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On Shannon — McMillan's Limit Theorem for Pairs of Stationary Random Processes*

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The aim of the present paper is to prove that the conditions for validity of the generalized version of the Shannon-McMillan's limit theorem for pairs of stationary random processes as given in Perez (1972) are not only sufficient but (in a certain sense) also necessary. Moreover, these conditions were replaced by similar ones, concerning finite-dimensional spaces.

1. INTRODUCTION

The classical Shannon-McMillan's fundamental limit theorem concerns the asymptotic behaviour of the probability of n-letter blocks produced by a discrete stationary source. In the case of abstract-alphabet sources we replace naturally the above probability by some probability density. Instead of individual sources we, thus, need now pairs of sources, the one dominating (in the sense of absolute continuity) the other for every *finite n*.

Generalized versions of Shannon-McMillan's limit theorem as developed by the author from 1956 and followed by other authors (for a summary of this development cf. Perez (1962)) concern, thus, the asymptotic behaviour of the corresponding Radon-Nikodym density resp. of the *entropy density*, as called by M. S. Pinsker the logarithm of the former. Since in the essence this theory contributes to the study of the asymptotic behaviour of the likelihood ratio, it is important as well for Information Theory as for Mathematical Statistics (for instance, in generalizing the Chernoff's result on the asymptotic discernibility of two random processes; cf. Perez (1972)).

The aim of the present paper is to prove that the conditions for the validity of the generalized version of the Shannon-McMillan's limit theorem for pairs of stationary random processes as given in Perez (1972) are not only sufficient but (in a certain

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sense) also necessary. Moreover, these conditions were replaced by similar ones concerning finite-dimensional spaces.

This version arised in trying to overcome an insufficiency of Perez (1962), the Lemma 2.2 of this paper being valid only under the condition of projectivity of the system of measures it concerns, i.e. namely for dominating process of Markovian type. In the general stationary case it, thus, arised the need to replace this lemma by an appropriate one (cf. Lemma 2.1 of Perez (1972)). In the essence, the object of my present paper is a further study of this Lemma, representing the crucial point in the proof of the generalized version of the Shannon-McMillan's limit theorem considered.

2. FORMULATION

Let $(X_0, \mathfrak{X}_0)$ be a measurable space (abstract alphabet). Let us denote by $(X_{r,s}, \mathfrak{X}_{r,s})$ for $r < s$ the measurable space $\bigtimes_{i=r+1}^{s} (X_i, \mathfrak{X}_i), (X_i, \mathfrak{X}_i) = (X_0, \mathfrak{X}_0)$, $i = r+1, r+2, \ldots$ $\ldots, s$, corresponding to letter sequences $x_{r,s} = (x_{r+1}, x_{r+2}, \ldots, x_s)$, $r, s = -\infty, \ldots$ $\ldots, -1, 0, 1, \ldots, +\infty$. In the special case $s = r + 1$ we shall denote $(X_{r,s}, \mathfrak{X}_{r,s})$ by $(X_s, \mathfrak{X}_s)$: it is the measurable space of the s-th "coordinate" x_s of $x = \ldots x_{-1}, x_0$, $x_1, \ldots$. The measurable space of double infinite sequences will be denoted by $(X, \mathfrak{X})$:

$$(X, \mathfrak{X}) = \bigtimes_{i=-\infty}^{i=+\infty} (X_i, \mathfrak{X}_i)\,.$$

Let P and Q be two stationary probability measures on $(X, \mathfrak{X})$. By $P_{r,s}$ and $Q_{r,s}$ we shall denote the restrictions of P and Q on $\mathfrak{X}_{r,s}$, respectively.

Let for every *finite* $n = 1, 2, \ldots$

$$P_{0n} \ll Q_{0n} \tag{2.1}$$

and denote by $f_{0n}(x)$ the corresponding Radon-Nikodym density.

In the sequel we shall be interested on conditions for which the following fundamental statement is valid.

Statement. The sequence

$$\left\{\frac{1}{n} \log f_{0n}(x)\right\}_{n\geq 1} \tag{2.2}$$

converges in P-mean (to a function $h(x)$ which is invariant with respect to the shift transformation T).

Obviously, if the statement holds then

$$\lim_{n\to\infty} \frac{1}{n} \int \log f_{0n} \,\mathrm{d}P \equiv \lim_{n\to\infty} \frac{1}{n} H(P_{0n}, Q_{0n}) = \int h(x) \,\mathrm{d}P = \hbar < \infty \,, \tag{2.3}$$

i.e., the generalized (relative) entropy rate of P with respect to Q *exists* and is *finite*. In other words, *condition*

$$\lim_{n\to\infty} \frac{1}{n} H(P_{0n}, Q_{0n}) = \hbar < \infty \tag{i}$$

is *necessary* for the validity of the Statement.

In the sequel we shall suppose that the conditional probability functions on $\mathfrak{X}_1$ given $x_{-n,0}$ (i.e. measurable with respect to the σ-algebra $\mathfrak{X}_{-n,0}$) corresponding to P, $p_P(\cdot/x_{-n,0})$, and to Q, $p_Q(\cdot/x_{-n,0})$, are *regular* for $n = 1, 2, \ldots$.

Let us introduce the probability measure $PQ_{-n,0,1}$ on $\mathfrak{X}_{-n,1}$ generated by $P_{-n,0}$ and $p_Q(\cdot/x_{-n,0})$ in the following manner: for $E \in \mathfrak{X}_{-n,0}$ and $F \in \mathfrak{X}_1$ define the set function*)

$$PQ_{-n,0,1}(E \times F) = \int_E p_Q(F/x_{-n,0}) \,\mathrm{d}P_{-n,0} \,. \tag{2.4}$$

$PQ_{-n,0,1}$ is then defined as the unique extension on the whole σ-algebra $\mathfrak{X}_{-n,1}$ of the above set function.

It, obviously, holds for $n = 1, 2, \ldots$ (cf. (2.1))

$$P_{-n,1} \ll PQ_{-n,0,1} \tag{2.5}$$

Let us denote by $g_n(x)$ the corresponding Radon-Nikodym derivative.

Then one may write (putting $f_{00} \equiv 1$) for $n = 1, 2, \ldots$

$$\frac{1}{n} \log f_{0n}(x) = \frac{1}{n} \sum_{k=0}^{n-1} (\log f_{0k+1}(x) - \log f_{0k}(x)) = \frac{1}{n} \sum_{k=0}^{n-1} \log g_k(T^k x) \tag{2.6}$$

since

$$\frac{f_{0k+1}(x)}{f_{0k}(x)} = g_k(T^k x) \quad (\text{by taking } g_0(x) = f_{01}(x)) \,.$$

Given the stationarity of P, on the base of (2.6) one obtains

$$\frac{1}{n} \int \log f_{0n} \,\mathrm{d}P \equiv \underbrace{\frac{1}{n} H(P_{0n}, Q_{0n})} = \frac{1}{n} \sum_{k=0}^{n-1} \int \log g_k(T^k x) \,\mathrm{d}PT^k = \tag{2.7}$$

*) Note that $p_Q(\cdot/x_{-n,0})$ exists not only a.s. $[Q_{-n,0}]$ but also a.s. $[P_{-n,0}]$, $n = 1, 2, \ldots$ (finite) since by assumption $P_{-n,0} \ll Q_{-n,0}$ for every n finite.

$$= \frac{1}{n}\sum_{k=0}^{n-1}\int \log g_k(x)\,\mathrm{d}P \equiv \frac{1}{n}\sum_{k=0}^{n-1} H(P_{-k,1}, PQ_{-k,0,1})\,.$$

By the first sum Cesaro theorem, if the following limit exists

$$\lim_{n\to\infty} H(P_{-n,1}, PQ_{-n,0,1}) = \hbar$$

then also

$$\lim_{n\to\infty} \frac{1}{n} H(P_{0n}, Q_{0n}) = \hbar\,.$$

In the sequel, condition (i) shall be replaced by assumption

(i′) $$\lim_{n\to\infty} H(P_{-n,1}, PQ_{-n,0,1}) = \hbar < \infty\,.$$

Further, provided that the sequence $\{\log g_n\}_{n\geqq 1}$ converges in P-mean to some limit $G(x)$, on the base of (2.6) and the stationarity of P one may write for some integrable function $h(x)$

(2.8) $$\int\left|\frac{1}{n}\log f_{0n}(x) - h(x)\right|\mathrm{d}P = \int\left|\frac{1}{n}\sum_{k=0}^{n-1}\log g_k(T^k x) - h(x)\right|\mathrm{d}P \leqq$$

$$\leqq \int\left|\frac{1}{n}\sum_{k=0}^{n-1}(\log g_k(T^k x) - G(T^k x))\right|\mathrm{d}P + \int\left|\frac{1}{n}\sum_{k=0}^{n-1}G(T^k x) - h(x)\right|\mathrm{d}P \leqq$$

$$\leqq \frac{1}{n}\sum_{k=0}^{n-1}\int\left|\log g_k(x) - G(x)\right|\mathrm{d}P + \int\left|\frac{1}{n}\sum_{k=0}^{n-1}G(T^k x) - h(x)\right|\mathrm{d}P\,.$$

If now, $h(x)$ is the limit in P-mean of $(1/n)\sum_{k=0}^{n-1} G(T^k x)$, which according to the well-known ergodic theorem exists since $G(x)$ is P-integrable, one obtains from (2.8) that also the sequence $\{1/n \log f_{0n}(x)\}_{n\geqq 1}$ converges in P-mean to $h(x)$ so that the Statement is fulfilled.

Our attention may, thus, be concentrated on suitable conditions ensuring the convergence in P-mean of the sequence $\{\log g_n(x)\}_{n\geqq 1}$.

Indeed, this is the object of Lemma 2.2 of Perez (1962) (valid for the case of projectivity of the system of probability measures $\{PQ_{-n,0,1}\}_{n\geqq n_0}$, i.e. for the case of Markovian Q) which in the general stationary case was replaced by Lemma 2.1 in Perez (1972).

Lemma 2.1 (Perez (1972)). If

(1) $$p_Q(\cdot/x_{-\infty,0}) = \lim_{n\to\infty} p_Q(\cdot/x_{-n,0}) \quad \text{a.s.} \quad [P]$$

(so that $PQ_{-\infty,0,1}$ is a probability measure)

(2)
$$H(P_{-\infty,1}, PQ_{-\infty,0,1}) < \infty ,$$

(3)
$$\lim_{n\to\infty} \int \left| \log \frac{\mathrm{d}p_Q(x_1/x_{-\infty,0})}{\mathrm{d}p_Q(x_1/x_{-n,0})} \right| \mathrm{d}P = O \text{ *)}$$

then the sequence $\{\log g_n(x)\}_{n\geqq 1}$ converges in P-mean. In particular,

(2.9)
$$\lim_{n\to\infty} H(P_{-n,1}, PQ_{-n,0,1}) = H(P_{-\infty,1}, PQ_{-\infty,0,1}) =$$
$$= \lim_{n\to\infty} \frac{1}{n} H(P_{0,n}, Q_{0,n}) = \hbar .$$

Proof. (new)

$$\Delta_n = \int \left| \log \overbrace{\frac{\mathrm{d}P_{-\infty,1}}{\mathrm{d}PQ_{-\infty,0,1}}}^{\text{exists under (2)}} - \log \frac{\mathrm{d}P_{-n,1}}{\mathrm{d}PQ_{-n,0,1}} \right| \mathrm{d}P =$$
$$= \int \left| \log \frac{\mathrm{d}p_P(x_1/x_{-\infty,0})}{\mathrm{d}p_Q(x_1/x_{-\infty,0})} - \log \frac{\mathrm{d}p_P(x_1/x_{-n,0})}{\mathrm{d}p_Q(x_1/x_{-n,0})} \right| \mathrm{d}P =$$
$$= \int \left| \log \frac{\mathrm{d}p_P(x_1/x_{-\infty,0})}{\mathrm{d}p_P(x_1/x_{-n,0})} - \log \frac{\mathrm{d}p_Q(x_1/x_{-\infty,0})}{\mathrm{d}p_Q(x_1/x_{-n,0})} \right| \mathrm{d}P \leqq$$
$$\leqq \int \left| \log \frac{\mathrm{d}p_P(x_1/x_{-\infty,0})}{\mathrm{d}p_P(x_1/x_{-n,0})} \right| \mathrm{d}P + \int \left| \log \frac{\mathrm{d}p_Q(x_1/x_{-\infty,0})}{\mathrm{d}p_Q(x_1/x_{-n,0})} \right| \mathrm{d}P .$$

One, thus, under (3) derives that $\Delta_n \to 0$ iff

$$\lim_{n\to\infty} \int \left| \log \frac{\mathrm{d}p_P(x_1/x_{-\infty,0})}{\mathrm{d}p_P(x_1/x_{-n,0})} \right| \mathrm{d}P = O$$

or, equivalently, iff

(2.10)
$$\lim_{n\to\infty} \int \log \frac{\mathrm{d}p_P(x_1/x_{-\infty,0})}{\mathrm{d}p_P(x_1/x_{-n,0})} \,\mathrm{d}P = O ,$$

*) By introducing the probability measure $P\tilde{Q}^{(\infty)}_{-n,0,1}$ on $\mathfrak{X}_{-\infty,1}$ by $P\tilde{Q}^{(\infty)}_{-n,0,1}(E \times F) = \int_E p_Q(F/x_{-n,0})\,\mathrm{d}P_{-\infty,0}$ for $E \in \mathfrak{X}_{-\infty,0}$, $F \in \mathfrak{X}_1$, as extension of $PQ_{-n,0,1}$, condition (3) may be written (implicitely assuming: $PQ_{-\infty,0,1} \ll P\tilde{Q}^{(\infty)}_{-n,0,1}$)

(3')
$$\lim_{n\to\infty} \int \left| \log \frac{\mathrm{d}PQ_{-\infty,0,1}}{\mathrm{d}P\tilde{Q}^{(\infty)}_{-n,0,1}} \right| \mathrm{d}P = O$$

since the integral in (2.10) represents a generalized entropy, so that, according to the well-known Pinsker's inequality,

$$\int \left| \log \frac{\mathrm{d}p_P(x_1/x_{-\infty,0})}{\mathrm{d}p_P(x_1/x_{-n,0})} \right| \mathrm{d}P \leqq \int \log \frac{\mathrm{d}p_P(x_1/x_{-\infty,0})}{\mathrm{d}p_P(x_1/x_{-n,0})} \mathrm{d}P + $$
$$+ \Gamma \sqrt{\left(\int \log \frac{\mathrm{d}p_P(x_1/x_{-\infty,0})}{\mathrm{d}p_P(x_1/x_{-n,0})} \mathrm{d}P \right)}.$$

Combining (2.10) with assumption (3) *it reduces to prove*

$$\lim_{n\to\infty} \int \left(\log \frac{\mathrm{d}p_P(x_1/x_{-\infty,0})}{\mathrm{d}p_P(x_1/x_{-n,0})} - \log \frac{\mathrm{d}p_Q(x_1/x_{-\infty,0})}{\mathrm{d}p_Q(x_1/x_{-n,0})} \right) \mathrm{d}P = 0$$

or

$$\lim_{n\to\infty} H(P_{-n,1}, PQ_{-n,0,1}) = H(P_{-\infty,1}, PQ_{-\infty,0,1}), \tag{2.11}$$

what is also a necessary condition for $\Delta_n \to O$ since it means

$$\delta_n = \int \left(\log \frac{\mathrm{d}P_{-\infty,1}}{\mathrm{d}PQ_{-\infty,0,1}} - \log \frac{\mathrm{d}P_{-n,1}}{\mathrm{d}PQ_{-n,0,1}} \right) \mathrm{d}P \to O.$$

If, now, $PQ^{(\infty)}_{-n,1}$ denotes the restriction of $PQ_{-\infty,0,1}$ on $\mathfrak{X}_{-n,1}$, since $P_{-n,1} \ll PQ^{(\infty)}_{-n,1}$ (resulting from $P_{-\infty,1} \ll PQ_{-\infty,0,1}$ which holds under assumption (2) $H(P_{-\infty,1}, PQ_{-\infty,0,1}) < \infty$) one may write

$$\log \frac{\mathrm{d}P_{-n,1}}{\mathrm{d}PQ_{-n,0,1}} = \log \frac{\mathrm{d}P_{-n,1}}{\mathrm{d}PQ^{(\infty)}_{-n,1}} + \log \frac{\mathrm{d}PQ^{(\infty)}_{-n,1}}{\mathrm{d}PQ_{-n,0,1}}, \tag{2.12}$$

so that

$$\delta_n = \int \left(\log \frac{\mathrm{d}P_{-\infty,1}}{\mathrm{d}PQ_{-\infty,0,1}} - \log \frac{\mathrm{d}P_{-n,1}}{\mathrm{d}PQ^{(\infty)}_{-n,1}} - \log \frac{\mathrm{d}PQ^{(\infty)}_{-n,1}}{\mathrm{d}PQ_{-n,0,1}} \right) \mathrm{d}P.$$

Since, on the other hand, under (2)

$$\int \left| \log \frac{\mathrm{d}P_{-\infty,1}}{\mathrm{d}PQ_{-\infty,0,1}} - \log \frac{\mathrm{d}P_{-n,1}}{\mathrm{d}PQ^{(\infty)}_{-n,1}} \right| \mathrm{d}P \to O \tag{2.12'}$$

(cf. Lemma 2.2 of Perez (1962)) it results: $\delta_n \to 0$ and, thus, under (2) and (3)

$$\Delta_n \to 0 \quad \text{iff} \quad A_n = \int \log \frac{\mathrm{d}PQ^{(\infty)}_{-n,1}}{\mathrm{d}PQ_{-n,0,1}} \mathrm{d}P \to O. \tag{2.13}$$

Let us denote by B_n the quantity figuring in (3) (without $|\ |$)

$$B_n = \int \log \frac{\mathrm{d}p_Q(x_1/x_{-\infty,0})}{\mathrm{d}p_Q(x_1/x_{-n,0})}\,\mathrm{d}P = \int \log \frac{\mathrm{d}PQ_{-\infty,0,1}}{\mathrm{d}P\tilde{Q}^{(\infty)}_{-n,0,1}}\,\mathrm{d}P \qquad \text{(cf. (3'))}\,. \tag{2.14}$$

We obtain immediately

$$A_n = H(P_{-n,1}, PQ_{-n,0,1}) - H(P_{-n,1}, PQ^{(\infty)}_{-n,1}) \tag{2.15}$$

$$B_n = H(P_{-\infty,1}, P\tilde{Q}^{(\infty)}_{-n,0,1}) - H(P_{-\infty,1}, PQ_{-\infty,0,1}) \tag{2.16}$$

Denoting

$$\frac{\mathrm{d}PQ_{-\infty,0,1}}{\mathrm{d}P\tilde{Q}^{(\infty)}_{-n,0,1}} = s_n \tag{2.17}$$

we obtain the expressions (cf. (2.13) and (2.14))

$$A_n = \int \log \mathscr{E}_P\{s_n \mid \mathfrak{X}_{-n,1}\}\,\mathrm{d}P \tag{2.18}$$

$$B_n = \int \log s_n\,\mathrm{d}P = \int \mathscr{E}_P\{\log s_n \mid \mathfrak{X}_{-n,1}\}\,\mathrm{d}P\,, \tag{2.19}$$

where by $\mathscr{E}_P\{\cdot \mid \mathfrak{X}_{-n,1}\}$ we denote the conditional expectation corresponding to P and measurable with respect to $\mathfrak{X}_{-n,1}$.

By *Jensen's inequality*, since log is a *concave* function, one obtains

$$\mathscr{E}_P\{\log s_n \mid \mathfrak{X}_{-n,1}\} \leqq \log \mathscr{E}_P\{s_n \mid \mathfrak{X}_{-n,1}\} \tag{2.20}$$

so that

$$A_n \geqq B_n \tag{2.21}$$

From (2.15) and (2.12') we obtain

$$\varlimsup_{n\to\infty} A_n = \varlimsup_{n\to\infty} H(P_{-n,1}, PQ_{-n,0,1}) - H(P_{-\infty,1}, PQ_{-\infty,0,1}) \tag{2.22}$$

$$\varliminf_{n\to\infty} A_n = \varliminf_{n\to\infty} H(P_{-n,1}, PQ_{-n,0,1}) - H(P_{-\infty,1}, PQ_{-\infty,0,1})$$

Since, one the other hand,

$$H(P_{-n,1}, PQ_{-n,0,1}) \leqq H(P_{-\infty,1}, P\tilde{Q}^{(\infty)}_{-n,0,1})\,,$$

it follows from (2.16) and (2.22) that

$$\varlimsup_{n\to\infty} A_n \leqq \varlimsup_{n\to\infty} B_n \tag{2.23}$$

$$\underline{\lim}_{n\to\infty} A_n \leqq \underline{\lim}_{n\to\infty} B_n$$

Now, comparing (2.21) and (2.23) we obtain

$$\overline{\lim}_{n\to\infty} A_n = \overline{\lim}_{n\to\infty} B_n \tag{2.24}$$

$$\underline{\lim}_{n\to\infty} A_n = \underline{\lim}_{n\to\infty} B_n$$

By assumption (3), however, we have $\lim_{n\to\infty} B_n = O$ and, thus, from (2.24) we obtain $\lim_{n\to\infty} A_n = O$, what, according to (2.13), completes the proof of the lemma (cf. (2.11)).

Q.E.D.

In the next section we shall prove some stronger results.

3. SOME NEW RESULTS

The following theorem represents a stronger version of the Lemma 2.1 (Perez (1972)) proved in a new way in section 2.

Theorem 1. For well-defined probability measure $PQ_{-\infty,0,1}$ for which it holds

$$H(P_{-\infty,1}, PQ_{-\infty,0,1}) < \infty \tag{i}$$

the condition

$$\lim_{n\to\infty} \int \left| \log \frac{\mathrm{d}PQ_{-\infty,0,1}}{\mathrm{d}P\tilde{Q}^{(\infty)}_{-n,0,1}} \right| \mathrm{d}P = O \tag{ii}$$

(where $P\tilde{Q}^{(\infty)}_{-n,0,1}$ is the extension on $\mathfrak{X}_{-\infty,1}$ of $PQ_{-n,0,1}$ defined by

$$P\tilde{Q}^{(\infty)}_{-n,0,1}(E \times F) = \int_E p_Q(F/x_{-n,0})\,\mathrm{d}P_{-\infty,0}$$

for $E \in \mathfrak{X}_{-\infty,0}$, $F \in \mathfrak{X}_1$) is *necessary* and *sufficient* for the convergence in P-mean of the sequence

$$\left\{\log \frac{\mathrm{d}P_{-n,1}}{\mathrm{d}PQ_{-n,0,1}}\right\}_{n\geqq 1}.$$

Proof. The sufficiency was already proved in Section 2. As to the necessity of (ii) it is proved as follows. We have (cf. (2.12))

$$\Delta_n = \int \left| \log \frac{\mathrm{d}P_{-\infty,1}}{\mathrm{d}PQ_{-\infty,0,1}} - \log \frac{\mathrm{d}P_{-n,1}}{\mathrm{d}PQ_{-n,0,1}} \right| \mathrm{d}P = \tag{3.1}$$

$$= \int \left| \log \frac{\mathrm{d}P_{-\infty,1}}{\mathrm{d}PQ_{-\infty,0,1}} - \log \frac{\mathrm{d}P_{-n,1}}{\mathrm{d}PQ^{(\infty)}_{-n,1}} - \log \frac{\mathrm{d}PQ^{(\infty)}_{-n,1}}{\mathrm{d}PQ_{-,n,0,1}} \right| \mathrm{d}P .$$

Let us introduce the probability measure $\tilde{P}^{(\infty)}_{-n,1}$ on $\mathfrak{X}_{-\infty,1}$ by extending $P_{-n,1}$ as follows.

(3.2) $$\tilde{P}^{(\infty)}_{-n,1}(E) = \int_E \frac{\mathrm{d}P_{-n,1}}{\mathrm{d}PQ^{(\infty)}_{-n,1}} \, \mathrm{d}PQ_{-\infty,0,1}$$

$E \in \mathfrak{X}_{-\infty,1}$. As, in Section 2, $PQ^{(\infty)}_{-n,1}$ is the restriction of $PQ_{-\infty,0,1}$ on $\mathfrak{X}_{-n,1}$.

It holds for $n = 1, 2, \ldots$

(3.3) $$P_{-\infty,1} \ll \tilde{P}^{(\infty)}_{-n,1} .$$

Indeed, let

$$C_n = \left\{ x : \frac{\mathrm{d}P_{-n,1}}{\mathrm{d}PQ_{-n,0,1}} = 0 \right\} \in \mathfrak{X}_{-n,1} .$$

Then

$$\tilde{P}^{(\infty)}_{-n,1}(C_n) = P_{-\infty,1}(C_n) = P_{-n,1}(C_n) = O .$$

If for some set $E \in \mathfrak{X}_{-\infty,1}$ it holds $P_{-\infty,1}(E) > O$ then

$$P_{-\infty,1}(E) = P_{-\infty,1}(E - C_n) + P_{-\infty,1}(E \cap C_n) = P_{-\infty,1}(E - C_n) > O .$$

Since, on the other hand, $P_{-\infty,1} \ll PQ_{-\infty,0,1}$ (according to assumption (i)) it follows that

$$PQ_{-\infty,0,1}(E - C_n) > O .$$

But then

$$\tilde{P}^{(\infty)}_{-n,1}(E) = \tilde{P}^{(\infty)}_{-n,1}(E - C_n) = \int_{E - C_n} \frac{\mathrm{d}P_{-n,1}}{\mathrm{d}PQ^{(\infty)}_{-n,1}} \, \mathrm{d}PQ_{-\infty,0,1} > O$$

and, thus, (3.3) is proved, the corresponding Radon-Nikodym derivative being

(3.4) $$\frac{\mathrm{d}P_{-\infty,1}}{\mathrm{d}\tilde{P}^{(\infty)}_{-n,1}} = \frac{\mathrm{d}P_{-\infty,1}}{\mathrm{d}PQ_{-\infty,0,1}} \Big/ \frac{\mathrm{d}P_{-n,1}}{\mathrm{d}PQ^{(\infty)}_{-n,1}} .$$

Thus, (3.1) may be written

(3.5) $$\Delta_n = \int \left| \log \frac{\mathrm{d}P_{-\infty,1}}{\mathrm{d}\tilde{P}^{(\infty)}_{-n,1}} - \log \frac{\mathrm{d}PQ^{(\infty)}_{-n,1}}{\mathrm{d}PQ_{-n,0,1}} \right| \mathrm{d}P .$$

If $\Delta_n \to 0$, then in particular

(3.6) $$\lim_{n \to \infty} \int \left| \log \frac{\mathrm{d}PQ^{(\infty)}_{-n,1}}{\mathrm{d}PQ_{-n,0,1}} \right| \mathrm{d}P = O$$

 since it holds also

$$\int\left|\log \frac{dPQ^{(\infty)}_{-n,1}}{dPQ_{-n,0,1}}\right| dP \leqq \Delta_n + \int\left|\log \frac{dP_{-\infty,1}}{d\tilde{P}^{(\infty)}_{-n,1}}\right| dP \tag{3.7}$$

and

$$\lim_{n\to\infty} \int\left|\log \frac{dP_{-\infty,1}}{d\tilde{P}^{(\infty)}_{-n,1}}\right| dP = O\,, \tag{3.8}$$

due to the fact that the *entropy*

$$H(P_{-\infty,1}, \tilde{P}^{(\infty)}_{-n,1}) = \int \log \frac{dP_{-\infty,1}}{d\tilde{P}^{(\infty)}_{-n,1}}\, dP = H(P_{-\infty,1}, PQ_{-\infty,0,1}) - H(P_{-n,1}, PQ^{(\infty)}_{-n,1})$$

converges to zero under assumption (i).

(*Remark.* What is just proved (cf. (3.5) and (3.6)) says more: A necessary and sufficient condition for $\Delta_n \to O$ is (3.6) under assumption (i).)

From (3.6) in particular it follows

$$\lim_{n\to\infty} A_n = \lim_{n\to\infty} \int \log \frac{dPQ^{(\infty)}_{-n,1}}{dPQ_{-n,0,1}}\, dP = O$$

so that by (2.24) [proved under (i)]

$$\lim_{n\to\infty} B_n = \lim_{n\to\infty} \int \log \frac{dPQ_{-\infty,0,1}}{dP\tilde{Q}^{(\infty)}_{-n,0,1}}\, dP = O \tag{3.9}$$

$$= \lim_{n\to\infty} \int \log s_n\, dP \qquad \text{(cf. (2.17))}$$

Thus, for proving (ii), i.e.

$$\lim_{n\to\infty} \int |\log s_n|\, dP = O \tag{ii}$$

it is sufficient (cf. (3.9)) to prove that

$$\lim_{n\to\infty} \int_{s_n \geqq 1} \log s_n\, dP = O\,. \tag{3.10}$$

On the base of (3.6), i.e. of

$$\lim_{n\to\infty} \int \log \mathscr{E}_P\{s_n/\mathfrak{X}_{-n,1}\}|\, dP = O$$

we have

$$(3.11) \qquad \lim_{n\to\infty} \int_{\mathscr{E}_P\{\log s_n/\mathfrak{X}_{-n,1}\} \geqq O} \log \mathscr{E}_P\{s_n/\mathfrak{X}_{-n,1}\}\, \mathrm{d}P = O \qquad (\text{cf. } (2.20))\,.$$

But (Jensen's inequality and log concave)

$$O \leqq \int_{s_n \geqq 1} \log s_n \, \mathrm{d}P \leqq \int_{\mathscr{E}_P\{\log s_n/\mathfrak{X}_{-n,1}\} \geqq O} \mathscr{E}_P\{\log s_n/\mathfrak{X}_{-n,1}\}\, \mathrm{d}P \leqq$$

$$\leqq \int_{\mathscr{E}_P\{\log s_n/\mathfrak{X}_{-n,1}\} \geqq O} \log \mathscr{E}_P\{s_n/\mathfrak{X}_{-n,1}\}\, \mathrm{d}P \to O$$

according to (3.11). Thus, (3.10) is proved and this completes the proof of the theorem.

Q.E.D.

Theorem 2. Under $P_{0n} \equiv Q_{0n}$ $(1 \leqq n < \infty)$, a necessary and sufficient condition for the convergence in the P-mean of the sequence

$$\left\{\log \frac{\mathrm{d}P_{-n,1}}{\mathrm{d}PQ_{-n,0,1}}\right\}_{n \geqq 1}$$

is the fulfillement of

$$(\mathrm{j}) \qquad \lim_{n\to\infty} H(P_{-n,1}, PQ_{-n,0,1}) = \hbar < \infty$$

and

$$(\mathrm{jj}) \qquad \lim_{m>n\to\infty} \int \left|\log \frac{\mathrm{d}P\tilde{Q}^{(\infty)}_{-m,0,1}}{\mathrm{d}P\tilde{Q}^{(\infty)}_{-n,0,1}}\right| \mathrm{d}P = O$$

(where $P\tilde{Q}^{(\infty)}_{-n,0,1}$ is the probability measure on $\mathfrak{X}_{-\infty,1}$ obtained by extension of $PQ_{-n,0,1}$ and defined through

$$P\tilde{Q}^{(\infty)}_{-n,0,1}(E \times F) = \int_E p_Q(F/x_{-n,0})\, \mathrm{d}P_{-\infty,0}$$

for $E \in \mathfrak{X}_{-\infty,0}$, $F \in \mathfrak{X}_1$).

Proof. For proving the *sufficiency* of the pair (j) and (jj) we may proceed as in the proof of Lemma 2.1 in Section 2. The only difference is that instead of $\Delta_n \to 0$ we have to prove that $(m > n)$

$$\Delta_{n,m} = \int \left|\log \frac{\mathrm{d}P_{-m,1}}{\mathrm{d}PQ_{-m,0,1}} - \log \frac{\mathrm{d}P_{-n,1}}{\mathrm{d}PQ_{-n,0,1}}\right| \mathrm{d}P \to O$$

for $n \to \infty$ (and, thus, $m \to \infty$).

Under (jj), it reduces to prove

$$\lim_{n,m\to\infty} \delta_{n,m} = O\,,$$

$$\delta_{n,m} = \int\left(\log \frac{\mathrm{d}P_{-m,1}}{\mathrm{d}PQ_{-m,0,1}} - \log \frac{\mathrm{d}P_{-n,1}}{\mathrm{d}PQ_{-n,0,1}}\right)\mathrm{d}P =$$

$$= H(P_{-m,1}, PQ_{-m,0,1}) - H(P_{-n,1}, PQ_{-n,0,1})\,,$$

what takes place according to (j).

As to the *necessity* of the pair (j) and (jj), it may be reduced to the necessity proof of Theorem 1, where the role of $PQ_{-\infty,0,1}$ is played by the following probability measure: $\Delta_{nm} \to 0$ means that there is a limit function $G(x)$ such that

$$\lim_{n\to\infty} \int\left|\log \frac{\mathrm{d}P_{-n,1}}{\mathrm{d}PQ_{-n,0,1}} - G\right| \mathrm{d}P = O\,. \tag{3.12}$$

Observing that under assumption $P_{0n} \equiv Q_{0n}$ for $n = 1, 2, \ldots$

$$P_{-n,1} \equiv PQ_{-n,0,1}$$

(and not only $P_{-n,1} \ll PQ_{-n,0,1}$ used up to now) we *define* $PQ_{-\infty,0,1}$ by

$$PQ_{-\infty,0,1}(E) = \int_E \mathrm{e}^{-G}\,\mathrm{d}P_{-\infty,1} \tag{3.13}$$

for $E \in \mathfrak{X}_{-\infty,1}$.

Then

$$G = \log \frac{\mathrm{d}P_{-\infty,1}}{\mathrm{d}PQ_{-\infty,0,1}}$$

and (3.12) gives in particular

$$\lim_{n\to\infty} H(P_{-n,1}, PQ_{-n,0,1}) = H(P_{-\infty,1}, PQ_{-\infty,0,1})\,.$$

Now, this limit is moreover necessarily finite if $\Delta_{n,m} \to 0$. Thus, for the $PQ_{-\infty,0,1}$ as defined by (3.13) condition (i) of Theorem 1 is satisfied. But under this condition (equivalent under our special definition of $PQ_{-\infty,0,1}$ to condition (j)) the condition (ii) (written for this $PQ_{-\infty,0,1}$) is necessary for $\Delta_n \to 0$ (written for this $PQ_{-\infty,0,1}$) or $\Delta_{n,m} \to 0$. But condition (ii) implies condition (jj) since

$$\int\left|\log \frac{\mathrm{d}P\tilde{Q}^{(\infty)}_{-m,0,1}}{\mathrm{d}P\tilde{Q}^{(\infty)}_{-n,0,1}}\right| \mathrm{d}P \leq \int\left|\log \frac{\mathrm{d}PQ_{-\infty,0,1}}{\mathrm{d}P\tilde{Q}^{(\infty)}_{-m,0,1}}\right| \mathrm{d}P + \int\left|\log \frac{\mathrm{d}PQ_{-\infty,0,1}}{\mathrm{d}P\tilde{Q}^{(\infty)}_{-n,0,1}}\right| \mathrm{d}P\,.$$

(Remark that all the $P\tilde{Q}^{(\infty)}_{-n,0,1}$'s are equivalent to $P_{-\infty,1}$ and, thus, equivalent also to $PQ_{-\infty,0,1}$ as defined by (3.13), so that all the densities above exist).

Thus, condition (jj) is also necessary. This completes the proof that conditions (j) (jj) are also necessary. Q.E.D.

4. REMARKS

1) On the base of definition (3.13) we obtain

$$(4.1)\qquad PQ_{-\infty,0,1}(E \times F) = \int_{E\times F} e^{-G}\,dP_{-\infty,1} =$$

$$= \int_{E\times F} e^{-G}\,dp_P(x_1/x_{-\infty,0})\,dP_{-\infty,0} = \int_{E\times F} dp_Q(x_1/x_{-\infty,0})\,dP_{-\infty,0} =$$

$$= \int_E p_Q(F/x_{-\infty,0})\,dP_{-\infty,0}$$

for $E \in \mathfrak{X}_{-\infty,0}$ and $F \in \mathfrak{X}_1$ *defining*

$$(4.2)\qquad p_Q(F/x_{-\infty,0}) = \int_F e^{-G(x_1,x_{-\infty,0})}\,dp_P(x_1/x_{-\infty,0})\,.$$

Thus, we see that $PQ_{-\infty,0,1}$ as defined by (3.13) has the form of $PQ_{-n,0,1}$ so that, by extension, it is denoted in a similar way.

2) The question arises:

In what extent $p_Q(\cdot/x_{-\infty,0})$ as defined by (4.2) corresponds to $p_Q(\cdot/x_{-\infty,0}) = \lim p_Q(\cdot/x_{-n,0})$ a.s. [P] as taken in condition (1) of Lemma 2.1 (Section 2).

We have [P]

$$\frac{dp_Q(x_1/x_{-n,0})}{dp_P(x_1/x_{-n,0})} = \frac{dPQ_{-n,0,1}}{dP_{-n,1}} = \frac{1}{g_n}\,.$$

Thus,

$$(4.3)\qquad p_Q(F/x_{-n,0}) = \int_F \frac{1}{g_n}\,dp_P(x_1/x_{-n,0}) = \int_F e^{-\log g_n}\,dp_P(x_1/x_{-n,0})\,.$$

Now, under $\{\log g_n\}_{n\geq 1}$ converges in P-mean to $G = \log g$ we have, in particular, $\log g_n \to \log g$ in P-probability $\Rightarrow g/g_n \to 1$ in P-probability.

Since, on the other hand,

$$p_P(\cdot/x_{-n,0}) \to p_P(\cdot/x_{-\infty,0}) \quad \text{a.s. } [P] \text{ or}$$

$$\frac{\mathrm{d}p_P(x_1/x_{-\infty,0})}{\mathrm{d}p_P(x_1/x_{-n,0})} \to 1 \qquad \text{a.s. } [P]$$

it follows that

$$\frac{\mathrm{d}p_Q(x_1/x_{-\infty,0})}{\mathrm{d}p_Q(x_1/x_{-n,0})} = \frac{g}{g_n} \frac{\mathrm{d}p_P(x_1/x_{-\infty,0})}{\mathrm{d}p_P(x_1/x_{-n,0})} \to 1 \quad \text{in } P\text{-probability}$$

or

$$\log \frac{\mathrm{d}p_Q(x_1/x_{-\infty,0})}{\mathrm{d}p_Q(x_1/x_{-n,0})} \to 0 \quad \text{in } P\text{-probability}\,.$$

But this is derivable directly from the necessary condition (ii) or (jj):

(ii)
$$\int \left| \log \frac{\mathrm{d}p_Q(x_1/x_{-\infty,0})}{\mathrm{d}p_Q(x_1/x_{-n,0})} \right| \mathrm{d}P \to O\,.$$

It seems that condition (1) in Lemma 2.1 (section 2) is superfluous. Condition (3) must however be formulated as follows: "*let there exists a.s.* $[P]$ *a conditional probability function* $p_Q(\cdot/x_{-\infty,0})$ *such that* (3) *resp.* (ii) *holds*". *Condition* (2) *resp.* (i) *must then follow condition* (3).

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