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COALITIONAL STABILITY AND RATIONALITY IN COOPERATIVE GAMES

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We propose a new solution concept which characterizes stable agreements in cooperative games. Although it is a refinement of an earlier solution concept introduced by McKelvey, Ordeshook and Winer [6], there are some similitudes with other solutions of cooperative games. The new concept may be defined both for transferable and non-transferable utility games and it is shown that it exists for very large classes of games.

1. INTRODUCTION

Traditionally, the formal definition of cooperative games involves coalitions and payoffs, and in this framework a solution must satisfy some stability conditions. But all solution theories begin by defining a set of *rational outcomes*. Thus, the classical solution concepts as core and von Neumann-Morgenstern solutions are sets of *imputations*, payoff vectors satisfying the individual rationality and a strong form of collective rationality. Other solutions, as the kernel and various bargaining sets are formed by *payoff configurations* which subdivide the global payoff vectors into coalitionally rational payoffs associated with the disjoint coalitions of a partition of the players set. In this last situation, a solution predicts the coalitions which will be formed and the payoffs for each such coalition. In a somewhat different approach a solution predicts configurations in which the coalitions are not necessarily disjoint. In this case, each player makes his option for one or more coalitions which seems to be profitable for him. In fact, such solution predicts potentially reasonable coalitions and payoffs for the negociation which will take place before players will commit themselves to the effective coalitions. This is the case of the *competitive solutions* of McKelvey, Ordeshook and Winer [6], but also of other solutions proposed by Albers [1], Cross [5], Bennett [3].

The new concept of *uniform competitive solution* closely follows the definition of the earlier notion of *competitive solution*. One major property of such a solution is the existence under very general conditions. As well as the original concept, an uniform competitive solution is a stable set of proposals, but it also satisfies the rationality requirement in a stronger form than the competitive solutions.

As usually, in the present paper, the set of players will be denoted by $N =$

$\{1, \ldots, n\}$ and for any coalition $C \subseteq N, \mathbb{R}^C$ stands for the set of the $|C|$-dimensional vectors. The projection of $x \in \mathbb{R}^N$ onto the subspace $\mathbb{R}^C$ will be denoted by x_C and for any subset $A \subseteq \mathbb{R}^N$ we will write $pr_C A$ for the set $\{x_C \mid x \in A\}$. (For a singleton coalition $C = \{i\}$ we will write simply $pr_i A$). Also, $x(C)$ denotes the sum $\sum_{i \in C} x_i$. If x, y are two vectors in the same Euclidean space, then $x \geq y$ denotes the usual ordering defined by $x_i \geq y_i$ for all i, $x > y$ means $x \geq y$, $x \neq y$ and $x \gg y$ will be used when $x_i > y_i$ for all i.

A n-person cooperative game in the characteristic function form is defined by the pair (N, V), where V is a set-valued function on 2^N which associates with each coalition S a subset $V(S)$ of $\mathbb{R}^S$. Conventionally, $V(S)$ represents the set of effective payoffs of S i.e. the set of all possible utility vectors that can be obtained by the members of the coalition S. The most general situation described by the pair (N, V) is of the non-transferable utility (NTU) games, but any transferable utility (TU) game may be also represented as above. Traditionally, a TU game is defined by its characteristic function $\nu : 2^N \mapsto \mathbb{R}$, where $\nu(S)$ is interpreted as the total payoff that the coalition S can make independent of actions of the players outside S. Obviously, in the TU case V can be defined as

$$V(S) = \{x \in \mathbb{R}^S \mid x(S) \leq \nu(S), x \geq a_S\} \tag{1}$$

where $a \in \mathbb{R}^N$ is a given vector. As usually, we will take a to be 0.

2. COMPETITIVE AND UNIFORM COMPETITIVE SOLUTIONS

Although the definition and the general properties of the solutions studied in the present paper are not dependent of any additional assumptions, we will consider the pair (N, V) satisfying the following three conditions:

(i) $V(C)$ is a closed (possibly empty) subset of $\mathbb{R}^C$ for every $C \in 2^N$.

(ii) For every $k \in N$, $v_k = \sup(V(\{k\} \cap pr_k V(N)) < \infty$.

(iii) For every coalition $C \in 2^N$, the set $\{x \in V(C) \mid x_k \geq v_k$ for all $k \in C\}$ is bounded from above.

Some other usual properties of the characteristic function, will be employed in the next.

(iv) $x \in V(C),\ y \in \mathbb{R}^C,\ y \leq x \Rightarrow y \in V(C)$.

(v) $C \subset D \Rightarrow V(C) \subseteq pr_C V(D)$.

Condition (iv), referred to as "comprehensiveness", allows for the free disposal of utility for any coalition.

Property (v) will be called "weak monotonicity" and insures that an utility which is effective for a coalition is still available for its members if they commit to a larger coalition.

For the main existence result of Section 3 the game (N, V) will be subjected to a less usual condition:

(vi) If $x, y \in V(S)$, for some $S \in 2^N$ such that $x_k > y_k$ for some $k \in S$ and $x_i = y_i$ for all $i \neq k$ then there exists $z \in V(S)$ such that $z_k = y_k$ and $z_i > x_i$ for all $i \in S \setminus \{k\}$.

This condition states that, within the set of effective payoffs of a given coalition it is always possible to improve the utility of all other players if one player accepts to diminish his own utility.

Note that any TU game satisfies the conditions (i) – (iv) and (vi).

For the remainder of this section it is no matter if the pair (N, V) will be a TU or a NTU game.

Definition 1. A proposal of the game (N, V) is a pair (x, S), where $S \in 2^N, S \neq \emptyset$ and $x \in V(S) \cap pr_S V(N)$.

Intuitively, a proposal represents an offer that a coalition could make. The payoff x is effective for S but it must be feasible too, i. e. it can be extended up to a payoff vector of $V(N)$.

For the following discussion, we will consider a finite collection $\mathcal{S}$ of proposals with no coalition associated with more than one proposal: $\mathcal{S} = \{(u^C, C) \mid C \in \mathcal{C}\}$ where $\mathcal{C} \subseteq 2^N$.

Definition 2 (McKelvey, Ordeshook, Winer [6]). $\mathcal{S}$ is a competitive solution (c.s.) if it satisfies the following two conditions:

$$\text{There are no } C, D \in \mathcal{C} \text{ such that } u^C_{C \cap D} > u^D_{C \cap D}. \tag{2}$$

$$\text{If } (x, S) \text{ is a proposal such that } x_{S \cap C} \gg u^C_{S \cap C} \text{ for some } C \in \mathcal{C} \text{ then there exist } D \in \mathcal{C} \text{ such that } u^D_{S \cap D} \gg x_{S \cap D}. \tag{3}$$

Definition 3 (Stefanescu [7]). S is an uniform competitive solution (u.c.s.) if the following conditions are satisfied:

$$u^C_{C \cap D} = u^D_{C \cap D} \text{ for every } C, D \in \mathcal{C} \tag{4}$$

$$\text{If } (x, S) \text{ is a proposal such that } x_{S \cap C} > u^C_{S \cap C} \text{ for some } C \in \mathcal{C} \text{ then there exist } D \in \mathcal{C} \text{ and } k \in S \cap D \text{ such that } u^D_k > x_k. \tag{5}$$

In summary both the competitive solutions and the uniform competitive solutions are stable configurations of proposals. This stability is internal (condition (2), respectively (4)) and external (condition (3), respectively (5)). The classical domination relation is replaced here by the preference relation of the pivotal players of two proposals (the players which belong to both coalitions associated with the considered proposals). As it will be shown in the next section these solutions respond to the rationality principles. It would be also interesting to point out the relationships with other solution concepts. Particularly, the core and the aspirations are closely related of the c.s., respectively, u.c.s.

Definition 4. The core of the game (N,V) is the set:

$$C(N,V)=\{u\in\ V(N)\,|\,\text{there is no proposal } (x,S) \text{ such that } x\gg u_S\}.$$

Definition 5. An aspiration is a payoff vector $u\in\mathrm{IR}^N$ satisfying the following two conditions:

$$\bigcup\{C\,|\,u_C\in V(C)\}=N. \tag{6}$$

$$\text{If } (x,S) \text{ is a proposal such that } x\geq u_S \text{ then } x=u_S. \tag{7}$$

3. GENERAL NTU GAMES

Let $\mathcal{S}$ be either a c.s. or an u.c.s. and set $K=\bigcup_{C\in\mathcal{C}}C$.

If $K=N$ the solution will be said *complete.*

In the general case, let us define the $|K|$-dimensional vector w by:

$$w_k=\max\{u_k^C\,|\,k\in C,C\in\mathcal{C}\}$$

and call w the *ideal payoff vector associated with* $\mathcal{S}$.

The next result establishes the individual rationality of both solutions defined in Section 2.

Proposition 3.1. For any c.s. or u.c.s. $w_k\geq v_k$ for all $k\in K$.

The coalitional rationality is expressed in somewhat different terms for u.c.s. and c.s.

Proposition 3.2. Let $(u^C,C)\in\mathcal{S}$. If $\mathcal{S}$ ia an u.c.s. (c.s.) then u^C is a Pareto-optimum (weak Pareto-optimum) of $V(C)\cap pr_C V(N)$.

It was firstly shown in McKelvey, Ordeshook and Winer [6] that if the core is nonempty then a single-proposal competitive solution always exists. This result can be extended to the u.c.s. and a converse implication also holds.

Proposition 3.3. Let (N,V) be any cooperative game such that $C(N,V)\neq\emptyset$. If $u\in C(N,V)$ then $\mathcal{S}=\{(u,N)\}$ is a c.s.

Proposition 3.4. Assume (N,V) be a cooperative game satisfying the conditions (iv) and (vi). If $C(N,V)\neq\emptyset$ and $u\in C(N,V)$ then $\mathcal{S}=\{(u,N)\}$ is an u.c.s.

Proposition 3.5. If $\mathcal{S} = \{(u, N)\}$ is either a (complete) u.c.s. or a c.s. then $u \in C(N, V)$. Consequently, $C(N, V) \neq \emptyset$.

As an important consequence of the previous results it follows that any sufficient condition for the nonemptiness of the core is a sufficient condition for the existence of the c.s. Particularly, balanced or (ordinal) convex games admits c.s. The same conclusion holds for the u.c.s. if the NTU games satisfy conditions (iv) and (vi).

In the following we will pay more attention for the complete uniform competitive solutions (c.u.c.s.). A new characterization of a c.u.c.s. follows from the two next propositions.

Proposition 3.6. Let $\mathcal{S} = \{(u^C, C) \mid C \in \mathcal{C}\}$ be a c.u.c.s. and w the associated ideal payoff. Then:

$$w_C \in (V(C) \cap pr_C V(N)) \text{ for all } C \in \mathcal{C}. \tag{8}$$

$$\text{If } (x, S) \text{ is a proposal and } x \geq w_S \text{ then } x = w_S. \tag{9}$$

Note that for the u.c.s., $w_C = u^C$ for every $C \in \mathcal{C}$. Therefore, the converse of the previous proposition immediately follows.

Proposition 3.7. Assume $\mathcal{C}$ be a collection of coalitions whose union is N and $w \in \mathbb{R}^N$ such that the conditions (8) and (9) are satisfied. Then, $\mathcal{S} = \{(w_C, C) \mid C \in \mathcal{C}\}$ is a c.u.c.s. Moreover, w is the ideal payoff associated with $\mathcal{S}$.

Finally the relationships between the c.u.c.s. and the aspirations can be established.

Proposition 3.8. Assume (N, V) be a NTU game satisfying (iv) and (vi). If $\mathcal{S}$ is a c.u.c.s. then the associated ideal payoff w is an aspiration.

Proposition 3.9. If the game (N, V) satisfies (v) and u is an aspiration then $\mathcal{S} = \{(u_C, C) \mid C \in \mathcal{C}\}$ where $\mathcal{C} = \{C \mid u_C \in (V(C) \cap pr_C V(N))\}$ is a c.u.c.s.

The main result of this section establishes the existence of the u.c.s. Moreover, the completeness of this solution can be also guaranteed.

Theorem 3.10. Let (N, V) be a cooperative game in the characteristic function form satisfying the properties (i)–(vi). If $V(N) \neq \emptyset$ then the game admits complete uniform competitive solutions (c.u.c.s.).

Proof. Note firstly that the set of coalitions $\mathcal{W} = \{C \in 2^N \mid V(C) \neq \emptyset\}$ is nonempty and closed with respect to the set-inclusion relation, i.e. it satisfies the following property:

$$C \in \mathcal{W},\ C \subset D \Rightarrow D \in \mathcal{W}.$$

Pick $T \in \mathcal{W}$ such that $|T| = \min\{|C|\,|C \in \mathcal{W}\}$. Obviously, $v_k = -\infty$ for all $k \in T$ and then it follows by (i) and (iii) that $V(T)$ is closed and bounded from above. Then there exists a Pareto optimum point of $V(T) \cap pr_T(N)$, say z.

Set $M = N \setminus T$. The "reduced game" (M, V_M) is defined by:

$$V_M(C) = \bigcup_{S \subseteq T} \{x \in \mathbb{R}^C \,|(x, z_S) \in V(C \cup S)\}$$

for any $\emptyset \neq C \subseteq M$.

One can easily verify that the game (M, V_M) satisfies the conditions (i)–(vi).

Suppose that it admits a c.u.c.s. and let $y \in \mathbb{R}^M$ its associated ideal payoff vector. Obviously, y toghether the set $\mathcal{C}$ of the coalitions involved in the proposals of the solution satisfy the conditions (8) and (9) of Proposition 3.6. Of course, for each $C \in \mathcal{C}$ there is $S \subseteq T$ such that $(y_C, z_S) \in V(C \cup S)$. Set $C' = C \cup S$ and $\mathcal{C}' = \{C' = C \cup S \,|\, C \in \mathcal{C}\}$. The crucial step of the proof consists by showing that the n-vector $u = (y, z)$ and the set $\mathcal{C}'$ satisfy the conditions (8) and (9). Then, by Proposition 3.7, the original game admits a c.u.c.s.

Since (8) immediately follows from (v) and the definition of $\mathcal{C}'$ we must verify condition (9).

To the contrary, assume (x, E) be a proposal of the game (N, V) such that $x > u_E$. Obviously, $E \cap M \neq \emptyset$, i.e. $E = C \cup S$ with $C \neq \emptyset, C \subseteq M$ and $S \subseteq T$. Since $(x_C, z_S) \in V(E)$ it follows that $x_C \in V_M(C)$. So that it is impossible to have $x_C > u_C = y_C$, otherwise the definition of y would be violated. Therefore, $x_C = y_C$ and $x_C > z_S$. Let $k \in S$ be such that $x_k > z_k$. Then, by (vi) there is $w \in V(E)$ such that $w_i > x_i = y_i$ for every $i \in C$ and $w_j \geq z_j$ for all $j \in S$. Hence, $w_C \in V_M(C)$ and $w_C \gg y_C$, a contradiction.

Now the theorem can be proved in two steps. In the first step, considering $\mathcal{W} = 2^N \setminus \{\emptyset\}$ show, by induction on $n = |N|$, that (N, V) has a c.u.c.s.

For $n = 1, V(\{1\})$ is a nonvoid compact and $\{(v_1, \{1\})\}$ is a c.u.c.s. Assume that every game with at most $n-1$ players has a c.u.c.s. and let (N, V) be a game with $|N| = n$. Set $M = N \setminus \{n\}$ and consider the game (M, V_M). Since $V_M(C) \neq \emptyset$ for all $C \neq \emptyset, C \subseteq M$, it follows by induction that (M, V_M) has a c.u.c.s. Taking $T = \{n\}$ in the above it will follow that (N, V) has a c.u.c.s. too.

The second step concerns with the general case. Since for the reduced game (M, V_M) it is obvious that $\mathcal{W}_M = \{C \subseteq M \,|\, V_M(C) \neq \emptyset\} = 2^M \setminus \{\emptyset\}$, the conclusion of the first step can be used. Then, the desired result follows from the first part of the proof. □

4. TU GAMES

As it was already mentioned, any TU game satisfies all conditions (i)–(iv) and (vi). Hence most results stated in the previous section holds for the TU games without special assumptions.

Since the existence of the c.u.c.s. is dependent of the property (v) it seems that the conclusion of the Theorem 3.8 would be also dependent of the monotonicity of the game. Obviously if the characteristic function ν is non-decreasing then the TU game has a c.u.c.s. However, as it was shown in Stefanescu [8], the monotonicity of ν is not necessary.

Theorem 4.1 (Stefanescu [8]). If the characteristic function of a TU game is non-negative, then a c.u.c.s. always exists.

Despite the existing similarities between theirs definitions the c.s. and u.c.s. are different notions. We can exemplify this assertion by producing two simple examples.

Example 1 $n = 3$. $\nu(\{1\}) = \nu(\{2\}) = 1.05$; $\nu(\{3\}) = 0$; $\nu(\{1,2\}) = \nu(N) = 2$; $\nu(\{1,3\}) = \nu(\{2,3\}) = 0.9$.

The set of proposals: $\mathcal{S} = \{((0.9, 1.1), \{1,2\}), ((1.1, 0.9, 0), N)\}$ is a c.s. but not a u.c.s. (condition (4) is violated).

Example 2 $n = 3$. $\nu(\{i\}) = 1$, $i = 1,2,3$; $\nu(\{i,j\}) = 2$ if $\{i,j\} \subset \{1,2,3\}$; $\nu(N) = 3$.

Of course $\mathcal{S} = \{((1,1), \{1,2\}), ((1,1), \{1,3\}), ((1,1), \{2,3\})\}$ is an u.c.s. which is not a c.s. Indeed, $((1.5, 1.5, 0), N)$ is a proposal which is strictly prefered to the first proposal of $\mathcal{S}$ by the pivotal players, and the axiom (3) is not verified.

The last result of this section give a necessary and sufficient condition for the existence of the c.s. It is interesting to note that for the class of TU games involved here the c.s. exists only when it coincides with an u.c.s. We will consider in the following the TU games whose characteristic functions are strictly superadditive i.e. satisfy the condition:

$$C, D \neq \emptyset, \ C \cap D = \emptyset \Rightarrow \nu(C) + \nu(D) < \nu(C \cup D).$$

Theorem 4.2 (Stefanescu [8]). Assume ν be strictly superadditive and non-negative. Then the TU game (N, V) admits a c.s. if and only if $C(N, V) \neq \emptyset$. Moreover, in this case every c.s. is an u.c.s. at the same time.

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