(d) } g^* = -\zeta^{-1}g$$

Proof. Let h be any $\gcd(a, a^*)$. We shall find a new gcd g with the property required, $h = eg$ where e is a unit.

First, we shall prove $h^* = \gcd(a^*, b^*)$ follows from $h = \gcd(a, b)$. Actually, from $a = a_0h$, $b = b_0h$ by taking the conjugates $a^* = a_0^*h^*$, $b^* = b_0^*h^*$ we see that h^* is a common divisor of a^*, b^*. To prove that it is the greatest one, let us suppose that k is another common divisor of a^*, b^*. By taking the conjugates we see that k^* is a common divisor of a, b. So k^* divides h, k divides h^*, h^* is the greatest one.

Second, applying this for a, a^* we see that h^* is $\gcd(a, a^*)$ as well as h is. Therefore h, h^* are associated: $h^* = uh$ where u is a unit. By substituting here its own conjugate, $h^* = uu^*h^*$, $uu^* = 1$. We see that u is not only unit ($1/u$ exists) but unitary ($1/u = u^*$). All units are $K\zeta^r$ (K nonzero real number, r integer), all unitaries are $\pm\zeta^r$. For a new gcd, it holds $e^*g^* = ueg$, $g^* = u(e/e^*)\,g$. All elements e/e^* are ζ^{2s} with integer s. For a given factor $u = \pm\zeta^r$, a compensating factor ζ^{2s} always can be found so that ue/e^* has one of four values: 1, -1, ζ, $-\zeta$. $\square$

Lemma 3. Let $g = \gcd(a, a^*)$ satisfy one of the conditions (a), (b), (c), (d) in Lemma 2. Then g can be expressed:

$$\text{a) } g = w^*a + wa^* \quad \text{b) } g = -w^*a + wa^*$$
$$\text{c) } g = \zeta w^*a + wa^* \quad \text{d) } g = -\zeta w^*a + wa^*$$

where w is a tsp.

Proof. Because the ring of tsp's is a principal ideal ring, g can be expressed as $g = u^*a + va^*$ where u, v are tsp's. We have:

$$\text{a) } g^* = g\,, \quad g = \frac{g + g^*}{2} = \frac{(v + u)^*}{2}a + \frac{v + u}{2}a^*$$

$$\text{b) } g^* = -g\,, \quad g = \frac{g - g^*}{2} = -\frac{(v - u)^*}{2}a + \frac{v - u}{2}a^*$$

c) $g^* = \zeta^{-1}g, \quad g = \dfrac{g + \zeta g^*}{2} = \zeta\dfrac{(v + \zeta u)^*}{2}a + \dfrac{v + \zeta u}{2}a^*$

d) $g^* = -\zeta^{-1}g, \quad g = \dfrac{g - \zeta g^*}{2} = -\zeta\dfrac{(v - \zeta u)^*}{2}a + \dfrac{v - \zeta u}{2}a^*$ □

THE NON-SYMMETRIC EQUATION THEORY

The equation (2) is investigated first. It will be used later for the symmetric case but the equation has other applications as well.

Theorem 1. Let a, b be polynomials. Consider the equation with two unknowns

$$ax^* + b^*y = 0 \tag{6}$$

Let $g = \gcd(a, b^*)$ such that $a_0 = a/g$, $b_0 = b/g^*$ are polynomials (it is always possible and $\eta g \leqq \eta a$). The general solution in the ring of tsp's is

$$x = b_0(u + v^*), \quad y = -a_0(u^* + v) \tag{7}$$

where u, v are arbitrary polynomials. The solution in the ring of polynomials has the following properties:

1) $$\partial u \leqq \eta a - \eta g, \quad \partial v \leqq \eta b + \partial g$$

2) for a nonzero solution

$$\partial b - \eta b - (\partial g - \eta g) \leqq \partial x \leqq \partial b + \eta a, \quad \partial a - \eta a - (\partial g - \eta g) \leqq \partial y \leqq \partial a + \eta b$$

3) the general solution is a subspace of dimension $\eta a + \eta b + \partial g - \eta g + 1$.

Proof. First of all, we shall prove the possibility of choosing $g : \partial g^* = -\eta g$, $\eta g^* = -\partial g$, $\partial a_0 = \partial a - \partial g$, $\eta a_0 = \eta a - \eta g$, $\partial b_0 = \partial b + \eta g$, $\eta b_0 = \eta b + \partial g$. Expressions a_0, b_0 will be polynomials ($\eta a_0 \geqq 0$, $\eta b_0 \geqq 0$) if $\eta g \leqq \eta a$, $\partial g \geqq -\eta b$. This is always possible, e.g. by $\eta g = 0$, $\partial g \geqq 0$.

By cancellation with g, (6) turns into an equivalent equation

$$a_0x^* + b_0^*y = 0 \tag{8}$$

where a_0, b_0^* are coprime. The general solution in the ring of tsp's is $x = b_0q$, $y = -a_0q^*$ where q is an arbitrary tsp. Firstly, by substitution we see that every such couple satisfies the equation. Secondly, we shall prove that every solution is of such form. The second term in (8) is divisible by b_0^*, so must be the first one. But a_0, b_0^* are coprime, hence (in a principal ideal domain) x^* must be divisible by b_0^*, x by b_0. For the same reason, y is divisible by a_0. Let us write $x = b_0q$, $y = a_0r$. By substituting into (8) we have $r = -q^*$. Denoting $\{q\}_+ = u$, $\{q\}_- = v$, we have the form to be proved.

For x, y to be polynomials, it must be

$$(9) \qquad \partial v \leqq \eta b_0 = \eta b + \partial g\,, \quad \partial u \leqq \eta a_0 = \eta a - \eta g$$

For a nonzero solution it is

$$\partial b_0 - \eta v \leqq \partial x \leqq \partial b_0 + \partial u\,, \quad \partial a_0 - \eta u \leqq \partial y \leqq \partial a_0 + \partial v$$

From here by means of $\eta v \leqq \partial v \leqq \eta b + \partial g$, $\eta u \leqq \partial u \leqq \eta a - \partial g$ we obtain

$$\partial b - \eta b - (\partial g - \eta g) \leqq \partial x \leqq \partial b + \eta a\,, \quad \partial a - \eta a - (\partial g - \eta g) \leqq \partial y \leqq \partial a + \eta b$$

The fact that the general solution is a subspace follows from linearity of the equation. According to (9) the solution contains at most $(\eta a - \eta g + 1) + (\eta b + \partial g + 1)$ arbitrary coefficients but the coefficient v_0 contributes nothing more than u_0. Hence the dimension is at most $\eta a + \eta b + \partial g - \eta g + 1$. Taking

$$u = \zeta^i\,, \quad v = 0\,, \quad i = 0, 1 \ldots \eta a - \eta g$$
$$u = 0\,, \quad v = \zeta_j\,, \quad j = 1 \quad \ldots \eta b + \partial g$$

we obtain a set of linearly independent solutions which is a basis and the dimension is really $\eta a + \eta b + \partial g - \eta g + 1$. □

Theorem 2. Let a, b, c, d be given polynomials. The equation

$$(10) \qquad ax^* + b^*y = c + d^*$$

is solvable in the ring of polynomials iff $g = \gcd(a, b^*)$ divides $c + d^*$. Every solution satisfies:

for $\partial c \leqq \partial a$: $\quad \partial y \leqq \partial a + \eta b$
for $\partial c > \partial a$: $\quad \partial y = \partial c + \eta b$
for $\partial d \leqq \partial b$: $\quad \partial x \leqq \partial b + \eta a$
for $\partial d > \partial b$: $\quad \partial x = \partial d + \eta a$

Among all solutions, the unique "y-minimal" one exists defined by:

for $\partial c \leqq \partial a$: $\quad \partial y < \partial a - \eta a - (\partial g - \eta g)$
for $\partial c > \partial a$: coefficients $y_i = 0$ for $\partial a - \eta a - (\partial g - \eta g) \leqq i \leqq \partial a + \eta b$

Similarly the unique "x-minimal" solution exists:

for $\partial d \leqq \partial b$: $\quad \partial x < \partial b - \eta b - (\partial g - \eta g)$
for $\partial d > \partial b$: coefficients $x_i = 0$ for $\partial b - \eta b - (\partial g - \eta g) \leqq i \leqq \partial b + \eta a$

Proof. The condition is necessary: let some x, y satisfies (10), the left-hand side is divisible by g, so must be the right-hand one. Conversely, let g divides $c + d^*$. In the principal ideal ring of tsp's, g can be expressed as

$$g = pa + qb^*$$

where p, q are tsp's. We have

$$c + d^* = rg = rpa + rqb^*$$

We have found a solution in the ring of tsp's:

$$x = r^*p^*, \quad y = rq$$

We shall construct a solution in the ring of polynomials. According to Lemma 1,

$$x = -b_0s^* + x_0, \quad y = a_0t^* + y_0$$

According to Theorem 1, the couple

$$x_h = b_0(s^* + t), \quad y_h = -a_0(s + t^*)$$

satisfies the homogeneous equation. Hence

$$x' = x + b_0(s^* + t) = x_0 + b_0t$$
$$y' = y - a_0(s + t^*) = y_0 - a_0s$$

is a polynomial solution.

Let us investigate the degrees. We shall use the identity $\partial(ax^*) \leqq \partial a$ in (10):

1) for $\partial c \leqq \partial a$ both the right-hand side and the first term of the left-hand one are at most ∂a, so must be the second term:

$$\partial(b^*y) \leqq \partial a, \quad \partial y \leqq \partial a + \eta b$$

2) for $\partial c > \partial a$ the second term of the left-hand side must be of the same degree as the right-hand side:

$$\partial(b^*y) = \partial c, \quad \partial y = \partial c + \eta b$$

Similarly for $\partial d \leqq \partial b$ and $\partial d > \partial b$.

Let us investigate the y-minimal solution. For a while, we suppose a, b^* coprime.

1) For $\partial c \leqq \partial a$, we shall look for a solution x', y' satisfying $\partial y' < \partial a - \eta a$. Let x, y be a solution, $\partial y \geqq \partial a - \eta a$. We take the quotient q and the remainder y_0 in the division $y = \zeta^{-\eta a}aq + y_0$. The couple $x_h = b\zeta^{\eta a}q^*$, $y_h = -a\zeta^{-\eta a}q$ satisfies the homogeneous equation according to Theorem 1. The x_h is really polynomial because $\partial y \leqq \partial a + \eta b$, $\partial q = \partial y - (\partial a - \eta a) \leqq \eta a + \eta b$. Hence

$$x' = x + b\zeta^{\eta a}q^*, \quad y' = y - a\zeta^{-\eta a}q = y_0 .$$

is also a solution and $\partial y' < \partial a - \eta a$. Such a solution is unique: Let x'_1, y'_1 and x'_2, y'_2 be two such solutions. Then $x' = x'_1 - x'_2$, $y' = y'_1 - y'_2$ satisfies the homogeneous equation and $\partial y' < \partial a - \eta a$ which contradicts Theorem 1.

2) For $\partial c > \partial a$, we shall prove that polynomials c', d', y_2, $\partial c' \leqq \partial a$ exist such that every solution x, y of (10) can be expressed

$$y = y_1 + \zeta^{\partial a + \eta b + 1}y_2 \quad \partial y_1 \leqq \partial a + \eta b$$

where y_1 is a solution of

$$ax^* + b^*y_1 = c' + d'^* \tag{11}$$

We shall prove this by induction. For $\partial c \leqq \partial a$ it must be $y_2 = 0, c' = c, d' = d$, the assertion holds. For $\partial c > \partial a$, by matching the highest terms in (10) we have

$$y_{\partial c + \eta b} = \frac{c_{\partial c}}{b_{\eta b}}$$

We can express

$$y = y' + \frac{c_{\partial c}}{b_{\eta b}} \zeta^{\partial c + \eta b}, \quad \partial y' < \partial y$$

By substituting it into (10) we have

$$ax^* + b^*y' = c' + d'^*$$

where

$$c' + d'^* = c + d^* - \frac{c_{\partial c}}{b_{\eta b}} \zeta^{\partial c + \eta b} b^*$$

We see that y' satisfies an equation with $\partial c' < \partial c$. The induction assumption can be used

$$y' = y_1' + \zeta^{\partial a + \eta b + 1} y_2', \quad y = y_1 + \zeta^{\partial a + \eta b + 1} y_2$$

where

$$y_1 = y_1', \quad y_2 = y_2' + \zeta^{\partial c - \partial a - 1} \frac{c_{\partial c}}{b_{\eta b}}$$

So the needed form is found. For y-minimal solution of (10) we take the y_1-minimal solution of (11). It is $\partial y_1 < \partial a - \eta a$, hence coefficients $y_i = 0$ for $\partial a - \eta a \leqq i \leqq \leqq \partial a + \eta b$. The solution is unique because y_2 was constructed in the only possible way and y_1 is the only y_1-minimal solution of (11).

Now let us consider the general case of non-coprime a, b^*.

$$a = a_0 g, \quad b = b_0 g^*, \quad c + d^* = (c_0 + d_0^*)\, g, \quad \partial g^* = -\eta g,$$

$$\eta g^* = -\partial g$$

$$\partial a_0 = \partial a - \partial g, \quad \eta a_0 = \eta a - \eta g, \quad \eta b_0 = \eta b + \partial g, \quad \partial c_0 = \partial c - \partial g$$

The equation turns by cancellation into

$$a_0 x^* + b_0^* y = c_0 + d_0^*$$

For the y-minimal solution we have proved

$$\partial c_0 \leqq \partial a_0: \quad \partial y < \partial a_0 - \eta a_0$$
$$\partial c_0 > \partial a_0: \quad y_i = 0 \quad \text{for} \quad \partial a_0 - \eta a_0 \leqq i \leqq \partial a_0 + \eta b_0$$

It yields

$$\begin{aligned}
\partial c \leqq \partial a : &\quad \partial y < \partial a - \eta a - (\partial g - \eta g) \\
\partial c > \partial a : &\quad y_i = 0 \quad \text{for} \quad \partial a - \eta a - (\partial g - \eta g) \leqq i \leqq \partial a + \eta b
\end{aligned}$$

Properties of the x-minimal solution could be proved in the same way. □

THE NON-SYMMETRIC EQUATION SOLUTION METHOD

In applications, there is usually need for finding certain particular solution of (10). One way is to compute the expression for general solution and then to select the particular one according to specified conditions. Often the y-minimal or x-minimal solution is asked for; in that case a direct computation is possible as follows.

For $\partial c > \partial a$ (both for y-minimal or x-minimal solution problem) we express

$$y = y' + \frac{c_{\partial c}}{b_{\eta b}} \zeta^{\partial c + \eta b}$$

By substituting it into (10) we have

$$ax^* + b^* y' = c' + d'^* \tag{12}$$

where

$$c' + d'^* = c + d^* - \frac{c_{\partial c}}{b_{\eta b}} \zeta^{\partial c + \eta b} b^* , \quad \partial c' < \partial c$$

We have replaced the original equation by the new one of the same form but with lower $\partial c'$. It can be shown that if x, y' is y'-minimal or x-minimal for (12) then x, y is respectively y-minimal or x-minimal for (10). It can also be shown that $\partial d \leqq \leqq \partial b \Rightarrow \partial d' \leqq \partial b$.

Similarly for $\partial b > \partial d$ we have

$$x = x' + \frac{d_{\partial d}}{a_{\eta a}} \zeta^{\partial d + \eta a}$$

$$ax' + b^* x = c' + d'^*$$

$$c' + d'^* = c + d^* - \frac{d_{\partial d}}{a_{\eta a}} \zeta^{-\partial d - \eta a} a$$

$$\partial d' < \partial d , \quad \partial c \leqq \partial a \Rightarrow \partial c' \leqq \partial a .$$

By repeating this process we shall come to the case $\partial c' \leqq \partial a$, $\partial d' \leqq \partial b$. For such an equation, suppose the y-minimal solution is to be found. We multiply (10) by $\zeta^{\partial b}$:

$$a\zeta^{-\eta a} \zeta^{\partial b + \eta a} x^* + b^* \zeta^{\partial b} y = \zeta^{\partial b}(c + d^*)$$

and denote the polynomials

$$\tilde{a} = a\zeta^{-\eta a} , \quad \tilde{b} = b^* \zeta^{\partial b} , \quad \tilde{c} = \zeta^{\partial b}(c + d^*)$$

Because for every solution of (10) $\partial x \leqq \partial b + \eta a$ holds, $\tilde{x} = \zeta^{\partial b + \eta a} x^*$ is a polynomial. Every solution x of (10) can be found via some solution $\tilde{x}$ of the polynomial equation

$$(13) \qquad \tilde{a}\tilde{x} + \tilde{b}y = \tilde{c}\,, \quad x = \zeta^{\partial b + \eta a}\tilde{x}^*$$

(Converse is not true: not every solution $\tilde{x}$ of (13) leads to a solution of (10); for $\partial\tilde{x} > \partial b + \eta a$ the x is not a polynomial). The y-minimal solution of (13) leads to the y-minimal solution of (10).

When the x-minimal solution is to be found we multiply the conjugate of (10) by $\zeta^{\partial a}$:

$$a^*\zeta^{\partial a}x + b\zeta^{-\eta b}\zeta^{\partial a + \eta b}y^* = \zeta^{\partial a}(c^* + d)$$

and denote

$$\tilde{a} = a^*\zeta^{\partial a}\,, \quad \tilde{b} = b\zeta^{-\eta b}\,, \quad \tilde{d} = \zeta^{\partial a}(c^* + d)$$

We have

$$(14) \qquad \tilde{a}x + \tilde{b}\tilde{y} = \tilde{d}\,, \quad y = \zeta^{\partial a + \eta b}\tilde{y}^*$$

The x-minimal solution of (14) leads to the x-minimal solution of (10).

For solving polynomial equations (13), (14) effective methods are known [4].

EXAMPLES

1) $a = 1 + 2\zeta\,, \quad b = 1 + 3\zeta\,, \quad c + d^* = 1\,,$
   x-minimal: $x = -0{\cdot}2\,, \quad y = 0{\cdot}4\zeta$
   y-minimal: $x = 0{\cdot}6\zeta\,, \quad y = -0{\cdot}2\zeta$

2) $a = 1 + 2\zeta\,, \quad b = 1 + 3\zeta\,, \quad c + d^* = \zeta$
   x-minimal: $x = 0{\cdot}6\,, \quad y = -0{\cdot}2\zeta$
   y-minimal: $x = 0{\cdot}5 - 0{\cdot}3\zeta\,, \quad y = -0{\cdot}1$

3) $a = 1 + 2\zeta\,, \quad b = 1 + 3\zeta\,, \quad c + d^* = \zeta^2$
   x-minimal: $x = -1{\cdot}8\,, \quad y = 0{\cdot}6\zeta + \zeta^2$
   y-minimal: $x = -1{\cdot}5 + 0{\cdot}9\zeta\,, \quad y = -0{\cdot}3 + \zeta^2$

THE SYMMETRIC EQUATION THEORY

Theorem 3. Consider the equation

$$ax^* + a^*x = 0$$

where $g = \gcd(a, a^*)$ is chosen to satisfy one of the conditions in Lemma 2. Then $a_0 = a/g$ is a polynomial and the general solution in the ring of tsp's is:

a) for $g^* = g \qquad x = a_0(t - t^*)$
b) for $g^* = -g \qquad x = a_0(t + t^*)$

c) for $g^* = \zeta^{-1} g \quad x = a_0(\zeta t - t^*)$
d) for $g^* = -\zeta^{-1} g \quad x = a_0(\zeta t + t^*)$

where t is an arbitrary polynomial. The solution in the ring of polynomials has $\partial t \leqq \eta a + \partial g$. For cases a, c, d for nonzero solution $\partial a - \partial g < \partial x \leqq \partial a + \eta a$ holds, the general solution is a subspace of dimension $\eta a + \partial g$. For case b for nonzero solution $\partial a - \partial g - 1 < \partial x \leqq \partial a + \eta a$ holds, the general solution is a subspace of dimension $\eta a + \partial g + 1$.

Proof.

a) $\eta g = -\partial g$, $\partial a_0 = \partial a - \partial g$, $\eta a_0 = \eta a + \partial g$, a_0 is a polynomial, a_0, a_0^* coprime. By cancellation with g the equation turns into an equivalent equation

$$a_0 x^* + a_0^* x = 0 \tag{15}$$

It is easily seen that every $x = a_0(t - t^*)$ satisfies (15). Conversely, we shall prove that every solution is of that form. The first term in (15) is divisible by a_0, so must be the second one. But a_0, a_0^* are coprime, hence (in a principal ideal domain) x must be divisible by a_0. We write $x = a_0 q$ where is q a tsp. By substituting this into (15) we have $q^* = -q$. We express q by (4)

$$q = \{q\}_+ + \{q\}_-^* , \quad q^* = \{q\}_- + \{q\}_+^*$$

The condition leads to

$$\{q\}_+ + \{q\}_- = -\{q\}_+^* - \{q\}_-^*$$

The left-hand side is a polynomial in ζ, the right-hand one in ζ^{-1}. Hence it is just an absolute term, $q_0 = -q_0$, $q_0 = 0$. We have $\{q\}_+ + \{q\}_- = 0$ and denote $\{q\}_+ = t$, $x = a_0 q = a_0(\{q\}_+ + \{q\}_-^*) = a_0(t - t^*)$.

The x will be a polynomial if $\eta x = \eta a_0 - \partial t \geqq 0$, $\partial t \leqq \eta a + \partial g$. For $\partial t = 0$ we have $t - t^* = 0$, for a nonzero solution it must be $0 < \partial t \leqq \eta a + \partial g$. From here $\partial x = \partial a_0 + \partial t$, $\partial a - \partial g < \partial x \leqq \partial a + \eta a$.

The fact that the general solution is a subspace follows from linearity of the equation. The polynomial x contains at most $(\partial a + \eta a) - (\partial a - \partial g) = \eta a + \partial g$ coefficients, dimension is at most $\eta a + \partial g$. Taking $t = \zeta^i$, $i = 1 \dots \eta a + \partial g$ we obtain a set of linearly independent solutions which is a basis and the dimension is really $\eta a + \partial g$.

b) $g^* = -g$, again $\eta g = -\partial g$, $\partial a_0 = \partial a - \partial g$, $\eta a_0 = \eta a + \partial g$, the equation obtained by cancellation is $a_0 x^* - a_0^* x = 0$. The general solution is $x = a_0 q$ where $q^* = q$. Expression by means of $\{q\}_+$, $\{q\}_-$ yields $\{q\}_+ - \{q\}_- = \{q\}_+^* - \{q\}_-^*$. Both sides are zero, we have $\{q\}_- = \{q\}_+$ and denote $\{q\}_+ = t$, $x = a_0(t + t^*)$. Again $\eta x = \eta a_0 - \partial t \geqq 0$, $\partial t \leqq \eta a + \partial g$. In this case even for $\partial t = 0$ we have nonzero solutions, hence $0 \leqq \partial t \leqq \eta a + \partial g$, $\partial a - \partial g \leqq \partial x \leqq \partial a + \eta a$, dimension $\eta a + \partial g + 1$, set of base solutions again by taking $t = \zeta^i$.

c) $g^* = \zeta^{-1}g$, $\partial g \geqq 1$, $\eta g = -\partial g + 1$, $\partial a_0 = \partial a - \partial g$, $\eta a_0 = \eta a + \partial g - 1$. The equation obtained by cancellation is $a_0 x^* + \zeta^{-1} a_0^* x = 0$, its general solution $x = a_0 q$, $q^* = -\zeta^{-1} q$. By substituting here the expression for q from (5), we have

$$[q]_- + \zeta^{-1}[q]_+ = -q_0 - q_0\zeta^{-1} - [q]_+^* - \zeta^{-1}[q_-^*]$$

Both sides must be just absolute terms. So $[q]_- + \zeta^{-1}[q]_+ = -q_0$, we can denote $[q]_+ = \zeta t$ and we have $x = a_0([q]_+ + q_0 + [q]_-^*) = a_0(\zeta t - t^*)$. As for the degrees, we have $\eta x = \eta a_0 - \partial t \geqq 0$, $0 \leqq \partial t \leqq \eta a + \partial g - 1$. From here, $\partial x = \partial a_0 + \partial t + 1$, $\partial a - \partial g + 1 \leqq \partial x \leqq \partial a + \eta a$, dimension $\eta a + \partial g$.

d) $g^* = -\zeta^{-1} g$, $a_0 x^* - \zeta^{-1} a_0 x = 0$, $x = a_0 q$, $q^* = \zeta^{-1} q$, $[q]_+ = \zeta t$, $x = a_0(\zeta t + t^*)$. □

Theorem 4. Let a be a polynomial, b a tsp. The equation

$$ax^* + a^*x = b \tag{16}$$

is solvable in the ring of polynomials iff $b^* = b$ and $g = \gcd(a, a^*)$ divides b. Every solution satisfies:

for $\partial b \leqq \partial a$: $\quad \partial x \leqq \partial a + \eta a$
for $\partial b > \partial a$: $\quad \partial x = \partial b + \eta a$

Among all solutions, the unique "minimal" one exists defined by:

$$\left.\begin{array}{l} \text{a) } g^* = g \\ \text{c) } g^* = \zeta^{-1}g \\ \text{d) } g^* = -\zeta^{-1}g \end{array}\right\} \quad \begin{array}{ll} \text{1) } \partial b \leqq \partial a: & \partial x \leqq \partial a - \partial g \\ \text{2) } \partial b > \partial a: & x_i = 0 \ \text{ for } \ \partial a - \partial g < i \leqq \partial a + \eta a \\ & \end{array}$$

$$\text{b) } g^* = -g \qquad \begin{array}{ll} \text{1) } \partial b \leqq \partial a: & \partial x \leqq \partial a - \partial g - 1 \\ \text{2) } \partial b > \partial a: & x_i = 0 \ \text{ for } \ \partial a - \partial g - 1 < i \leqq \partial a + \eta a \end{array}$$

Proof. Both conditions are necessary: let x be a solution, the left-hand side is symmetric, so must be the right-hand one. The left-hand side is divisible by g, so must be the right-hand one. Conversely, let $b^* = b$, $b = b_0 g$. We express g according to Lemma 3:

$$\begin{array}{llll} \text{a) } g^* = g, & b_0^* = b_0, & g = w^*a + wa^* \\ \text{b) } g^* = -g, & b_0^* = -b_0, & g = -w^*a + wa^* \\ \text{c) } g^* = \zeta^{-1}g, & b_0^* = \zeta b_0, & g = \zeta w^*a + wa^* \\ \text{d) } g^* = -\zeta^{-1}g, & b_0^* = -\zeta b_0, & g = -\zeta w^*a + wa^* \end{array}$$

In all four cases, we obtain $b_0 g = ab_0^* w^* + a^* b_0 w$, i.e. $x = b_0 w$ is a solution. But it is a tsp; we shall construct a polynomial solution by means of Lemma 1. Let $a = a_0 g$, $x = a_0 q^* + x_0$. The expression $x_h = a_0(q - q^*)$ satisfies the homogeneous equation, hence $x' = x + a_0(q - q^*) = x_0 + a_0 q$ is a polynomial solution.

Let us investigate the degrees. We use the identity $\partial(ax^*) \leqq \partial a$ in (16).

1) For $\partial b \leqq \partial a$ the right-hand side and the first term of the left-hand one is at most ∂a, so must be the second term:

$$\partial(a^* x) \leqq \partial a\,, \quad \partial x \leqq \partial a + \eta a$$

2) For $\partial b > \partial a$ the second term of the left-hand side must be of the same degree as the right-hand side:

$$\partial(a^* x) = \partial b\,, \quad \partial x = \partial b + \eta a$$

Let us investigate the minimal solution for $\partial b \leqq \partial a$.

a)
$$g^* = g\,, \quad \eta g = -\partial g\,, \quad b_0^* = b_0\,,$$
$$\partial a_0 = \partial a - \partial g\,, \quad \eta a_0 = \eta a + \partial g\,, \quad \partial b_0 = \partial b - \partial g$$

By cancellation with g

$$a_0 x^* + a_0^* x = b_0\,, \quad \partial b_0 \leqq \partial a_0$$

We shall look for a solution x' with $\partial x' \leqq \partial a_0$. Let x be a solution, $\partial x > \partial a_0$. We take the quotient and the remainder in the division $x = a_0 q + x_0$. The expression $x_h = a_0(q - q^*)$ satisfies the homogeneous equation according to Theorem 3 and it is a polynomial because $\partial x \leqq \partial a_0 + \eta a_0$, $\partial q = \partial x - \partial a_0 \leqq \eta a_0$. Hence $x' = x - a_0(q - q^*) = x_0 + a_0 q^*$ is also a solution and $\partial x' \leqq \partial a_0 = \partial a - \partial g$. This solution is unique: let x_1' and x_2' be two such solutions, $\partial x_1' \leqq \partial a_0$, $\partial x_2' \leqq \partial a_0$. Then $x' = x_1' - x_2'$ satisfies the homogeneous equation and $\partial x' \leqq \partial a_0$ which contradicts Theorem 3.

b)
$$g^* = -g\,, \quad \eta g = -\partial g\,, \quad b_0^* = b_0\,,$$
$$\partial a_0 = \partial a - \partial g\,, \quad \eta a_0 = \eta a + \partial g\,, \quad \partial b_0 = \partial b - \partial g$$

the equation obtained by cancellation is

$$a_0 x^* - a_0^* x = b_0\,, \quad \partial b_0 \leqq \partial a_0$$

We shall look for a solution $\partial x' < \partial a$. Let x be a solution, $\partial x \geqq \partial a_0$. We take the quotient and the remainder in $x = a_0 q + x_0$. We denote q_0 the absolute term of the polynomial q. The expression $x_h = a_0(q + q^* - q_0)$ satisfies the homogeneous equation and it is a polynomial. Hence $x' = x - a_0(q + q^* - q_0) = x_0 - a_0 \cdot (q^* - q_0)$ is a also a solution, in this case $\partial x' < \partial a_0 = \partial a - \partial g$. Such solution is again unique.

c)
$$g^* = \zeta^{-1} g\,, \quad b_0^* = \zeta b_0\,, \quad \partial g \geqq 1\,, \quad \eta g = -\partial g + 1\,,$$
$$\partial a_0 = \partial a - \partial g\,, \quad \eta a_0 = \eta a + \partial g - 1\,, \quad \partial b_0 = \partial b - \partial g$$

The cancelled equation is

$$a_0 x^* + \zeta^{-1} a_0^* x = b_0\,, \quad \partial b_0 \leqq \partial a_0$$

We shall look for $\partial x' \leqq \partial a_0$. The division is in this case $x = a_0 \zeta q + x_0$, $\partial x x \leqq \partial a_0$.

The solution of homogeneous equation is $x_h = a_0(\zeta q - q^*)$, it is a polynomial because $\partial x \leqq \partial a_0 + \eta a_0$, $\partial q = \partial x - \partial a_0 - 1 < \eta a_0$. The new solution is $x' = x - a_0(\zeta q - q^*) = x_0 + a_0 q^*$, $\partial x' \leqq \partial a_0 = \partial a - \partial g$, again unique.

d) $g^* = -\zeta^{-1} g$, $b_0^* = -\zeta b$, cancelled equation $a_0 x^* - \zeta^{-1} a_0^* x = b_0$, solution of homog. equation $x_h = a_0(\zeta q + q^*)$, further as in c).

Now we are coming to the case $\partial b > \partial a$. We shall prove that polynomials b', x_2, $\partial b' \leqq \partial a$ exist such that every solution x of (16) can be expressed as

$$x = x_1 + \zeta^{\partial a + \eta a + 1} x_2$$

where x_1 is a solution of

$$ax_1^* + a^* x_1 = b' \tag{17}$$

We shall prove this by induction. For $\partial b \leqq \partial a$ it must be $x_2 = 0$, $b' = b$ the assertion holds. For $\partial b > \partial a$, by matching the highest terms in (16) we have

$$x_{\partial b + \eta a} = \frac{b_{\partial b}}{a_{\eta a}}$$

We can express

$$x = x' + \frac{b_{\partial b}}{a_{\eta a}} \zeta^{\partial b + \eta a}, \quad \partial x' < \partial x$$

By substituting it into (16) we have

$$ax'^* + a^* x' = b'$$

where

$$b' = b - \frac{b_{\partial a}}{a_{\eta a}} \left(\zeta^{\partial b + \eta a} a^* + \zeta^{-\partial b - \eta a} a\right)$$

We see that x' satisfies an equation with $\partial b' < \partial b$. The induction assumption can be used

$$x' = x_1' + \zeta^{\partial a + \eta a + 1} x_2', \quad x = x_1 + \zeta^{\partial a + \eta a + 1} x_2$$

where

$$x_1 = x_1', \quad x_2 = x_2' + \zeta^{\partial b - \partial a - 1} \frac{b_{\partial b}}{a_{\eta a}}$$

So the needed form is found. For the minimal solution of (16) we take the minimal solution of (17). In cases a, c, d it is $\partial x_1 \leqq \partial a - \partial g$, hence the coefficients $x_i = 0$ for $\partial a - \partial g < i \leqq \partial a + \eta a$. In case b the lower bound is one less. The solution is unique because x_2 was constructed in the only possible way and x_1 is the only minimal solution of (17). □

THE SYMMETRIC EQUATION SOLUTION METHOD

For finding the minimal solution, we can use the same method as in the corresponding continuous case [2]: replacing the symmetric equation by a non-symmetric one, solving it an restoring the symmetry. The symmetry restoring is here more complicated because the minimal solution of the non-symmetric equation is not in the same time minimal for the symmetric equation.

Let us do it for (16). The case $\partial b > \partial a$ can be transformed to the case $\partial b \leqq \partial a$ as in the proof of Theorem 4, so we suppose $\partial b \leqq \partial a$. Consider the equation

$$ax^* + a^* y = b$$

with two unknown polynomials and find the x-minimal solution x', y', $g = \gcd(a, a^*)$ and $a_0 = a/g$, $b_0 = b/g$. Four cases are distinguished:

a) $g^* = g$, $a_0 x^* + a_0^* y = b_0$. From Theorem 2, $\partial x' < \partial a_0 - \eta a_0$, $\partial y' \leqq \partial a_0 + \eta a_0$. Generally $\partial x' \neq \partial y'$ and $x' = y'$ cannot be required. For the minimal solution of (16), it must be $\partial x \leqq \partial a_0$. The couple $x_h = a_0 q^*$, $y_h = -a_0 q$ where q is any polynomial, satisfies the homogeneous equation $a_0 x^* + a_0^* y = 0$. Hence $x = x' + a_0 q^*$, $y = y' - a_0 q$ satisfies (16), $\partial x \leqq \partial a_0$. The condition $x = y$ yields

$$q + q^* = \frac{y' - x'}{a_0}, \quad q = \left\{\frac{y' - x'}{a_0}\right\}_+$$

which leads to the minimal solution.

b) $g^* = -g$, $a_0 x^* - a_0^* y = b_0$, $x = x' + a_0 \zeta^{-1} q^*$, $y = y' + a_0 \zeta q$, $\partial x < \partial a_0$,

$$\zeta q - \zeta^{-1} q^* = \frac{y' - x'}{a_0}, \quad q = \zeta^{-1} \left\{\frac{y' - x'}{a_0}\right\}_+$$

c) $g^* = \zeta^{-1} g$, $a_0 x^* + \zeta^{-1} a_0^* y = b_0$, $x = x' + a_0 q^*$, $y = y' - a_0 \zeta q$,

$$q^* + \zeta q = \frac{y' - x'}{a_0}, \quad q = \zeta^{-1} \left[\frac{y' - x'}{a_0}\right]_+$$

d) $g^* = -\zeta^{-1} g$, $a_0 x^* - \zeta^{-1} a_0^* y = b_0$, $x = x' + a_0 q^*$, $y = y' + a_0 \zeta q$,

$$q^* - \zeta q = \frac{y' - x'}{a_0}, \quad q = -\zeta^{-1} \left[\frac{y' - x'}{a_0}\right]_+$$

The most important case of (16) is that of a stable polynomial a, $\partial b \leqq \partial a$. Stability of a means $a(\zeta) \neq 0$ for $|\zeta| \leqq 1$. For that case, $g = 1$ and the minimal solution is $\partial x \leqq \partial a$. The properties of stable polynomials make another algorithm possible which utilizes the symmetry and needs about half computational operations. It was described in [1] in terms of special matrices. Let us formulate it here in polynomial terms.

For stable polynomials, it is always $a_0 \neq 0$, $|a_{\partial a}/a_0| \neq 1$. In (16), if $\partial b > \partial a$ then the substitution

$$x = x' + \frac{b_{\partial b}}{a_0} \zeta^{\partial b}$$

leads to an equivalent equation

$$a x'^* + a^* x' = b'$$

where

$$b' = b - \frac{b_{\partial b}}{a_0} \left(\zeta^{\partial b} a^* + \zeta^{-\partial b} a \right), \quad \partial b' < \partial b$$

Thus the equation is replaced by another one of the same form but with lower degree of b.

If $\partial b \leqq \partial a$ then we make the substitution

$$x = x' - \frac{a_{\partial a}}{a_0} \zeta^{\partial a} x'^*$$

Taking the conjugate and considering the vector substitution $(x, x^*) \leftrightarrow (x', x'^*)$ we can see that its Jacoby's determinant is $1 - (a_{\partial a}/a_0)^2 \neq 0$. Hence the substitution is one-to-one and leads to an equivalent equation

$$a' x'^* + a'^* x' = b$$

where

$$a' = a - \frac{a_{\partial a}}{a_0} \zeta^{\partial a} a^*$$

Thus the equation is replaced by another one of the same form but with lower degree of a. It follows from the stability theory that a' is also stable.

By repeating both steps we come to the case $\partial a = 0$ which can be solved directly. By performing all substitutions backwards we obtain the solution x of the original equation. The forward part of the algorithm is the same as in discrete stability test. The procedure is easily mechanized for a computer or a calculator.

EXAMPLES

The simplest examples are most illustrative.

The homogeneous equation $\zeta^4 x^* + \zeta^{-4} x = 0$ has the general solution $x = \zeta^4 (t - t^*)$ where t is any polynomial $1 \leqq \partial t \leqq 4$ i.e.

$$x = -K_4 - K_3 \zeta - K_2 \zeta^2 - K_1 \zeta^3 + K_1 \zeta^5 + K_2 \zeta^6 + K_3 \zeta^7 + K_4 \zeta^8$$

For various right-hand sides, we have the minimal solutions:

$$\begin{array}{llll} b = 2 & x = \zeta^4 & b = \zeta^4 + \zeta^{-4} & x = 1 \\ b = \zeta + \zeta^{-1} & x = \zeta^3 & b = \zeta^5 + \zeta^{-5} & x = \zeta^9 \\ b = \zeta^2 + \zeta^{-2} & x = \zeta^2 & b = \zeta^6 + \zeta^{-6} & x = \zeta^{10} \\ b = \zeta^3 + \zeta^{-3} & x = \zeta & b = \zeta^7 + \zeta^{-7} & x = \zeta^{11} \end{array}$$

The case with a common factor: $a = (1 - 2\zeta)(1 - 0{\cdot}5\zeta) = 1 - 2{\cdot}5\zeta + \zeta^2$. General solution of homogeneous equation: $x = \zeta(t - t^*)$ where $\partial t = 1$, i.e. $x = -K + K\zeta^2$. Minimal solution for $b = \zeta^{-1} - 2{\cdot}5 + \zeta$, $x = 0{\cdot}5\zeta$.

The common factor of type b): $a = 1 - \zeta^2$, $g = \zeta - \zeta^{-1}$, general $x = K_1 + K_2\zeta + K_1\zeta^2$, $b = \zeta^2 - 2 + \zeta^{-2}$, minimal $x = -1$.

Type c): $a = 1 + \zeta$, $g = 1 + \zeta$, general $x = K - K\zeta$, $b = \zeta^{-1} + 2 + \zeta$, minimal $x = 1$.

Type d): $a = 1 - \zeta$, $g = 1 - \zeta$, general $x = K + K\zeta$, $b = \zeta^{-1} - 2 + \zeta$, minimal $x = -1$.

CONCLUSIONS

In both parts of the paper, linear polynomial equations with a special kind of symmetry were investigated. The equations have direct application in some problems of control theory. Besides of that, the mathematical apparatus of polynomial equations emerging in recent years, seems to be very elegant, powerful and worth of development as such. Perhaps it will be used also in another branches similarly as the vector and tensor calculus is now used much broader than originally intended.

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