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LINEAR QUADRATIC CONTROL
State Space vs. Polynomial Equations[1]

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A class of linear quadratic control problems, the state space solution of which is well known, is solved here using the method of polynomial equations. The discussion includes linear regulator, state estimator, observer for a linear functional of state, and finally the case of linear quadratic control with incomplete and/or noisy measurements. The emphasis is placed on relating the two design techniques and on demonstrating the basic features of the polynomial equation approach. This provides further insight as well as simple and efficient algorithms for control system design.

INTRODUCTION

In recent years, we have witnessed a growing presence of algebra in systems and control theory. Algebra is now recognized as a natural and powerful tool in studying the structure and dynamical behaviour of linear constant systems, especially through polynomial models.

These polynomial models can also be used to advantage when solving optimal control problems. The basic idea is to reduce the design procedure to the solution of a polynomial (Diophantine) equation. The first attempts to employ the polynomial equations in regulator synthesis are due to Volgin [19], Chkhartishvili [6], Åström [2], Peterka [16] and Kučera [10, 11, 12].

No doubt this polynomial approach is natural and effective when the system or the process to be controlled is specified by its external model, see Åström [3] and Kučera [13]. The purpose of this paper is to demonstrate that the method of

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polynomial equations can also be applied to systems described by internal models, as an attractive alternative to the state space techniques. To make the relations between the two approaches clear and explicit, the discussion concentrates on the standard building blocks of state space design: linear regulation, state estimation and observer design. Their solutions are translated into the polynomial parlance and then used to derive simple, uniform and efficient procedures for the solution of linear quadratic control problems with incomplete and/or noisy measurements. To keep the presentation simple and instructive, we shall restrict ourselves to single-input single-output systems: the generalization to the multiterminal case is addressed at the end of the paper.

LINEAR REGULATOR

Consider a reachable system

$$\dot{x} = Ax + Bu\,, \quad x(0) = x_0$$

where $x \in R^n$ and $u \in R$, together with the cost

$$Q = \int_0^\infty (x'Wx + u^2)\,\mathrm{d}t$$

where $W \geqq 0$. It is desired to find a control law, relating u to x, which makes the resultant system asymptotically stable and minimizes Q for every x_0.

This control law is linear

$$u = -Kx$$

where

(1) $$K = B'P$$

and P is the solution of the algebraic Riccati equation

(2) $$A'P + PA - PBB'P + W = 0$$

such that $A - BK$ is a stability matrix.

To obtain the solution by polynomial methods, we shall follow Kalman [9], Brockett [5], MacFarlane [15] and Kučera [14]; a similar approach was reported by Shaked [17]. Denote s the differential operator and write

(3) $$(sI - A)^{-1}\,B = \frac{\bar{b}(s)}{a(s)}$$

where $a(s)$ is the characteristic polynomial of A. By reachability, the $a(s)$ and $\bar{b}(s)$ are right coprime. Then define a stable monic polynomial $g(s)$ by the relation

(4) $$a(-s)\,a(s) + \bar{b}'(-s)\,W\,\bar{b}(s) = g(-s)\,g(s)\,.$$

This is called the spectral factorization and the $g(s)$, if it exists, is unique.

We first establish the identity

$$(5)\qquad [a(-s) + \bar{b}'(-s)\,K']\,[a(s) + K\,\bar{b}(s)] = a(-s)\,a(s) + \bar{b}'(-s)\,W\,\bar{b}(s)\,.$$

Add and subtract sP from equation (2) to obtain

$$(-sI - A')\,P + P(sI - A) = W - PBB'P\,.$$

Premultiply the result by $B'(-sI - A')^{-1}$ and postmultiply it by $(sI - A)^{-1}\,B$. Then (5) follows on using (1) and (3).

Next observe that the characteristic polynomial of $A - BK$ equals $a(s) + K\,\bar{b}(s)$. Indeed, (3) implies

$$(sI - A)\,\bar{b}(s) = B\,a(s)$$

and the claim is immediate when adding $BK\,\bar{b}(s)$ to both sides above. Closed loop stability then implies that $a(s) + K\,\bar{b}(s)$ must be a stable polynomial. We have thus proved the following result.

Lemma 1. The linear regulator gain K satisfies the equation

$$a(s) + K\,\bar{b}(s) = g(s)$$

with $g(s)$ defined in (4).

Thus $g(s)$ is the characteristic polynomial of the closed-loop system matrix $A - BK$.

Since $a(s)$ and $\bar{b}(s)$ are right coprime, the existence of K hinges on the existence of $g(s)$. Let $d(s)$ be the greatest common right divisor of $a(s)$ and $W\,\bar{b}(s)$. Then $g(s)$ exists if and only if $d(s)$ has no purely imaginary root; this corresponds to the absence of purely imaginary eigenvalues of the matrix

$$\begin{bmatrix} -A & BB' \\ W & A' \end{bmatrix}.$$

If K does exist, it is unique.

STATE ESTIMATION

Consider a process modeled by an observable system

$$\dot{x} = Ax + v$$

$$y = Cx + w$$

where $y \in R$, $x \in R^n$ and $v \in R^n$, $w \in R$ are two independent zero-mean Gaussian white random processes with intensities $V \geqq 0$ and 1, respectively. Find an estimate $\hat{x}$ of x, generated from y by an asymptotically stable system, such that the expectation

$$\mathsf{E}(\hat{x} - x)'\,U(\hat{x} - x)$$

is minimized in steady state for every $U \geqq 0$.

The optimal estimator is linear and given by

$$\dot{\hat{x}} = (A - LC)\,\hat{x} + Ly$$

where

$$L = PC'$$

and P satisfies the algebraic Riccati equation

$$AP + PA' - PC'CP + V = 0$$

with $A - LC$ constrained to be a stability matrix.

This is a dual of the linear regulator problem. Write

$$C(sI - A)^{-1} = \frac{\bar{c}(s)}{a(s)}$$

where $a(s)$ is the characteristic polynomial of A. By observability, the $a(s)$ and $\bar{c}(s)$ are left coprime. Then define a stable monic polynomial $f(s)$ by the relation

$$a(s)\,a(-s) + \bar{c}(s)\,V\,\bar{c}'(-s) = f(s)f(-s)\,. \tag{6}$$

The polynomial solution is now obtained by dual arguments; we have

Lemma 2. The state estimator gain L satisfies the equation

$$a(s) + \bar{c}(s)\,L = f(s)$$

with $f(s)$ defined in (6).

Again, the L exists if and only if $d(s)$, the greatest common left divisor of $a(s)$ and $\bar{c}(s)V$, is devoid of purely imaginary roots. This corresponds to the absence of purely imaginary eigenvalues of the matrix

$$\begin{bmatrix} -A' & C'C \\ V & A \end{bmatrix}.$$

If L does exist, it is unique. The $f(s)$ is then the characteristic polynomial of the estimator matrix $A - LC$.

OBSERVING A FUNCTIONAL OF STATE

Consider a reachable and observable system

$$\dot{x} = Ax + Bu \tag{7}$$
$$y = Cx$$

where $y \in R$, $x \in R^n$, $u \in R$ and the state x is not available for measurement. Let a control law be specified in the form

$$u = -Kx\,. \tag{8}$$

Since the state of (7) is not directly available to implement (8), a system of the form

$$\dot{z} = Fz + G_1 y + G_2 u \tag{9}$$
$$h = Hz + Jy$$

with $z \in R^m$ is constructed in such a way that h approximates Kx from available data. We say that (9) is an asymptotic observer for Kx if and only if there exists an $m \times n$ matrix T such that F is a stability matrix and

$$TA - FT = G_1 C$$
$$TB = G_2$$
$$HT + JC = K .$$

There are two distinctive choices which guarantee an arbitrary characteristic polynomial of F for any K. These are a full order ($m = n$) observer obtained when taking T nonsingular, and a least order ($m = n - 1$) observer obtained when T complements C to a nonsingular matrix.

A direct polynomial solution is due to Wolovich [21]. Write

$$(sI - A)^{-1} B = \frac{\bar{b}(s)}{a(s)}$$

and

$$C(sI - A)^{-1} B = \frac{b(s)}{a(s)}$$

where $a(s)$ is the characteristic polynomial of A. By reachability and observability, the polynomials $a(s)$ and $b(s)$ are coprime. Introduce a pseudostate x_p by

$$u = a(s)\, x_p \tag{10}$$
$$y = b(s)\, x_p .$$

The state x is then

$$x = \bar{b}(s)\, x_p$$

and we define a polynomial $k(s)$ via

$$Kx = K\, \bar{b}(s)\, x_p = k(s)\, x_p . \tag{11}$$

To determine h from u and y we consider a system governed by

$$f(s)\, h = q(s)\, y + r(s)\, u . \tag{12}$$

Inserting (10) and (11) into (12), it is readily established that

$$f(s)\,(h - Kx) = [a(s)\, r(s) + b(s)\, q(s) - k(s) f(s)]\, x_p .$$

Thus (12) is an asymptotic observer for Kx if and only if $f(s)$ is a stable polynomial and

$$a(s)\,r(s) + b(s)\,q(s) = k(s)\,f(s) \tag{13}$$

$$\deg r(s) \leqq \deg f(s)$$

$$\deg q(s) \leqq \deg f(s)\,.$$

The last two inequalities are to guarantee that (12) is a dynamical system.

Consider the least degree solution $q(s)$, $r(s)$ of (13) with respect to $q(s)$; it satisfies $\deg q(s) < n$, where $n = \deg a(s)$. The conditions above are to be fulfilled for any $k(s)$, hence we must take $\deg f(s) \geqq n - 1$ to satisfy the last inequality. Since neither $\deg b(s)$ nor $\deg k(s)$ exceeds $n - 1$, it is seen from (13) that this choice will also satisfy the other inequality. To summarize, we have

Lemma 3. The system governed by (12) is an asymptotic observer for Kx **if and only if** $f(s)$ **is a stable monic polynomial of degree no less than** $n - 1$ **and** $q(s)$, $r(s)$ **are polynomials satisfying the equation (13) and the constraint** $\deg q(s) < n$.

Thus $f(s)$ is the characteristic polynomial of F. Clearly $\deg f(s) = n - 1$ yields a least order observer with arbitrary dynamics whereas $\deg f(s) = n$ results in a full order observer. The $a(s)$ and $b(s)$ being coprime, an observer can always be found for any $k(s)$ and any $f(s)$. Moreover, for any chosen $f(s)$ the polynomials $q(s)$ and $r(s)$ are unique.

DETERMINISTIC CONTROL

Consider a reachable and observable system

$$\dot{x} = Ax + Bu\,, \quad x(0) = x_0 \tag{14}$$

$$y = Cx$$

where $y \in R$, $x \in R^n$, $u \in R$ along with the cost

$$Q = \int_0^\infty (x'Wx + u^2)\,\mathrm{d}t$$

where $W \geqq 0$. It is assumed that the state x of (14) is not accessible and our objective is to find a least order dynamical controller, processing y instead of x, which makes the overall system asymptotically stable while minimizing Q for every x_0.

Since the control law

$$u = -Kx \tag{15}$$

resulting from the regulator problem cannot be directly implemented, the optimal controller would in general depend on x_0. A natural approach is to construct a least order asymptotic observer (9) for Kx and then implement the control law

$$u = -h \tag{16}$$

in which Kx is replaced by its estimate h. The overall system then obeys the equation

$$\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} A - BK & -BH \\ 0 & F \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} \tag{17}$$

where $e = z - Tx$.

The polynomial solution of the deterministic control problem with incomplete measurements can now be obtained by combining the polynomial solutions to the regulator and observer problems. Write

$$(sI - A)^{-1} B = \frac{\bar{b}(s)}{a(s)}$$

and

$$C(sI - A)^{-1} B = \frac{b(s)}{a(s)}$$

where $a(s)$ is the characteristic polynomial of A. By reachability and observability, the polynomials $a(s)$ and $b(s)$ are coprime.

Theorem 1. The observer-based control law for the deterministic problem is given by

$$p(s)\, u = -q(s)\, y$$

where the polynomials $p(s)$ and $q(s)$ satisfy the equation

$$a(s)\, p(s) + b(s)\, q(s) = f(s)\, g(s) \tag{18}$$

and the constraint $\deg q(s) < n$. The $f(s)$ is any desired stable monic polynomial of degree $n - 1$ and $g(s)$ is a stable monic polynomial defined by

$$a(-s)\, a(s) + \bar{b}'(-s)\, W \bar{b}(s) = g(-s)\, g(s)\,.$$

Proof. By Lemma 1, the gain K in (15) satisfies

$$a(s) + K\, \bar{b}(s) = g(s) \tag{19}$$

where $g(s)$ is defined in (4). Using Lemma 3, the least order observer for Kx is

$$f(s)\, h = q(s)\, y + r(s)\, u \tag{20}$$

where

$$a(s)\, r(s) + b(s)\, q(s) = k(s) f(s) \tag{21}$$

and $\deg q(s) < n$. By definition,

$$k(s) = K\, \bar{b}(s) \tag{22}$$

and $f(s)$ is an arbitrary but stable monic polynomial of degree $n - 1$.

The loop is now closed according to (16). Combining (16) and (20), the observer-based control law is given by

$$[r(s) + f(s)]\, u = -q(s)\, y .$$

Adding $a(s)\, f(s)$ to equation (21) gives

$$a(s)\,[r(s) + f(s)] + b(s)\, q(s) = f(s)\,[a(s) + k(s)] .$$

Since $a(s) + k(s) = g(s)$ in view of (19) and (22), our claim follows on identifying $p(s) = r(s) + f(s)$. □

Clearly, the deterministic control problem is solvable if and only if the associated regulator problem is solvable. The characteristic polynomial of the closed loop system matrix in (17) is a product of $f(s)$, the characteristic polynomial of F, and $g(s)$, the characteristic polynomial of $A - BK$. Once these polynomials are determined, the optimal control law can directly be obtained by solving equation (18).

STOCHASTIC CONTROL

Consider a controlled process modeled by a reachable and observable system

(23)
$$\begin{aligned} \dot{x} &= Ax + Bu + v \\ y &= Cx + w \end{aligned}$$

where $y \in R$, $x \in R^n$, $u \in R$ and $v \in R^n$, $w \in R$ are two independent zero-mean Gaussian white random processes with intensities $V \geqq 0$ and 1, respectively. In addition, the cost

$$\bar{Q} = \mathsf{E}(x'Wx + u^2)$$

is given. It is assumed that the state x of (23) is not accessible, and our aim is to find a dynamical controller, processing y instead of x, which makes the resultant system asymptotically stable while minimizing $\bar{Q}$ in steady state.

Invoking the separability of estimation and control, we first solve the linear regulator problem for the system

$$\dot{x} = Ax + Bu$$

and the cost

$$Q = \int_0^{\infty} (x'Wx + u^2)\, \mathrm{d}t$$

to obtain the control law

$$u = -Kx .$$

Then we solve the state estimation problem for the process

$$\begin{aligned} \dot{x} &= Ax + v \\ y &= Cx + w \end{aligned}$$

to obtain the estimator

$$\dot{\hat{x}} = (A - LC)\,\hat{x} + Ly + Bu\,. \tag{24}$$

Finally we implement the control law

$$u = -K\hat{x}$$

in which x is replaced by its estimate $\hat{x}$. The closed loop system then obeys the equation

$$\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} A - BK & -BK \\ 0 & A - LC \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} + \begin{bmatrix} I & 0 \\ -I & L \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix} \tag{25}$$

in which $e = \hat{x} - x$.

The polynomial solution of the stochastic control problem with incomplete and noisy measurements is now at hand. Write

$$(sI - A)^{-1} B = \frac{\bar{b}(s)}{a(s)}, \quad C(sI - A)^{-1} = \frac{\bar{c}(s)}{a(s)}$$

and

$$C(sI - A)^{-1} B = \frac{b(s)}{a(s)}$$

where $a(s)$ is the characteristic polynomial of A. By reachability and observability, the polynomials $a(s)$ and $b(s)$ are coprime.

Theorem 2. The optimal control law for the stochastic problem is given by

$$p(s)\,u = -q(s)\,y$$

where the polynomials $p(s)$ and $q(s)$ satisfy the equation

$$a(s)\,p(s) + b(s)\,q(s) = f(s)\,g(s) \tag{26}$$

and the constraint $\deg q(s) < n$. The $f(s)$ and $g(s)$ are stable monic polynomials defined by

$$\begin{aligned} a(s)\,a(-s) + \bar{c}(s)\,V\bar{c}'(-s) &= f(s)f(-s) \\ a(-s)\,a(s) + \bar{b}'(-s)\,W\bar{b}(s) &= g(-s)\,g(s)\,. \end{aligned}$$

Proof. The state estimator (24) is just a special full order observer of Kx in which $h = z$. It is obtained from (9) by setting $T = I$, $G_1 = L$. In view of Lemma 2, the characteristic polynomial, $f(s)$, of the estimator matrix $F = A - LC$ is given by (6). The proof then proceeds as in Theorem 1. □

We can conclude by noting that the stochastic control problem is solvable if and only if the associated regulator and estimator problems are both solvable. The solution is then unique. The characteristic polynomial of the overall system matrix appearing in (25) is again a product of $f(s)$, the characteristic polynomial of $A - LC$, and $g(s)$, the characteristic polynomial of $A - BK$. They are now both of degree n.

Once these polynomials are determined via the spectral factorizations, the optimal control law can directly be obtained by solving equation (26). Similar results in terms of rational transfer functions were reported by Shaked [18].

CONCLUSIONS

The method of polynomial equations has been described as an alternative to the standard state space design of constant linear quadratic controllers. The emphasis has been placed on relating the two techniques and on demonstrating the simplicity of the polynomial approach.

An interesting feature of this approach is that it closely associates the design of optimal controllers with the pole shifting technique. The spectral factorization is used to specify the characteristic polynomial of the optimal system and the controller is then constructed so as to assign this polynomial.

In addition to providing further insight, this procedure is computationally attractive. There are efficient algorithms which perform the spectral factorization, see Vostrý [20], and the polynomial equations can easily be solved using the extended Euclidean algorithm, see Blankinship [4], Kučera [13] or Ježek [7].

Finally, let us note that all the results presented in this paper hold true for discrete-time systems. We just have to replace the differential operator s by the advance operator z and modify the relations for the spectral factorization appropriately; that is, we use z^{-1} in place of $-s$ and take the spectral factors of degree n. The results also generalize nicely for multivariable systems by employing the notion of matrix fractions. The multivariable regulator and estimator problems are discussed by Anderson and Moore [1], Kailath [8] and Kučera [14], the observer problem by Wolovich [21] and Kailath [8]. The main difficulty arising in the multivariable observer design is associated with the choice of $F(s)$, a matrix counterpart for the observer polynomial $f(s)$, which would lead to the observer of specified order with a desired dynamics. Once this problem is solved, Theorems 1 and 2 are ready to apply.

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REFERENCES

[1] B. D. O. Anderson and J. B. Moore: Linear Optimal Control. Prentice-Hall, New York 1971.

[2] K. J. Åström: Introduction to Stochastic Control Theory. Academic, New York 1970.

[3] K. J. Åström: Algebraic system theory as a tool for regulator design. Report CODEN: LUTFD2/(TFRT-7164)/1-023/(1979), Lund Institute of Technology, Lund 1979.

[4] W. A. Blankinship: A new version of the Euclidean algorithm. Amer. Math. Monthly *70* (1963), 742–745.

[5] R. W. Brockett: Finite Dimensional Linear Systems. Wiley, New York 1970.

[6] L. P. Chkhartishvili: Synthesis of control systems minimizing the rms error (in Russian). Izv. Akad. Nauk SSSR Tekhn. Kibernet. *1* (1964), 143–153.

[7] J. Ježek: New algorithm for minimal solution of linear polynomial equations. Kybernetika *18* (1982), 6, 505—516.

[8] T. Kailath: Linear Systems. Prentice-Hall, Englewood Cliffs 1980.

[9] R. E. Kalman: When is a linear control system optimal? Trans. ASME Ser. D *86* (1964), 51.

[10] V. Kučera: Algebraic theory of discrete optimal control for single-variable systems. Kybernetika *9* (1973), 94—107, 206—221 and 291—312.

[11] V. Kučera: Algebraic approach to discrete linear control. IEEE Trans. Automat. Control *AC-20* (1975), 116—120.

[12] V. Kučera: Design of steady-state minimum variance controllers. Automatica *15* (1979), 411—418.

[13] V. Kučera: Discrete Linear Control: The Polynomial Equation Approach. Wiley, New York and Academia, Prague 1979.

[14] V. Kučera: New results in state estimation and regulation. Automatica *17* (1981), 745—748.

[15] A. G. J. MacFarlane: Return-difference matrix properties for optimal stationary Kalman-Bucy filter. Proc. IEE *118* (1971), 373—376.

[16] V. Peterka: On steady state minimum variance control strategy. Kybernetika *8* (1972), 219—232.

[17] U. Shaked: A general transfer function approach to linear stationary filtering and steady-state optimal control problems. Internat. J. Control *24* (1976), 741—770.

[18] U. Shaked: A general transfer function approach to the steady-state linear quadratic Gaussian stochastic control problem. Internat. J. Control *24* (1976), 771—800.

[19] L. N. Volgin: The Fundamentals of the Theory of Controllers (in Russian). Soviet Radio, Moscow 1962.

[20] Z. Vostrý: New algorithm for polynomial spectral factorization with quadratic convergence. Kybernetika *11* (1975), 415—422 and Kybernetika *12* (1976), 248—259.

[21] W. A. Wolovich: Linear Multivariable Systems. Springer-Verlag, Berlin—Heidelberg—New York 1974.

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