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EXISTENCE CONDITIONS FOR STABILIZING AND ANTISTABILIZING SOLUTIONS TO THE NONAUTONOMOUS MATRIX RICCATI DIFERENTIAL EQUATION

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Several necessary and sufficient conditions for the existence of stabilizing and antistabilizing solutions of the nonautonomous matrix Riccati differential equation are presented. Some properties of these solutions are investigated. The results are applied to the optimal stabilization problem.

1. INTRODUCTION

In this paper several necessary and sufficient conditions for existence of stabilizing and antistabilizing solutions to the nonautonomous matrix Riccati differential equation (RDE) are presented. The conditions are reduced to existence of a solution to the corresponding Riccati type matrix inequality, or to existence of exponential dichotomy for the associated Hamiltonian linear differential system, or to convergence of the Newton type iterative algorithm for construction of the stabilizing or antistabilizing solution.

Existence of extremal solutions to the nonautonoms RDE is a well-known fact (history of the problem and a list of references may be found in fundamental papers [1], [2]). In the present paper it is shown that, in the class of all bounded solutions to the nonautonomous RDE, the stabilizing solution is the maximal solution and the antistabilizing solution is the minimal one (for the autonomous RDE these properties of extremal solutions were reported in [3], [4]). Asymptotic properties of the extremal solutions to the nonautonomous RDE are investigated in the paper and the sets of the associated attracting solutions are indicated.

The stabilizing or antistabilizing solution to the RDE turns out to be very useful in many optimal control design and identification problems [5 – 10]. We shall apply the obtained results to the linear nonstationary control system optimal stabilization problem under a quadratic performance criterion of arbitrary form. This problem is of a great interest in the theory of optimization and invariance of linear control systems

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(cf. [7]). It is also useful for practical optimal controller design, for example, in the minimum energy ship steering problem (see [8]).

As it was shown in [9] (see also [10]), existence of an optimal stabilizing nonstationary feedback control (with respect to indefinite quadratic functional) is equivalent to existence of a stabilizing solution to the RDE. In [9], [10] a method of successive approximations for finding the stabilizing solution to the RDE and the nonstationary optimal stabilizing control is proposed, but its convergence was proved only for performance criterion of positive definite form. The method, also known as the Newton-Raphson iterative method (cf. [5]), results in constructing successive iterations defined as a unique bounded solution to the corresponding matrix Lyapunov differential equations. For a special case the method was suggested independently in [11] and developed in [12], [3]; see [5], [13] for further references.

The obtained necessary and sufficient conditions show the universal character of this iterative algorithm in the sense that convergence of the stabilizing iterations is equivalent to existence of an optimal stabilizing control (in the case that the criterion is of indefinite form). So convergence of this algorithm can serve not only as a criterion on existence of an optimal stabilizing control, but also as an effective method how to construct it. Another useful property of the obtained conditions is the possibility to characterize the class of all quadratic functionals for which an optimal stabilizing control exists. The family of these quadratic functionals consists of all functionals with such quadratic form that can be represented as a sum of positive definite form and derivative along the system trajectories of another quadratic form.

2. NOTATIONS AND DEFINITIONS

Let $\mathbb{C}_{n,m}$ denote the set of real continuous $n \times m$ matrix functions, bounded on $\mathbb{R}^1$, and $\mathbb{C}_n := \mathbb{C}_{n,n}$. The matrix inequality $Z \geqq 0$ for $Z \in \mathbb{C}_n$ denotes that $Z = Z^{\mathrm{T}}$ and $x^{\mathrm{T}}Zx \geqq 0$ for all $x \in \mathbb{R}^n$, $t \in \mathbb{R}^1$. The inequality $Z \geqq Y$ for $Y \in \mathbb{C}_n$ means that $Z - Y \geqq 0$. The symbol $^{\mathrm{T}}$ denotes vector and matrix transposing. I_n is the unit $n \times n$ matrix.

The Euclidian norm $|Z|$ for a symmetric matrix Z is the absolute value of the numerically largest eigenvalue of Z. Thus for every $Z = Z^{\mathrm{T}} \in \mathbb{C}_n$ we have $-sI_n \leqq \leqq Z(t) \leqq sI_n$, $t \in \mathbb{R}^1$, where $s = \sup_t |Z(t)|$. By

$$\mathbb{D}_n^+ := \{Z \in \mathbb{C}_n \mid \exists c \in (0, +\infty) : cI_n \leqq Z(t),\, t \in \mathbb{R}^1\},$$

$$\mathbb{D}_n^- := \{Z \in \mathbb{C}_q \mid \exists d \in (-\infty, 0) : Z(t) \leqq dI_n,\, t \in \mathbb{R}^1\}$$

we denote the sets of positive definite and negative definite on $\mathbb{R}^1$ matrix functions respectively.

Let $V_n^{-p}(t, a)$ and $V_n^{+q}(t, a)$ be real continuous $n \times n$ matrix functions defined by the following conditions: There exist constants $f, g, p, q \in (0, +\infty)$ such that the matrix inequalities

$$V_n^{-p}(t, a)^{\mathrm{T}} V_n^{-p}(t, a) \leqq f \exp(-p(t-a)) I_n ,$$
$$g \exp(q(t-a)) I_n \leqq V_n^{+q}(t, a)^{\mathrm{T}} V_n^{+q}(t, a)$$

hold for all $a \in \mathbb{R}^1$ and $t \in [a, +\infty)$.

Let $X_F(t, a)$ be a transition matrix of a linear differential system with the coefficient matrix F defined by equations

$$\frac{\mathrm{d}}{\mathrm{d}t} X_F(t, a) = F(t) X_F(t, a), \quad X_F(a, a) = I_n, \quad a \in \mathbb{R}^1, \quad t \in [a, +\infty).$$

By

$$\mathbb{E}_n^{-p} := \{F \in \mathbb{C}_n \mid X_F(t, a) = V_n^{-p}(t, a)\},$$
$$\mathbb{E}_n^{+q} := \{F \in \mathbb{C}_n \mid X_F(t, a) = V_n^{+q}(t, a)\},$$
$$\mathbb{E}_n^{-p,+q} := \{F \in \mathbb{C}_n \mid X_F(t, a) = V_n^{-p}(t, a) + V_n^{+q}(t, a)\},$$

we denote the sets of matrix coefficients of the exponentially stable, exponentially antistable and exponentially dichotomy nth order linear differential systems respectively.

The pair of matrices $A \in \mathbb{C}_n$, $D \in \mathbb{C}_{n,m}$ is called a stabilizable (antistabilizable) pair, if there exists a matrix function $M \in \mathbb{C}_{n,m}$ such that $A + DM^{\mathrm{T}} \in \mathbb{E}_n^{-p}$ ($A + DM^{\mathrm{T}} \in \mathbb{E}_n^{+q}$ respectively).

We consider the nonautonomous matrix RDE

$$K[Z](t) = 0, \quad t \in \mathbb{R}^1, \tag{2.1}$$
$$K[Z] := \frac{\mathrm{d}Z}{\mathrm{d}t} + A^{\mathrm{T}}Z + ZA + ZBZ + C,$$

and the associated Hamiltonian linear differential system

$$J \frac{\mathrm{d}z}{\mathrm{d}t} = H(t) z, \quad t \in \mathbb{R}^1, \tag{2.2}$$
$$J := \begin{Vmatrix} 0 & -I_n \\ I_n & 0 \end{Vmatrix}, \quad H := \begin{Vmatrix} C & A^{\mathrm{T}} \\ A & B \end{Vmatrix}, \quad z := \begin{Vmatrix} x \\ y \end{Vmatrix},$$

where $x, y \in \mathbb{R}^n$, $A, B, C \in \mathbb{C}_n$ and $B = B^{\mathrm{T}}$, $C = C^{\mathrm{T}}$.

The matrix function $\bar{Z} \in \mathbb{C}_n$ is called a stabilizing (antistabilizing) solution of the RDE (2.1), if $K[\bar{Z}](t) = 0$, $t \in \mathbb{R}^1$, and $A + B\bar{Z} \in \mathbb{E}_n^{-p}$ ($A + B\bar{Z} \in \mathbb{E}_n^{+q}$ respectively).

3. FORMULATION OF THE MAIN RESULTS

Theorem 1 (cf. [9]). Let $C \in \mathbb{D}_n^+$, $B = -DD^{\mathrm{T}}$, $D \in \mathbb{C}_{n,m}$ and (A, D) is a stabilizable pair (antistabilizable pair). Then there exists a sequence of matrix functions $Z_k \in \mathbb{D}_n^+$ ($Z_k \in \mathbb{D}_n^-$), for every $k = 1, 2, 3, \ldots$, defined as a unique (bounded on $\mathbb{R}^1$) solution of the matrix Lyapunov differential equation (LDE)

$$(3.1)\qquad \mathbb{L}^k[Z](t) = 0\,,\quad t \in \mathbb{R}^1\,,$$

$$\mathbb{L}^k[Z] := \frac{\mathrm{d}Z}{\mathrm{d}t} + (A + BZ_{k-1})^{\mathrm{T}} Z + Z(A + BZ_{k-1}) + C - Z_{k-1}BZ_{k-1} =$$

$$= \frac{\mathrm{d}Z}{\mathrm{d}t} + (A + DM_{k-1}^{\mathrm{T}})^{\mathrm{T}} Z + Z(A + DM_{k-1}^{\mathrm{T}}) + C + M_{k-1} M_{k-1}^{\mathrm{T}}\,,$$

where $M_k := -Z_k D$, $k = 1, 2, 3, \ldots$, and $M_0 \in \mathbb{C}_{n,m}$ is an arbitrary stabilizing $(A + DM_0^{\mathrm{T}} \in \mathbb{E}_n^{-p})$ (resp. antistabilizing $(A + DM_0^{\mathrm{T}} \in \mathbb{E}_n^{+q})$) matrix function. The sequence Z_k has the following properties

$$(3.2)\qquad \begin{gathered} Z_{k+1}(t) \leqq Z_k(t) \quad (Z_{k+1}(t) \geqq Z_k(t))\,,\quad t \in \mathbb{R}^1 \\ A + BZ_k \in \mathbb{E}_n^{-p} \quad (A + BZ_k \in \mathbb{E}_n^{+q})\,, \\ Z_k(t) \to \bar{Z}(t)\,,\quad \text{as}\quad k \to +\infty \end{gathered}$$

where the convergence is uniform on every finite subinterval of $\mathbb{R}^1$, and the limit matrix function $\bar{Z} \in \mathbb{D}_n^+$ $(\bar{Z} \in \mathbb{D}_n^-)$ is a stabilizing (resp. antistabilizing) solution to the RDE (2.1).

Theorem 2. Let $B = -DD^{\mathrm{T}}$, $D \in \mathbb{C}_{n,m}$, and (A, D) be a stabilizable pair (antistabilizable pair). Then the following statements are equivalent:

1) A stabilizing (resp. antistabilizing) solution to the RDE (2.1) $\bar{Z} = \bar{Z}^{\mathrm{T}} \in \mathbb{C}_n$ exists, defined by condition $A + B\bar{Z} \in \mathbb{E}_n^{-p}$ $(A + B\bar{Z} \in \mathbb{E}_n^{+q})$.

2) There exists a matrix function $Y = Y^{\mathrm{T}} \in \mathbb{C}_n$, satisfying condition $K[Y] \in \mathbb{D}_n^+$ $(K[Y] \in \mathbb{D}_n^-)$.

3) The Hamiltonian linear differential system (2.2) possesses an exponential dichotomy, i.e. $J^{-1}H \in \mathbb{E}_{2n}^{-p,+p}$ $(J^{-1}H \in \mathbb{E}_{2n}^{-q,+q})$.

4) For every matrix function $M_0 \in \mathbb{C}_{n,m}$ such that $A + DM_0^{\mathrm{T}} \in \mathbb{E}_n^{-p}$ $(A + DM_0^{\mathrm{T}} \in \mathbb{E}_n^{+q})$ the sequence of matrix functions $Z_k = Z_k^{\mathrm{T}} \in \mathbb{C}_n$ exists, where for every $k = 1, 2, 3, \ldots$ the matrix function $Z_k(t)$ is a unique bounded on $\mathbb{R}^1$ solution to the LDE (3.1), satisfying conditions (3.2), for which the matrix function $\bar{Z} = \bar{Z} \in \mathbb{C}_n$ is a stabilizing (resp. antistabilizing) solution to the RDE (2.1).

Theorem 3. Let the conditions of Theorem 2 be fulfilled and let the matrix function $\bar{Z} = \bar{Z}^{\mathrm{T}} \in \mathbb{C}_n$ be a stabilizing (antistabilizing) solution to the RDE (2.1). Then the following statements are true:

1) For any bounded on $\mathbb{R}^1$ solution to the RDE (2.1) $Z = Z^{\mathrm{T}} \in \mathbb{C}_n$, the matrix inequality $Z(t) \leqq \bar{Z}(t)$ $(\bar{Z}(t) \leqq Z(t))$ holds for all $t \in \mathbb{R}^1$.

2) For any solution to the RDE (2.1) $Z = Z^{\mathrm{T}}$, defined on the semiaxis $(-\infty, a]$ $([a, +\infty))$ and satisfying for some $r > 0$ the matrix inequality

$$(3.3)\qquad \begin{gathered} (\bar{Z} - Z)\, B + B(\bar{Z} - Z) \geqq 2p(r - 1)\, I_n\,,\quad t \in \mathbb{R}^1\,, \\ ((\bar{Z} - Z)\, B + B(\bar{Z} - Z) \leqq 2q(r - 1)\, I_n\,,\quad t \in \mathbb{R}^1)\,, \end{gathered}$$

we have

$$\lim_{t \to -\infty} (Z(t) - \bar{Z}(t)) = 0 \quad \left(\lim_{t \to +\infty} (Z(t) - \bar{Z}(t)) = 0 \right).$$

Theorem 2 gives three different necessary and sufficient conditions for existence of stabilizing or antistabilizing solution to the RDE. Statement 4) has a constructive nature and can be checked practically. It describes an iterative algorithm for constructing the stabilizing or antistabilizing solutions and shows that its convergence may be used as a criterion of their existence.

Indeed, let for some stabilizing matrix function M_0' stabilizing iterations Z_k' converge for $k \to \infty$ to the matrix function $\bar{Z}$, satisfying the stable condition $A + B\bar{Z} \in \mathbb{E}_n^{-p}$. Then $\bar{Z}$ is a stabilizing solution to the RDE. In the opposite case, if iterations Z_k' do not converge, or if the limit matrix function does not satisfy the stable condition, the stabilizing solution to the RDE does not exist. Actually, if we suppose that the stabilizing solution $\bar{Z}$ exists, then from condition 4) we conclude that iterations Z_k must convergence to $\bar{Z}$ for every initial stabilizing matrix function M_0, including M_0', and so we get a contradiction.

This iterative Newton type algorithm is known to have a quadratic rate of convergency (cf. [1], [5]), but its computational efficiency depends on the availability of an initial stabilizing matrix function and on the method of the numerical solution to the LDE [12–15]. The same conclusions are valid if we replace the stabilizing solutions by antistabilizing ones.

Theorem 3 confirms that the stabilizing and antistabilizing solutions, if they exist, are the maximal and minimal bounded on $\mathbb{R}^1$ solutions to the RDE respectively. Moreover, the stabilizing solution attracts other solutions for $t \to -\infty$, while the antistabilizing does so for $t \to +\infty$. Observe that the matrix inequality (3.3) will be fulfilled for any solution $Z(t)$ of the RDE (2.1) such that $Z(a) \geqq \bar{Z}(a)$ for some initial moment $a \in \mathbb{R}^1$, if $\bar{Z}(t)$ is a stabilizing solution, or $Z(a) \leqq \bar{Z}(a)$, if $\bar{Z}(t)$ is an antistabilizing solution respectively.

4. PROOF OF THE THEOREMS

Lemma 1. (cf. [14] p. 20). Let $A \in \mathbb{E}_n^{-p}$ $(A \in \mathbb{E}_n^{+q})$ and $C = C^{\mathrm{T}} \in \mathbb{C}_n$. Then the LDE

$$L[Z](t) = 0, \quad t \in \mathbb{R}^1, \tag{4.1}$$

$$L[Z] := \frac{\mathrm{d}Z}{\mathrm{d}t} + A^{\mathrm{T}}Z + ZA + C,$$

has the unique bounded on $\mathbb{R}^1$ solution $\bar{Z} = \bar{Z}^{\mathrm{T}} \in \mathbb{C}_n$, defined by the formula

$$\bar{Z}(t) = \int_t^{+\infty} X_A(s, t)^{\mathrm{T}} C(s) X_A(s, t) \, \mathrm{d}s, \quad t \in \mathbb{R}^1,$$

$$\left(\bar{Z}(t) = \int_{-\infty}^t X_A(s, t)^{\mathrm{T}} C(s) X_A(s, t) \, \mathrm{d}s, \quad t \in \mathbb{R}^1\right).$$

For any other solution to the LDE (4.1), say $Z(t)$, we have $Z(t) - \bar{Z}(t) \to 0$ as $t \to -\infty$ ($t \to +\infty$ resp.). Moreover, if $C \in \mathbb{D}_n^+$, then $\bar{Z} \in \mathbb{D}_n^+$ ($\bar{Z} \in D_n^-$ resp.), and if $C \in D_n^-$, then $\bar{Z} \in D_n^-$($\bar{Z} \in D_n^+$ resp.).

Lemma 2. (cf. [10] p. 62). The condition $A \in \mathbb{E}_n^{-p}$ holds if and only if there exist matrix functions $C \in \mathbb{D}_n^+$ and $\bar{Z} \in \mathbb{D}_n^+$ satisfying the LDE (4.1).

Lemma 1 contains, in the form needed below, some known facts about the LDE solutions. The proof is evident and may be found, for example, in [14, p. 20]. Lemma 2 is the well-known Lyapunov lemma on an exponential stability, see [10, p. 62] or [14, p. 16].

Theorem 1 for the case (A, D) is a stabilizable pair and a more special form of the RDE were proved in [9. p. 217], see also [10, p. 176]. In the case that (A, D) is an antistabilizable pair Theorem 1 can be proved in an analogical way. For the same reason Theorems 2 and 3 will be proved only for the stable case.

Proof of Theorem 2. We shall prove at first the implication 2) ⇒4) ⇒1) ⇒2), and then the equivalence of statements 1) and 3).

Let statement 2) be true, i.e. the matrix function $Y = Y^T \in \mathbb{C}_n$ exists, which satisfies condition $K[Y] \in \mathbb{D}_n^+$. Let us consider the following auxiliary matrix Riccati differential equation (ARDE)

$$K_Y[V](t) = 0\,, \quad t \in \mathbb{R}^1\,, \tag{4.2}$$

$$K_Y[V] := \frac{\mathrm{d}V}{\mathrm{d}t} + (A^T + YB)\,V + V(A + BY) + VBV + K[Y]\,.$$

As (A, D) is a stabilizable pair, there exists matrix function $M \in \mathbb{C}_n$ for which $A + DM^T \in \mathbb{E}_n^{-p}$. Then for $M_Y := M + YD$ we have $(A + BY) + DM_Y^T = A + DM^T \in \mathbb{E}_n^{-p}$, and $(A + BY, D)$ is a stabilizable pair, too.

We see now that all conditions of Theorem 1 are fulfilled for the ARDE (4.2), so the sequence of matrix functions V_k exists, which for every $k = 1,2,3,\ldots$ is the unique bounded on $\mathbb{R}^1$ solution of the auxiliary LDE

$$L_Y^k[V](t) = 0\,, \quad t \in \mathbb{R}^1\,,$$

$$L_Y^k[V] := \frac{\mathrm{d}V}{\mathrm{d}t} + (A + BY + BV_{k-1})^T V + V(A + BY + BV_{k-1}) + K[Y] - V_{k-1}BV_{k-1}\,,$$

and $V_k \to \bar{V}$ as $k \to +\infty$. Here $\bar{V} \in \mathbb{D}_n^+$ is a stabilizing solution to the ARDE (4.2) that means

$$K_Y[\bar{V}](t) = 0\,, \quad t \in \mathbb{R}^1\,, \quad A + B(Y + \bar{V}) \in \mathbb{E}_n^{-p}\,. \tag{4.3}$$

It is not difficult to check that $K_Y[V] = K[V + Y]$ and $L_Y^k[V] = L^k[Y + V]$. Denoting $\bar{Z} := Y + \bar{V}$ and $Z_k := Y + V_k$, $k = 1, 2, 3, \ldots$, we can see from (4.3) that $\bar{Z}$ is a stabilizable solution to the RDE (2.1), and statements 4) and 1) of Theorem 2 are proved.

The next step is to prove statement 2), when statement 1) is true and the stabilizable solution to the RDE (2.1) $\bar{Z}$ exists. As $A + B\bar{Z} \in \mathbb{E}_n^{-p}$, then according to Lemma 1 the LDE

$$\text{(4.4)} \qquad \frac{\mathrm{d}W}{\mathrm{d}t} + (A + B\bar{Z})^{\mathrm{T}} W + W(A + B\bar{Z}) + I_n = 0\,, \quad t \in \mathbb{R}^1\,,$$

has a unique bounded on $\mathbb{R}^1$ solution $\bar{W} \in \mathbb{D}_n^+$. Let $Y := \bar{Z} - c\bar{W}$ for arbitrary positive number c. Then using that $\bar{Z}$ is a solution to the RDE (2.1) and $\bar{W}$ is a solution to the LDE (4.4), we have

$$K[Y] = K[\bar{Z}] - c\,\frac{\mathrm{d}\bar{W}}{\mathrm{d}t} - c(A + B\bar{Z})^{\mathrm{T}}\,\bar{W} - c\bar{W}(A + B\bar{Z}) + c^2\bar{W}B\bar{W} = c(I_n + c\bar{W}B\bar{W})\,.$$

From here we conclude that for small enough positive c it will be $K[Y] \in \mathbb{D}_n^+$, and statement 2) is proved.

Now we shall prove an equivalence of statements 1) and 3). The proof is based on well-known results (cf. [2], [6]) binding solutions to the RDE (2.1) and the Hamiltonian system (2.2), and is reduced according to the scheme for autonomous case.

Let statement 1) be true and $\bar{Z} = \bar{Z}^{\mathrm{T}} \in \mathbb{C}_n$ be a stabilizable solution to the RDE (2.1). Then, it is a known fact (cf. [2], [6]) that the transition matrix $\bar{X}(t, a)$ defined by the equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\bar{X}(t, a) = (A + B\bar{Z})\,\bar{X}(t, a)\,, \quad \bar{X}(a, a) = I_n\,, \quad a \in \mathbb{R}^1\,, \quad t \in [a, +\infty)\,,$$

and the matrix function $\bar{Y}(t) := \bar{Z}(t)\,\bar{X}(t)$, generate n linearly independent solutions to the Hamiltonian system (2.2) $z_-^1(t), \ldots, z_-^n(t)$, which are $2n$-vector columns of the $2n \times n$ matrix function $\|\bar{X}(t)^{\mathrm{T}}, \bar{Y}(t)^{\mathrm{T}}\|^{\mathrm{T}}$. Due to conditions $A + B\bar{Z} \in \mathbb{E}_n^{-p}$ and $\bar{Z} \in \mathbb{C}_n$, the vectors $z_-^1(t), \ldots, z_-^n(t)$ form a family of exponentially decreasing solutions to the system (2.2).

To get the family of exponentially increasing solutions, let us consider the linear transformation $z = Sw$ with the Lyapunov matrix function S (S is called a Lyapunov matrix function, if $S, S^{-1}, \mathrm{d}S/\mathrm{d}t \in \mathbb{C}_n$) where

$$S := \begin{Vmatrix} I_n & 0 \\ \bar{Z} & I_n \end{Vmatrix}, \quad w = \begin{Vmatrix} u \\ v \end{Vmatrix}, \quad u, v \in \mathbb{R}^n\,.$$

Then the transformed system (2.2) takes the form

$$\text{(4.5)} \qquad \frac{\mathrm{d}u}{\mathrm{d}t} = (A + B\bar{Z})\,u + Bv\,,$$

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -\,(A + B\bar{Z})^{\mathrm{T}}\,v\,, \quad t \in \mathbb{R}^1\,.$$

The second equation is independent of the first variable u and has a coefficient matrix, conjugated with the exponentially stable matrix function $A + B\bar{Z}$. Therefore $-(A + B\bar{Z})^{\mathrm{T}} \in \mathbb{E}_n^{+p}$, and the second equation gives rise to n linearly independent exponentially increasing solutions $w_+^1(t), \ldots, w_+^n(t)$ to the system (4.5). Due to the properties of the Lyapunov matrix function $S(t)$, the same will hold for the solutions $z_+^i(t) = S(t)^{-1} w_+^i(t)$ of the Hamiltonian system (2.2).

The exponentially decreasing solutions $z_-^i(t)$ and exponentially increasing solutions $z_+^i(t)$, $i = 1, \ldots, n$, form a basis for the solutions to the linear differential system (2.2). Therefore every solution to (2.2) may be written as a sum of linear combinations of these two families of solutions, and its transition matrix in the form $Z_{J^{-1}H}(t, a) = V_{2n}^{-p}(t, a) + V_{2n}^{+p}(t, a)$, hence $J^{-1}H \in \mathbb{E}_{2n}^{-p,+p}$.

In the opposite side, let statement 3) be true and the system (2.2) have an exponential dichotomy. It is easy to see that if $z(t) = \|x(t)^{\mathrm{T}}, y(t)^{\mathrm{T}}\|^{\mathrm{T}}$ is a solution to the system (2.2), then $w(t) = \|-y(t)^{\mathrm{T}}, x(t)^{\mathrm{T}}\|^{\mathrm{T}}$ is a solution to the conjugate system $\mathrm{d}w/\mathrm{d}t = -(J^{-1}H)^{\mathrm{T}} w$, $t \in \mathbb{R}^1$. As both solutions satisfy the following equality for their Euclidian norms $|z(t)|^2 = |w(t)|^2$, $t \in \mathbb{R}^1$, the system (2.2) together with every exponentially increasing solution must have the corresponding exponentially decreasing solution.

Therefore n linearly independent exponentially decreasing solutions of (2.2) $z_-^i(t)$, $i = 1, \ldots, n$ exist, which form the $2n \times n$ matrix function $\|z_-^1(t), \ldots, z_-^n(t)\| =: \|X_-(t)^{\mathrm{T}}, Y_-(t)^{\mathrm{T}}\|^{\mathrm{T}}$. Let $X_-(t)$ and $Y_-(t)$ be the upper and lower $n \times n$ blocks of this matrix function. Suppose that $\det X_-(t) \neq 0$, $t \in \mathbb{R}^1$. Then the matrix function $\bar{Z}(t) := Y_-(t)\, X_-(t)^{-1} \in \mathbb{C}_n$ is a solution to the RDE (2.1) (cf. [2], [6]).

Consequently, the matrix function $X_-(t)$, consisting of exponentially decreasing solutions to (2.2), satisfies the equation

$$\frac{\mathrm{d}X}{\mathrm{d}t} = (A + B\bar{Z})\, X\,, \quad t \in \mathbb{R}^1\,.$$

Hence $A + B\bar{Z} \in \mathbb{E}_n^{-p}$, and $\bar{Z}$ is a stabilizable solution to the RDE (2.1).

To justify the assumption $\det X_-(t) \neq 0$, we note that if there exists a nonzero vector $c \in \mathbb{R}^n$, such that $X_-(t)\, c = 0$, then from the system (2.2) we can conclude that

$$\frac{\mathrm{d}w}{\mathrm{d}t} = -A^{\mathrm{T}} w\,, \quad D^{\mathrm{T}} w = 0\,, \quad t \in \mathbb{R}^1\,,$$

where $w(t) := Y_-(t)\, c$. The last equalities are in contradiction with the assumption on stabilizability of the pair (A, D). □

Proof of Theorem 3. To prove statement 1) we note that from Theorem 2 follows existence of a sequence of the matrix functions $Z_k = Z_k^{\mathrm{T}} \in \mathbb{C}_n$, which are the solutions to the LDE (3.1) and satisfy conditions (3.2). If $Z = Z^{\mathrm{T}} \in \mathbb{C}_n$ is a solution to the RDE (2.1), then the difference $Z_k - Z$ will be a solution to the following LDE

$$L^k[Z_k] - K[Z] = \frac{\mathrm{d}}{\mathrm{d}t}(Z_k - Z) + (A + BZ_{k-1})^{\mathrm{T}} (Z_k - Z) + (Z_k - Z)(A + BZ_{k-1}) + T_k = 0\,,$$

$t \in \mathbb{R}^1$, $k = 1, 2, 3, \ldots$. Here $T_k := (Z_{k-1} - Z)^{\mathrm{T}} DD^{\mathrm{T}}(Z_{k-1} - Z) \geqq 0$, $t \in \mathbb{R}^1$, and $A + BZ_{k-1} \in \mathbb{E}_n^{-p}$. From Lemma 1 we conclude that $Z_k - Z \geqq 0$, $t \in \mathbb{R}^1$, $k = 1, 2, 3, \ldots$. Then $\lim_{k \to +\infty} Z_k - Z = \bar{Z} - Z \geqq 0$, $t \in \mathbb{R}^1$, and statement 1) is proved.

Let now $\bar{Z} = \bar{Z}^{\mathrm{T}} \in \mathbb{C}_n$ be a stabilizing solution and $Z = Z^{\mathrm{T}} \in \mathbb{C}_n$ be an arbitrary solution to the RDE (2.1), and inequality (3.3) hold. Firstly observe that the difference $Z - \bar{Z}$ is a solution to LDE

$$(4.6) \quad \bar{L}[Y] := \frac{\mathrm{d}Y}{\mathrm{d}t} + (A + \tfrac{1}{2}B(Z + \bar{Z}))^{\mathrm{T}} Y + Y(A + \tfrac{1}{2}B(Z + \bar{Z})) = 0, \quad t \in \mathbb{R}^1 .$$

Hence, to prove statement 2) it suffices to show that $A + \frac{1}{2}B(Z + \bar{Z}) \in \mathbb{E}_n^{-p}$ and use Lemma 1.

From the condition $A + B\bar{Z} \in \mathbb{E}_n^{-p}$ and Lemma 2 we conclude that a matrix function $Z_+ = Z_+^{\mathrm{T}} \in \mathbb{D}_n^+$ exists, which is a solution to the LDE

$$\frac{\mathrm{d}Y}{\mathrm{d}t} + (A + B\bar{Z})^{\mathrm{T}} Y + Y(A + B\bar{Z}) + I_n = 0 , \quad t \in \mathbb{R}^1 ,$$

and satisfies the matrix inequality $Z_+(t) \geqq zI_n$, where $z \in (0, 1/p)$. This equation can be also written as

$$(4.7) \qquad \bar{L}[Y] + T[Y] = 0 , \quad t \in \mathbb{R}^1 .$$

where $\bar{L}[Y]$ is defined by (4.6), and $T[Y] := \frac{1}{2}((\bar{Z} - Z) BY + YB(\bar{Z} - Z)) + I_n$. From inequality (3.3) for $Y = Z_+$ we have

$$T[Z_+] \geqq \tfrac{1}{2}z((\bar{Z} - Z) B + B(\bar{Z} - Z)) + I_n \geqq (zp(r - 1) + 1) I_n \geqq rI_n , \quad t \in \mathbb{R}^1 ,$$

that means $T[Z_+] \in \mathbb{D}_n^+$.

Now we have two matrix functions Z_+, $T[Z_+] \in \mathbb{D}_n^+$, which satisfy the LDE (4.7). From Lemma 2 we conclude that $A + \frac{1}{2}B(Z + \bar{Z}) \in \mathbb{E}_n^{-p}$, and statement 2) is proved. □

6. THE OPTIMAL STABILIZATION PROBLEM

Let $F \in \mathbb{C}_n$, $G \in \mathbb{C}_{n,m}$ and $\mathbb{S}_{n,m}^- := \{M \in \mathbb{C}_{n,m} \mid F + GM^{\mathrm{T}} \in \mathbb{E}_n^{-p}\}$ be the set of stabilizing matrix functions. Using Lemma 1, for every $M \in \mathbb{S}_{n,m}^-$ we can construct the unique bounded on $\mathbb{R}^1$ solution to the following parametric LDE

$$L[Z, M] := \frac{\mathrm{d}Z}{\mathrm{d}t} + (F^{\mathrm{T}} + MG^{\mathrm{T}}) Z + Z(F + GM^{\mathrm{T}}) + C[M] = 0 , \quad t \in \mathbb{R}^1 ,$$

defined by the formula

$$\bar{Z}[M](t) = \int_t^{+\infty} X_{F+GM^{\mathrm{T}}}(s, t)^{\mathrm{T}} C[M](s) X_{F+GM^{\mathrm{T}}}(s, t) \, \mathrm{d}s , \quad t \in \mathbb{R}^1 ,$$

where $C[M] := P + QM^{\mathrm{T}} + MQ^{\mathrm{T}} + MRM^{\mathrm{T}}$, $P = P^{\mathrm{T}} \in \mathbb{C}_n$, $Q \in \mathbb{C}_{n,m}$, $R = R^{\mathrm{T}} \in \mathbb{C}_m$.

We shall focus our attention on searching an optimal stabilizing matrix function $\overline{M}$, which delivers minimum (in the sense of matrix inequalities) to the solutions $\overline{Z}[M]$ on the set $\mathbb{S}_{n,m}^{-}$. It is easy to see that this problem is equivalent to the standard linear quadratic infinite time optimal regulator problem.

Indeed, let us consider the control system

$$(5.1)\qquad \frac{\mathrm{d}x}{\mathrm{d}t} = F(t)\,x + G(t)\,u\,,\quad t \in \mathbb{R}^1\,,\quad x \in \mathbb{R}^n\,,\quad u \in \mathbb{R}^m\,,$$

with the stabilizing feedback controls

$$(5.2)\qquad u = u(t, x) = M(t)^{\mathrm{T}}\,x\,,\quad M \in \mathbb{S}_{n,m}^{-}$$

and the performance index

$$(5.3)\quad J(t_0, x_0, u) := \int_{t_0}^{+\infty} [x(t)^{\mathrm{T}} P(t)\,x(t) + x(t)^{\mathrm{T}} Q(t)\,u(t) + u(t)^{\mathrm{T}} Q(t)^{\mathrm{T}} x(t) + u(t)^{\mathrm{T}} R(t)\,u(t)]\,\mathrm{d}t\,.$$

The value of the performance index (5.3) on stabilizing control (5.2) can be expressed by a quadratic form with the $\overline{Z}[M]$ matrix:

$$J(t_0, x_0, M^{\mathrm{T}}x) = \int_{t_0}^{+\infty} x(t)^{\mathrm{T}}\,C[M]\,(t)\,x(t)\,\mathrm{d}t =$$

$$= x_0 \int_{t_0}^{+\infty} X_{F+GM^{\mathrm{T}}}(t, t_0)^{\mathrm{T}}\,C[M]\,(t)\,X_{F+GM^{\mathrm{T}}}(t, t_0)\,\mathrm{d}t\,x_0 = x_0^{\mathrm{T}}\,\overline{Z}[M]\,(t_0)\,x_0.$$

Therefore to search the minimum of the performance criterion (5.3) on the set of stabilizing control (5.2) is equivalent to searching the matrix function $\overline{M} \in \mathbb{S}_{n,m}^{-}$, for which the matrix inequality $\overline{Z}[\overline{M}] \leqq \overline{Z}[M]$ is satisfied for all $M \in \mathbb{S}_{n,m}^{-}$.

The following theorem contains two necessary and sufficient conditions for the solution of the above problem. The first condition is a consequence of the Bellman optimality principle. The second one, existence of the stabilizing solution to the corresponding RDE, is well-know. For the first time it has been proved in this form for nonstationary case apparently in [9].

Theorem 4 (cf. [14]). Let $R \in \mathbb{D}_m^{+}$ and (F, G) be a stabilizable pair. Then the following statements are equivalent:

1) An optimal stabilizing matrix function $\overline{M} \in \mathbb{S}_{n,m}^{-}$ exists such that $\overline{Z}[\overline{M}]\,(t) \leqq \overline{Z}[M]\,(t)$ for all $M \in \mathbb{S}_{n,m}^{-}$ and $t \in \mathbb{R}^1$.

2) Matrix functions $\overline{M} \in \mathbb{S}_{n,m}^{-}$ and $\overline{Z} = \overline{Z}^{\mathrm{T}} \in \mathbb{C}_n$ exist such that $L[\overline{Z}, M]\,(t) \geqq L[\overline{Z}, \overline{M}]\,(t) = 0$ for all $M \in \mathbb{S}_{n,m}^{-}$ and $t \in \mathbb{R}^1$.

3) The stabilizing solution $\overline{Z}$ of the EDR (2.1) with the matrix coefficients $A = F - GR^{-1}Q^{\mathrm{T}}$, $B = GR^{-1}G^{\mathrm{T}}$, $C = P - QR^{-1}Q^{\mathrm{T}}$ exists.

The optimal stabilizing matrix function is unique and defined by the formula $M = -(\overline{Z}G + Q)\,R^{-1}$, besides $\overline{Z} = \overline{Z}[\overline{M}]$.

Statement 3) of the theorem allows to use any condition reported in Theorem 2 to check whether the optimal stabilizing control exists. From statement 4) of Theorem 2 we get an iterative algorithm for constructing the optimal stabilizing control called the Newton-Raphson method in control theory (cf. [5], [6]). As it has been mentioned above, the algorithm turns out to be a criterion on existence of a non-stationary optimal control at the same time, in the sense that the stabilizing iterations convergence is equivalent to existence of the optimal stabilizing control.

Statement 2) of Theorem 2 highlights positivity of the functional (5.3), which is usually supposed to be fulfilled in the optimal regulator problem [3–6], [13]. Indeed, let us consider the functional

$$(5.4)\qquad J_Y(t_0, x_0, u) := J(t_0, x_0, u) + \int_{t_0}^{+\infty} \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{(5.1)} x(t)^{\mathrm{T}}\, Y(t)\, x(t)\, \mathrm{d}t =$$

$$= \int_{t_0}^{+\infty} x(t)^{\mathrm{T}}\, K[Y]\,(t)\, x(t) + |u(t) - (YG + Q)^{\mathrm{T}}\, x(t)|^2_{R(t)}\, \mathrm{d}t\,,$$

where $|v|^2_R := v^{\mathrm{T}}Rv$ for $v \in \mathbb{R}^m$, and $\mathrm{d}/\mathrm{d}t|_{(5.1)}$ denotes derivative along the solutions of the system (5.1).

From one side,

$$(5.5)\qquad J_Y(t_0, x_0, u) = J(t_0, x_0, u) - x_0^{\mathrm{T}}\, Y(t_0)\, x_0\,,$$

hence for every matrix function $Y = Y^{\mathrm{T}} \in \mathbb{C}_n$ the new functional (5.4) takes the minimal value in u at the same point as the initial functional (5.3). From the other side, if conditions of Theorem 2 are fulfilled and $Y = Y^{\mathrm{T}} \in \mathbb{C}_n$ is such a matrix function that $K[Y] \in \mathbb{D}_n^+$, then the corresponding functional (5.4) has a positive definite form under its integral sign.

So always when optimal stabilizing control exists, there exists a functional with positive definite quadratic form, which is equivalent to the initial functional (in the sense of (5.5)). Now we can describe the family of functionals for which optimal stabilizing control exists. The family is defined by the formula (5.4) and consists of all functionals whose form may be represented as a sum of positive definite form and derivative along system solutions of another quadratic form.

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