Krishnan Balachandran; P. Balasubramaniam

A note on controllability of nonlinear Volterra integrodifferential systems

Kybernetika, Vol. 28 (1992), No. 4, 284--291

Persistent URL: `http://dml.cz/dmlcz/124810`

KYBERNETIKA — VOLUME 28 (1992), NUMBER 4, PAGES 284–291

A NOTE ON CONTROLLABILITY OF NONLINEAR VOLTERRA INTEGRODIFFERENTIAL SYSTEMS

K. BALACHANDRAN AND P. BALASUBRAMANIAM

Sufficient conditions for complete controllability of nonlinear Volterra integrodifferential systems with implicit derivative are established. The results are generalization of the previous results and are obtained through the notions of condensing map and measure noncompactness of a set.

1. INTRODUCTION

The problem of controllability of dynamical systems described by nonlinear ordinary differential equations has been investigated by several authors with the help of fixed point theorems [6]. Dacka [8] introduced a new method of analysis to study the controllability of nonlinear systems with implicit derivative based on the measure of noncompactness of a set and Darbo's fixed-point theorem. This method has been extended to a larger class of dynamical systems by Balachandran [2, 3, 4]. Anichini *et al.* [1] studied the problem through the notions of condensing map and measure of noncompactness of a set. They used the fixed-point theorem due to Sadovskii [9]. In this paper, we shall study the controllability of nonlinear Volterra integrodifferential systems with implicit derivative, by suitably adopting the technique of Anichini *et al.* [1]. The results generalize the results of Balachandran [5].

2. MATHEMATICAL PRELIMINARIES

We first summarize some facts concerning condensing maps; for definitions and results about the measure of noncompactness and related topics, the reader can refer the paper of Dacka [8]. Let X be a subset of a Banach space. An operator $T : X \to X$ is called condensing if, for any bounded subset E in X with $\mu(E) \neq 0$, we have $\mu(T(E)) < \mu(E)$, where $\mu(E)$ denotes the measure of noncompactness of the set E.

We observe that, as a consequence of the properties of μ, if an operator T is the sum of a compact operator and a condensing operator, then T itself is a condensing operator. Further, if the operator $P : X \to X$ satisfies the condition $|Px - Py| \leq k\,|x - y|$ for $x, y \in X$, with $0 \leq k < 1$, then the operator P is a μ-contractive operator with constant k; that is, $\mu(T(E)) \leq k\mu(E)$ for any bounded set E in X. In this case, P has a fixed point property [9]. However, the condition $|Px - Py| \leq k|x-y|\,(x, y \in X)$ is insufficient to ensure that P is a condensing map or that P will admit a fixed point (see [7]). The fixed point property holds in the condensing case (see [9]).

Let $C_n(J)$ denote the space of continuous $\mathbb{R}^n$-valued functions on the interval J. For $x \in C_n(J)$ and $h > 0$, let

$$\theta(x,h) = \sup\{|x(t) - x(s)| : t, s \in J \text{ with } |t-s| \leq h\},$$

and write $\theta(E,h) = \sup_{x \in E} \theta(x,h)$, so that $\theta(E,\cdot)$ is the modulus of continuity of a bounded set E; and let Ω be the set of functions $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ that are right continuous and nondecreasing such that $\omega(r) < r$, for $r > 0$, put $J = [t_o, t_1]$.

Lemma 1 (cf. [9]). Let $X \subset C_n(J)$ and let β and γ be functions defined on $[0, t_1 - t_o]$ such that $\lim_{s\to 0}\beta(s) = \lim_{s\to 0}\gamma(s) = 0$. If a mapping $T : X \to C_n(J)$ is given such that it maps bounded sets into bounded sets and, it is such that

$$\theta(T(x),h) < \omega(\theta(x,\beta(h))) + \gamma(h) \text{ for all } h \in [0, t_1 - t_o] \text{ and } x \in X$$

with $\omega \in \Omega$, than T is a condensing mapping.

Lemma 2 (cf. [1,9]). Let $X \subset C_n(J)$, let $J = [0,1]$ and let $S \subset X$ be a bounded closed convex set. Let $H : J \times S \to X$ be a continuous operator such that, for any $\alpha \in J$, the map $H(\alpha,\cdot) : S \to X$ is condensing. If $x \neq H(\alpha,x)$ for any $\alpha \in J$ and any $x \in \partial S$ (the boundary of S), then $H(1,\cdot)$ has a fixed point. Finally it is possible to show that, for any bounded and equicontinuous set E in $C_n^1(J)$, the following relation holds:

$$\mu_{C_n^1}(E) \equiv \mu_1(E) = \mu(DE) \equiv \mu_{C_n}(DE)$$

where

$$DE = \{\dot{x} : x \in E\}.$$

3. MAIN RESULT

Consider the nonlinear Volterra integrodifferential system

$$\dot{x}(t) = A(t)x(t) + \int_{t_o}^{t} H(t,s)x(s)\mathrm{d}s + B(t)u(t) + f(t,x)(t), \dot{x}(t), u(t)) \tag{1}$$

where the state $x(t)$ is an n-vector and the control $u(t)$ is an m-vector. The entries of the matrix functions

$$A : J \to \mathbb{R}^{n^2},\ B : J \to \mathbb{R}^{nm} \text{ and } H : \Delta \to \mathbb{R}^{n^2}, \Delta = \{(t,s) : t_o \leq s \leq t \leq t_1\}$$

are assumed to be continuous, also $f : J \times \mathbb{R}^{2n+m} \to \mathbb{R}^n$ is a continuous n-vector function. The solution of the system (1) is given by

$$x(t) = R(t,t_o)x_o + \int_{t_o}^{t} R(t,s)B(s)u(s)\mathrm{d}s + \int_{t_o}^{t} R(t,s)f(s,x)(s), \dot{x}(s), u(s))\mathrm{d}s$$

where

$$\frac{\partial R(t,s)}{\partial s} + R(t,s)A(s) + \int_s^t R(t,\eta)H(\eta,s)d\eta = 0$$

$$R(t,t) = \text{ identity for } t_o \le s \le t \le t_1.$$

We say that system (1) is completely controllable if for any $x_o, x_1 \in \mathbb{R}^n$ there exists a continuous control function $u(t)$ defined on J such that the solution x of (1) satisfies $x(t_1) = x_1$. Define the controllability matrix

$$G(t_o,t) = \int_{t_o}^t R(t,s)B(s)B^*(s)R^*(t,s)\mathrm{d}s$$

where the star denotes the matrix transpose. The main result concerning the controllability of the system (1) is given in the following theorem.

Theorem 1. Suppose that the above conditions are satisfied for the system (1) and assume the additional conditions:

(i) $\limsup_{|x|\to\infty} \frac{|f(t,x,y,u)|}{|x|} = 0$

(ii) there exists a continuous nondecreasing function $\omega : \mathbb{R}^+ \to \mathbb{R}^+$, with $\omega(r) < r$, such that $|f(t,x,y,u) - f(t,x,z,u)| < \omega(|y-z|)$ for all $(t,x,y,u) \in J \times \mathbb{R}^{2n} \times \mathbb{R}^m$

(iii) the symmetric matrix $G(t_o,t_1)$ is nonsingular for some $t_1 > t_o$.

Then the system (1) is completely controllable on J.

Proof. Define the nonlinear transformation

$$T : C_m(J) \times C_n^1(J) \to C_m(J) \times C_n^1(J)$$

by

$$T(u,x)(t) = (T_1(u,x)(t),\ T_2(u,x)(t))$$

where the pair of operators T_1 and T_2 is defined by

$$T_1(u,x)(t) = -B^*(t)\Phi^*(t_1,t)G^{-1}(t_o,t_1)$$

$$\times \left[\int_{t_o}^{t_1} \Phi(t_1,s)f(s,x(s),\dot{x}(s),u(s))\mathrm{d}s - x_1 + \Phi(t_1,t_o)x_o\right]$$

$$T_2(u,x)(t) = \Phi(t,t_o)x_o + \int_{t_o}^t \Phi(t,s)B(s)T_1(u,x)(s)\mathrm{d}s$$

$$+\int_{t_o}^{t}\Phi(t,s)f(s,x(s),\dot{x}(s),T_1(u,x)(s))\mathrm{d}s$$

Since all the functions involved in the definition of the operator T are continuous, T is continuous. Moreover by direct differentiation with respect to t, a fixed point for the operator T gives rise to a control u and a corresponding function $x = x(u)$, solution of the system (1) satisfying $x(t_o) = x_o, x(t_1) = x_1$. Let

$$\begin{aligned}\eta^o &= (u^o, x^o) \in C_m(J) \times C_n^1(J),\\ \eta &= (u, x) \neq 0 \in C_m(J) \times C_n^1(J)\end{aligned}$$

and consider the equation

$$\eta^o = \eta - \alpha T(\eta),$$

where $\alpha \in [0,1]$. This equation can be equivalently written as

$$u = u^o + \alpha T_1(u,x) \tag{2}$$

$$x = x^o + \alpha T_2(u,x) \tag{3}$$

From condition (i), for any $\varepsilon > 0$ there exists $R > 0$ such that if $|x| > R$ then $|f(t,x,y,u)| < \varepsilon|x|$. Then from (2) we get

$$\begin{aligned}|u| &\leq |u^o| + |\alpha|\ |B|\ |\Phi|\ |G^{-1}|\ \{|\Phi|\,\varepsilon\,|x|\ |\delta + |x_1| + |\Phi|\ |x_o|\}\\ &\leq |u^o| + k_1 + |B|\ |\Phi|^2|G^{-1}|\,\varepsilon\,\delta|x|\end{aligned} \tag{4}$$

where $\delta = t_1 - t_o$ and $k_1 = |B|\ |\Phi|\ |G^{-1}|(|x_1| + |\Phi|\ |x_o|)$.
From this inequality and from (3), by applying the Gronwall lemma, we obtain

$$\begin{aligned}|x| &\leq [|x^o| + |\Phi|\ |x_o| + |T_1(u,x)|\ |\Phi|\ |B|\,\delta]\ \exp\,(|\Phi|\,\varepsilon\,\delta)\\ &\leq [|x^o| + |\Phi|\ |x_o| + (k_1 + |B|\ |\Phi|^2|G^{-1}|\,\varepsilon\,\delta|x|)|\Phi|\ |B|\,\delta]\exp\,(|\Phi|\,\varepsilon\,\delta)\end{aligned} \tag{5}$$

Note that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}T_2(u,x)(t) &= A(t)T_2(u,x)(t) + \int_{t_o}^{t} H(t,s)T_2(u,x)(s)\mathrm{d}s\\ &\quad + B(t)T_1(u,x)(t) + f(t,x(t),\dot{x}(t),T_1(u,x)(t))\end{aligned}$$

By application of the Gronwall lemma and by using the change of order of integration, we get

$$T_2(u,x) \leq [|B|\ |T_1(u,x)|\,\delta + \varepsilon\,\delta|x|]\ \exp\,(A_o) \tag{6}$$

where

$$A_o = \int_{t_o}^{t_1} |A(s) + \int_{s}^{t_1} H(\eta,s)\mathrm{d}\eta|\mathrm{d}s.$$

Taking the derivative with respect to t, we obtain from (3)

$$\dot{x} = \frac{dx^o}{dt} + \alpha \frac{d}{dt}(T_2(u,x)(t))$$

and that gives,

$$\begin{aligned}
|\dot{x}| &\leq |\dot{x}^o| + |A|\,|T_2(u,x)| + |H|\,|T_2(u,x)|\,\delta + |B|\,|T_1(u,x)| + \varepsilon|x| \\
&\leq |\dot{x}^o| + |T_1(u,x)|\,[(|A| + |H|\,\delta)|B|\,\delta\exp(A_o) + |B|\,] \\
&\quad + |x|\,[(|A| + |H|\,\delta)\varepsilon\,\delta\exp(A_o) + \varepsilon\,] \\
&= |\dot{x}^o| + k_2 + |x|\,[|B|^2\,|\Phi|^2\,|G^{-1}|\,\delta\varepsilon\,\{(|A| + |H|\,\delta)\,\delta\exp(A_o) + 1\} \\
&\quad + (|A| + |H|\,\delta)\varepsilon\,\delta\exp(A_o) + \varepsilon\,]
\end{aligned} \tag{7}$$

where

$$k_2 = k_1[|B|(|A| + |H|\,\delta)\,\delta\,\exp(A_o) + 1]$$

From (4) we get

$$|u| - |B|\,|\Phi|^2|G^{-1}|\,\varepsilon\,\delta\,|x| \leq |u^o| + k_1$$

and from (5), (6) and (7)

$$|x|\,[\exp(-|\Phi|\,\varepsilon\,\delta) - |B|^2\,|\Phi|^3\,|G^{-1}\,|\,\delta\cdot\varepsilon\delta] \leq k_3 + |x^o|$$

where

$$k_3 = |\Phi|\,|x_o| + k_1|B|\,|\Phi|\,\delta$$

and

$$\begin{aligned}
&|\dot{x}| - |x|\,[|B|^2\,|\Phi|^2\,|G^{-1}|\,\varepsilon\,\delta\,\{(|A| + |H|\,\delta)\,\delta\exp(A_o) + 1\} \\
&\qquad (|A| + |H|\delta)\,\varepsilon\delta\exp(A_o) + \varepsilon] \leq k_2 + |\dot{x}^o|
\end{aligned}$$

Taking the sums of all the above quantities we obtain

$$\begin{aligned}
&|u| - |x|\,\{|B|\,|\Phi|^2\,|G^{-1}|\,\varepsilon\,\delta - \exp(-|\Phi|\,\varepsilon\,\delta) + |B|^2\,|\Phi|^3\,|G^{-1}|\,\delta\cdot\varepsilon\,\delta \\
&\quad + |B|^2\,|\Phi|^2\,|G^{-1}|\,\varepsilon\,\delta\,[(|A| + |H|\,\delta)\,\delta\,\exp(A_o) + 1\,] \\
&\quad + (|A| + |H|\,\delta)\,\varepsilon\,\delta\exp(A_o) + \varepsilon\} + |\dot{x}| \\
&= |u| - \lambda\,|x| + |\dot{x}| \leq |u^o| + k_1 + k_3 + |x^o| + k_2 + |\dot{x}^o|
\end{aligned}$$

where

$$\begin{aligned}
\lambda = |B|\,|\Phi|^2\,|G^{-1}|\,\varepsilon\,\delta\,\{1 + |B|\,|\Phi|\,\delta + |B|\,[(|A| + |H\,|\delta)\,\delta\exp(A_o) + 1]\} \\
+\varepsilon + (|A|\,|H|\,\delta)\,\varepsilon\,\delta\exp(A_o) - \exp(-|\Phi|\,\varepsilon\,\delta)
\end{aligned}$$

Then, for suitable positive constants a, b, c we can write

$$|u| - [\varepsilon a - \exp(-\varepsilon\,b)]\,|x| + |\dot{x}| \leq |u^o| + |x^o| + |\dot{x}^o| + c,$$

so we divide by $|u| + |x| + |\dot{x}|$ and, from the arbitrariness of ε, we get the existence of a ball S in $C_m(J) \times C_n^1(J)$ sufficiently large such that

$$|\eta - \alpha T(\eta)| > 0 \quad \text{for} \quad \eta = (u, x) \in \partial S.$$

We want to show that T is a condensing map. To this aim, we note that $T_1 : C_m(J) \to C_m(J)$ is a compact operator and then, if E is a bounded set, $\mu(T_1(E)) = 0$. Then it will be enough to show that T_2 is a condensing operator. For that, let us consider the modulus of continuity of $DT_2(u, x)(\cdot)$. Now, for $t, s \in J$, we have

$$\begin{aligned} |D\,T_2(u,x)(t) - D\,T_2(u,x)(s)| \le\ & |A(t)\,T_2(u,x)(t) - A(s)\,T_2(u,x)(s)| \\ & + \Big|\int_{t_o}^{t} H(t,\eta)\,T_2(u,x)(\eta)\mathrm{d}\eta - \int_{t_o}^{s} H(s,\eta)\,T_2(u,x)(\eta)\mathrm{d}\eta\Big| \\ & + |B(t)\,T_1(u,x)(t) - B(s)\,T_1(u,x)(s)| \\ & + |f(t,x(t),\dot{x}(t),T_1(u,x)(t)) - f(s,x(s),\dot{x}(s),T_1(u,x)(s))| \end{aligned}$$

For the first three terms of the right side of the inequality we may give the upper estimate as $\beta_o(|t-s|)$ with $\lim_{h\to 0} \beta_o(h) = 0$ and it may be chosen independent of the choice of (u, x). For the fourth term we can give the following estimate:

$$\begin{aligned} & |f(t,x(t),\dot{x}(t),T_1(u,x)(t)) - f(s,x(s),\dot{x}(s),T_1(u,x)(s))| \\ \le\ & |f(t,x(t),\dot{x}(t),T_1(u,x)(t)) - f(t,x(t),\dot{x}(s),T_1(u,x)(t))| \\ & + |f(t,x(t),\dot{x}(s),T_1(u,x)(t)) - f(s,x(s),\dot{x}(s),T_1(u,x)(s))| \end{aligned}$$

For the first term we have the upper estimate $\omega(|\dot{x}(t) - \dot{x}(s)|)$ whereas for the second term we may find an estimate

$$\beta_1(|s-t|) \quad \text{with} \quad \lim_{h\to 0} \beta_1(h) = 0.$$

Hence

$$\theta(D\,T_2(u,x), h) \le \omega(\theta(DE, h)) + \beta(h)$$

where $\beta = \beta_o + \beta_1$. Therefore, by Lemma 1, we get

$$\theta_o(DT_2(E)) < \theta_o(DE)$$

Hence from

$$\begin{aligned} 2\mu_1(T_2(E)) = 2\mu(DT_2(E)) &= \theta_o(DT_2(E)) < \theta_o(DE) \\ &= 2\mu(DE) = 2\mu_1(E) \end{aligned}$$

it follows that $\mu_1(T_2(E)) < \mu_1(E)$. Then the existence of a fixed point of the operator T follows from Lemma 2; that is, there exist functions $u^* \in C_m(J)$ and $x^* \in C_n^1(J)$ such that

$$T(u^*, x^*) = (u^*, x^*),$$

that is,

$$u^*(t) = T_1(u^*, x^*)(t), \quad x^*(t) = T_2(u^*, x^*)(t)$$

These functions are the required solutions. Further, it is easy to verify that the function $x(\cdot)$ given above by the system (1) satisfies the boundary conditions $x(t_o) = x_o$ and $x(t_1) = x_1$. Hence the system (1) is completely controllable. □

Remark 1. If we assume that the nonlinear function in the equation (1) also satisfies the Lipschitz condition with respect to the state variable, then we can obtain the unique response determined by any control.

Remark 2. The result of Theorem 1 still holds if we replace the condition (i) by

$$|f(t, x, \dot{x}, u)| \leq \alpha(t)|x| + \beta(t)$$

where α and β are continuous functions.

4. EXAMPLE

We give an example of application of the above result to the following scalar nonlinear Volterra integrodifferential system.

$$\dot{x}(t) = (\mathrm{e}^{-3(t_1-t)} - 3)\, x(t) + 3\int_{t_o}^{t} \mathrm{e}^{-5(t-s)} x(s)\mathrm{d}s + \mathrm{e}^{-2t}u(t) + \frac{\log x}{\sqrt{1+u^2}} + \operatorname{arctg}\dot{x},$$

for $t_1 > t_o$.

We have here

$$A(t) = \mathrm{e}^{-3(t_1-t)} - 3, \quad H(t,s) = 3\mathrm{e}^{-5(t-s)}, \quad B = \mathrm{e}^{-2t}, \quad f = \frac{\log x}{\sqrt{1+u^2}} + \operatorname{arctg}\dot{x},$$

It has been easily seen that

$$R(t,s) = \mathrm{e}^{-2(t-s)} \quad \text{satisfies}$$

$$\frac{\partial R(t_1,s)}{\partial s} + R(t_1,s)A(s) + \int_s^{t_1} R(t_1,\eta)H(\eta,s)\mathrm{d}\eta = 0.$$

so that

$$\begin{aligned} G(t_o,t_1) &= \int_{t_o}^{t_1} \mathrm{e}^{-4t_1}\mathrm{d}s \\ &= \mathrm{e}^{-4t_1}(t_1 - t_o) > 0 \quad \text{for some } t_1 > t_o. \end{aligned}$$

Furthermore

$$\begin{aligned} |f(t,x,y,u) - f(t,x,z,u)| &= |\operatorname{arctg} y - \operatorname{arctg} z| \\ &< \operatorname{arctg} |y - z| \quad \text{if} \quad y \neq z \end{aligned}$$

and $\lim_{|x|\to\infty} \frac{|f(t,x,y,u)|}{|x|} = 0$, so the hypotheses of Theorem 1 are satisfied. Hence the system is completely controllable.

ACKNOWLEDGEMENT

The authors are thankful to the referee for the improvement of the paper.

(Received April 8, 1991.)

REFERENCES

[1] G. Anichini, G. Conti and P. Zecca: A note on controllability of certain nonlinear systems. Note di Mathematica *6* (1986), 1, 99–111.

[2] K. Balachandran: Global and local controllability of nonlinear systems. Proc. IEE – Part D, Control Theory and Applications *132* (1985), 1, 13–17.

[3] K. Balachandran: Controllability of a class of perturbed nonlinear systems. Kybernetika *24* (1981), 61–64.

[4] K. Balachandran: Controllability of perturbed nonlinear systems. IMA J. Math. Control Inform. *6* (1989), 2, 253–258.

[5] K. Balachandran: Controllability of nonlinear Volterra integrodifferential systems. Kybernetika *25* (1989), 6, 505–508.

[6] K. Balachandran and J. P. Dauer: Controllability of nonlinear systems via fixed point theorems. J. Optim. Theory Appl. *53* (1987), 3, 345–352.

[7] F. E. Browder: Nonexpansive nonlinear operators in Banach spaces. Proc. Nat. Acad. Sci. *54* (1965), 1041–1044.

[8] C. Dacka: On the controllability of a class of nonlinear systems. IEEE Trans. Automat. Control *AC-25* (1980), 2, 263–266.

[9] J. B. Sadovskii: Limit compact and condensing operators. Russian Math. Surveys *27* (1972), 1, 85–155.

Dr. K. Balachandran, P. Balasubramaniam, Department of Mathematics, Bharathiar University, Coimbatore 641 046, Tamil Nadu, India.