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MATRIX EQUATIONS ARISING IN REGULATOR PROBLEMS

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The two coupled linear equations

$$C_* X - Z_* B = A_{1*}$$
$$C_* Y + Z_* A = B_{1*}$$

in polynomial matrices are studied in detail. These equations are crucial in the theory and design of linear optimal dynamic regulators via frequency-domain methods.

The solvability of these equations is established under natural conditions. All solutions are characterized and then a specific solution is studied. Relation to other matrix equations is discussed.

1. INTRODUCTION

In order to motivate and justify the study of the above equations, we recall the formulation of a standard linear regulator problem.

Given a discrete-time, linear, l-output, m-input, n-dimensional system

$$x_{t+1} = Fx_t + Gu_t$$
$$y_t \quad = Hx_t$$

which is stabilizable as well as detectable. Find a regulator which generates u_t from $y_t, y_{t-1}, \ldots$ in such a way that the cost

$$J = \sum_{t=0}^{\infty} u_t^T u_t + y_t^T y_t$$

is minimized for every initial state x_0.

The time-domain solution consists of two steps. First the system's state is reconstructed from measurable data and then a state-variable feedback is applied. The central problem is to solve an algebaic Riccati equation. Such equations have been given much attention in the literature and are now well understood.

In contrast, the frequency-domain approach developed recently by Kučera and Šebek [7] makes use of spectral factorization and the solution of linear equations in polynomial matrices. Specifically, write the transfer matrix of the system in terms of matrix fractions

$$\begin{aligned} H(I_n - dF)^{-1}\, dG &= A^{-1}(d)\, B(d) \\ &= B_1(d)\, A_1^{-1}(d) \end{aligned}$$

where A, B and A_1, B_1 are polynomial matrices in the delay operator d. If D is a greatest common left divisor of A and B and D_1 is a greatest common right divisor of A_1 and B_1, then we assume that both det D and det D_1 are stable polynomials. Define a polynomial matrix C with det C stable by the relations

$$A_1^T(d^{-1})\, A_1(d) + B_1^T(d^{-1})\, B_1(d) = C^T(d^{-1})\, C(d)$$

This is called spectral factorization. Finally we solve the pair of equations

$$\begin{aligned} C^T(d^{-1})\, X(d) - Z^T(d^{-1})\, B(d) &= A_1^T(d^{-1}) \\ C^T(d^{-1})\, Y(d) + Z^T(d^{-1})\, A(d) &= B_1^T(d^{-1}) \end{aligned}$$

for polynomial matrices X, Y, and Z such that $Z(0) = 0$. Then $X^{-1}(d)\, Y(d)$ is the transfer matrix of an optimal regulator.

The spectral factorization is a well developed classical gadget and it is discussed elsewhere in the literature. The purpose of this paper is to investigate deeply the properties of the coupled matrix polynomial equations. Similar equations were obtained by Kučera [5], [6] when solving steady-state minimum variance control problems. The reference [5] also contains first results on solvability of these equations as well as some other observations. Here we give a complete theory in the hope to provide further insight and contribute to the progress of frequency-domain design techniques.

2. PRELIMINARIES

Let us first recall some mathematical concept from the theory of polynomial matrices in a single indeterminate d over the real field. More details can be found, for example, in the books by Gantmakher [2], MacDuffee [8], Barnett [1] and specifically Kučera [5].

A square polynomial matrix U is called *unimodular* if and only if det U is a nonzero real number. Polynomial matrices A and B are *equivalent* (we write $A \sim B$) if and only if there are unimodular matrices U_1, U_2 such that $A = U_1 B U_2$.

Let A, B and C be polynomial matrices and $A = BC$. Then B is a *left divisor* of A while C is a *right divisor* of A. Now consider two polynomial matrices A and B. A square polynomial matrix D is termed a *common left divisor* of A and B if and only if D is a left divisor of both A and B; if, furthermore, every other common left divisor of A and B is a left divisor of D, then D is a *greatest common left divisor* of A and B. It is known that D is a greatest common left divisor of A and B if and only if there is a unimodular matrix U such that

$$[A \ B] \, U = [D \ 0] \tag{1}$$

A greatest common right divisor of two polynomial matrices is defined in an entirely analogous fashion.

The polynomial matrices A and B are said to be relatively left prime if and only if their only common left divisors are unimodular matrices. The polynomial matrices A_1 and B_1 are said to be relatively right prime if and only if their only common right divisors are unimodular matrices.

A square polynomial matrix C is said to be *stable* if and only if $\det C$ has no root whose magnitude is less than or equal to 1, and the C is said to be *Hurwitz* if and only if $\det C$ has no root with magnitude less than 1.

As well-known theorem states that any polynomial matrix A can be reduced to *Smith form*

$$\mathcal{S}(A) = U_1 A U_2$$

by means of unimodular matrices U_1 and U_2. If rank $A = r$, the Smith form is a matrix having nonzero polynomials $a_1, \ldots, a_r$, possibly followed by zeros, on its leading diagonal and having zeros elsewhere. The polynomials $a_1, \ldots, a_r$, called the *invariant polynomials* of A, have the property that a_k divides a_{k+1} for $k = 1, \ldots$ $\ldots, r - 1$ and are determined uniquely up to nonzero real multiples. Two matrices have the same Smith form if and only if they are equivalent.

The following result, the proof of which can be found in Newman [9], gives the multiplicativity condition for Smith forms:

$$\mathcal{S}(AB) = \mathcal{S}(A) \, \mathcal{S}(B) \tag{2}$$

whenever A and B are square polynomial matrices with relatively prime determinants.

Further, given an arbitrary polynomial matrix

$$E = E_0 + E_1 d + \ldots + E_n d^n$$

we denote the zero-position coefficient E_0 by $\langle E \rangle$ and and define the conjugate matrix E_* by

$$E_* = E_0^T + E_1^T d^{-1} + \ldots + E_n^T d^{-n}$$

The E is said to be *proper* whenever E_n is invertible.

Finally for any $l \times m$ polynomial matrix A of rank m we define a Hurwitz polynomial matrix C by the relation

$$A_* A = C_* C$$

Such a C is called a *(right) spectral factor* of A and it is uniquely determined up to left orthogonal matrix multiples, see Kučera [5].

3. SPECIFICATIONS

This paper is devoted to the study of the following two coupled equations

$$C_* X - Z_* B = A_{1*} \tag{3a}$$

$$C_* Y + Z_* A = B_{1*} \tag{3b}$$

for polynomial matrices X, Y, and Z. The A, B and A_1, B_1 are respectively $l \times l$, $l \times m$ and $m \times m$, $l \times m$ given polynomial matrices related by

$$AB_1 = BA_1 \tag{4}$$

and such that both A and A_1 are invertible and D_1, the greatest common right divisor of A_1 and B_1, is a stable polynomial matrix. The C is a given $m \times m$ Hurwitz polynomial matrix satisfying

$$A_{1*}A_1 + B_{1*}B_1 = C_* C \tag{5}$$

which is to say the C is a spectral factor.

4. SOLVABILITY CONDITION

The equations (3) are special cases of the bilateral equation

$$EP + QF = G \tag{6}$$

for polynomial matrices P and Q, where E, F and G are respectively $l \times p$, $q \times m$ and $l \times m$ given polynomial matrices. This equation was studied in detail by Roth [10], Barnett [1], and Kučera [3, 5]. The following lemma states the general solvability condition.

Lemma 1. Equation (6) has a solution if and only if

$$\begin{bmatrix} E & 0 \\ 0 & F \end{bmatrix} \sim \begin{bmatrix} E & G \\ 0 & F \end{bmatrix}$$

By specializing this result one could obtain the necessary and sufficient condition for the existence of X, Y and Z in (3). Due to the particular structure of these equations, however, we can hope for deeper results. In fact, a simple sufficient condition is available which is motivated by the underlaying control-theoretic considerations and corresponds to system stabilizability.

To prove this condition we need the following lemma.

Lemma 2. Let E and F be respectively $m \times m$ and $l \times l$ polynomial matrices with relatively prime determinants. Then

$$\begin{bmatrix} E & 0 \\ 0 & F \end{bmatrix} \sim \begin{bmatrix} E & G \\ 0 & F \end{bmatrix}$$

for an arbitrary polynomial $m \times l$ matrix G.

Proof. Because of the relative primeness of det E and det F, relation (2) gives

$$\mathscr{S}\left(\begin{bmatrix} E & 0 \\ 0 & F \end{bmatrix}\right) = \mathscr{S}\left(\begin{bmatrix} I_m & 0 \\ 0 & F \end{bmatrix}\right) \mathscr{S}\left(\begin{bmatrix} E & 0 \\ 0 & I_l \end{bmatrix}\right) \tag{7}$$

and

$$\mathscr{S}\left(\begin{bmatrix} E & G \\ 0 & F \end{bmatrix}\right) = \mathscr{S}\left(\begin{bmatrix} I_m & 0 \\ 0 & F \end{bmatrix}\right) \mathscr{S}\left(\begin{bmatrix} E & G \\ 0 & I_l \end{bmatrix}\right) \tag{8}$$

Further

$$\begin{bmatrix} I_m & -G \\ 0 & I_l \end{bmatrix} \begin{bmatrix} E & G \\ 0 & I_l \end{bmatrix} = \begin{bmatrix} E & 0 \\ 0 & I_l \end{bmatrix} \tag{9}$$

and so

$$\mathscr{S}\left(\begin{bmatrix} E & 0 \\ 0 & I_l \end{bmatrix}\right) = \mathscr{S}\left(\begin{bmatrix} E & G \\ 0 & I_l \end{bmatrix}\right) \tag{10}$$

So combining the relations (7), (8) and (10) we have got

$$\mathscr{S}\left(\begin{bmatrix} E & 0 \\ 0 & F \end{bmatrix}\right) = \mathscr{S}\left(\begin{bmatrix} E & G \\ 0 & F \end{bmatrix}\right) \tag{11}$$

for an arbitrary G and this is equivalent to the assertion of the lemma. □

Now we are ready to state the principal result.

Theorem 1. Let the greatest common left divisor of A and B be stable. Then equations (3) have a solution.

Proof. Let us write equations (3) in the compact form

$$C_*[X \;\; Y] - Z_*[B \;\; -A] = [A_{1*} \;\; B_{1*}] \tag{12}$$

In this equation, polynomial matrices in both positive and negative powers of d occur. To remedy this situation, premultiply (12) by the matrix

$$R = \begin{bmatrix} d^{k_1} & & \\ & \ddots & \\ & & d^{k_m} \end{bmatrix} \tag{13}$$

where k_i is the degree of the i-th row of the composite matrix

$$[C_* \;\; A_{1*} \;\; B_{1*}]$$

Thus equation (12) is equivalent to

$$\bar{C}[X\ Y] - \bar{Z}[B\ -A] = [\bar{A}_1\ \bar{B}_1] \tag{14}$$

with

$$\begin{aligned} \bar{C} &= RC_* \\ \bar{Z} &= RZ_* \\ [\bar{A}_1\ \bar{B}_1] &= R[A_{1*}\ B_{1*}] \end{aligned} \tag{15}$$

Now, taking Lemma 1 into account, we need only to prove that a stable greatest common left divisor of matrices A and B implies the equivalence

$$\begin{bmatrix} \bar{C} & 0 & 0 \\ 0 & B & -A \end{bmatrix} \sim \begin{bmatrix} \bar{C} & \bar{A}_1 & \bar{B}_1 \\ 0 & B & -A \end{bmatrix} \tag{16}$$

Due to (1) there is an $(m + l) \times (l + m)$ unimodular matrix J,

$$J = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}$$

such that

$$BJ_{11} + AJ_{21} = D \tag{17}$$

$$BJ_{12} + AJ_{22} = 0 \tag{18}$$

where the $l \times l$ polynomial matrix D is a greatest common left divisor of B and A. The J_{12} and J_{22} are relatively right prime matrices.
Hence

$$\begin{bmatrix} \bar{C} & 0 & 0 \\ 0 & B & -A \end{bmatrix} \begin{bmatrix} I_m & 0 & 0 \\ 0 & J_{11} & J_{12} \\ 0 & -J_{21} & -J_{22} \end{bmatrix} = \begin{bmatrix} \bar{C} & 0 & 0 \\ 0 & D & 0 \end{bmatrix}$$

or

$$\begin{bmatrix} \bar{C} & 0 & 0 \\ 0 & B & -A \end{bmatrix} \sim \begin{bmatrix} \bar{C} & 0 & 0 \\ 0 & D & 0 \end{bmatrix} \tag{19}$$

and, similarly,

$$\begin{bmatrix} \bar{C} & \bar{A}_1 & \bar{B}_1 \\ 0 & B & -A \end{bmatrix} \sim \begin{bmatrix} \bar{C} & \bar{A}_1 J_{11} - \bar{B}_1 J_{21} & \bar{A}_1 J_{12} - \bar{B}_1 J_{22} \\ 0 & D & 0 \end{bmatrix}. \tag{20}$$

Now all matrices P, Q satisfying

$$AP + BQ = 0$$

are by (18) of the form

$$\begin{aligned} P &= J_{22} T \\ Q &= J_{12} T \end{aligned}$$

for some polynomial matrix T. It follows from (4) that A_1 and $-B_1$ must also have this form and hence

$$A_1 = J_{12}D_1$$
$$-B_1 = J_{22}D_1$$

where D_1 is a greatest common right divisor of theirs. When substituting this result into (5) we obtain $C = C_1D_1$ for some polynomial matrix C_1. Using (5) and (15) we get

$$\bar{A}_1 J_{12} - \bar{B}_1 J_{22} = \bar{C} C_1$$

so that

$$\begin{bmatrix} \bar{C} & \bar{A}_1 J_{11} - \bar{B}_1 J_{21} & \bar{C} C_1 \\ 0 & D & 0 \end{bmatrix} \begin{bmatrix} I_m & 0 & C_1 \\ 0 & I_l & 0 \\ 0 & 0 & -I_m \end{bmatrix} = \begin{bmatrix} \bar{C} & \bar{A}_1 J_{11} - \bar{B}_1 J_{21} & 0 \\ 0 & D & 0 \end{bmatrix}$$

or, taking (20) into account,

$$\text{(21)} \qquad \begin{bmatrix} \bar{C} & \bar{A}_1 & \bar{B}_1 \\ 0 & B & -A \end{bmatrix} \sim \begin{bmatrix} \bar{C} & \bar{A}_1 J_{11} - \bar{B}_1 J_{21} & 0 \\ 0 & D & 0 \end{bmatrix}$$

Now C is Hurwitz by definition. Hence a stable D implies that det $\bar{C}$ and det D are relatively prime polynomials and the hypothesis of Lemma 2 is satisfied. Using this lemma for matrices from (19) and (21), the proof is completed. □

5. GENERAL SOLUTION

Let us begin our discussion with the bilateral equation (6). It was shown by Kučera [3, 4] that any two solutions P, Q and P_0, Q_0 of (6) are related by

$$\text{(22)} \qquad P = P_0 + U_2 T U_4^{-1}$$
$$Q = Q_0 - U_1^{-1} V U_3$$

Moreover, if P_0, Q_0 is a solution of (6), then any P, Q from (22) is also a solution. Here U_1, U_2, U_3 and U_4 are unimodular matrices defined by

$$\mathscr{S}(E) = U_1 E U_2 = \begin{bmatrix} e_1 & & & & \\ & \ddots & & & \\ & & e_r & & \\ & & & 0 & \\ & & & & \ddots \end{bmatrix}$$

$$\mathscr{S}(F) = U_3 F U_4 = \begin{bmatrix} f_1 & & & & \\ & \ddots & & & \\ & & f_s & & \\ & & & 0 & \\ & & & & \ddots \end{bmatrix}$$

with $r = \text{rank } E$, $s = \text{rank } F$, and T, V are polynomial matrices, respectively $p \times m$ and $l \times q$, of the form

$$T = \begin{array}{cc} & \overbrace{\phantom{T_{11}}}^{s} \\ r\{ & \left[\begin{array}{c|c} T_{11} & 0 \\ \hline T_{21} & T_{22} \end{array}\right] \end{array} \qquad V = \begin{array}{cc} & \overbrace{\phantom{V_{11}}}^{s} \\ r\{ & \left[\begin{array}{c|c} V_{11} & V_{12} \\ \hline 0 & V_{22} \end{array}\right] \end{array}$$

The T_{11} has entries $t_{ij}f_j/d_{ij}$, the V_{11} has entries $e_i t_{ij}/d_{ij}$ where d_{ij} is a greatest common divisor of e_i and f_j whereas t_{ij} is an arbitrary polynomial. The T_{21}, T_{22} and V_{12}, V_{22} are arbitrary polynomial matrices of appropriate dimensions.

When specializing this general result for the coupled equations (3) we get the following theorem.

Theorem 2. Let the greatest common left divisor of A and B be stable. Then any two solutions X, Y, Z and X_0, Y_0, Z_0 of equations (3) are related by

$$\begin{aligned} X &= X_0 + WB \\ Y &= Y_0 - WA \\ Z_* &= Z_{0*} + C_* W \end{aligned}$$

where W is an $m \times l$ real matrix.

Moreover, if X_0, Y_0, Z_0 is a solution of equations (3), then any X, Y, Z given above is also a solution.

Proof. As in the proof of Theorem 1, any solution X, Y, Z of equations (3) can be recovered from some solution $\bar{X}$, $\bar{Y}$, $\bar{Z}$ of the equation

(23) $$\bar{C}[\bar{X}\ \bar{Y}] - \bar{Z}[B\ -A] = [\bar{A}_1\ \bar{B}_1]$$

using the relations

(24) $$\begin{aligned} X &= \bar{X} \\ Y &= \bar{Y} \\ Z_* &= R^{-1}\bar{Z} \end{aligned}$$

Now (22) can be used to express a solution of (23). Here

$$\begin{aligned} E &= \bar{C} \\ F &= -[B\ -A] \end{aligned}$$

Repeating the arguments used in the proof of Theorem 1, $\mathscr{S}(\bar{C})$ and $\mathscr{S}([B\ -A])$ have no common factor and that is why $d_{ij} = 1$ for every $i = 1, \ldots, m$, $j = 1, \ldots, l$. Moreover, both E and F have full rank in this case so that

$$\begin{aligned} T &= [T_{11}\ 0] = N\,\mathscr{S}(F) \\ V &= V_{11} = \mathscr{S}(E)\,N \end{aligned}$$

for an arbitrary $m \times l$ polynomial matrix N. Hence putting $W = U_2 N U_3$, the solution $\bar{X}$, $\bar{Y}$, $\bar{Z}$ of (23) is related to any other solution $\bar{X}_0$, $\bar{Y}_0$, $\bar{Z}_0$ by

$$[\bar{X} \;\; \bar{Y}] = [\bar{X}_0 \;\; \bar{Y}_0] + W[B \;\; -A]$$
$$\bar{Z} = \bar{Z}_0 + \bar{C}W$$

The claim then follows on using (24). For Z to be a polynomial matrix in d, the W must be restricted to real matrices only. □

6. OPTIMAL SOLUTION

In applications we usually face the problem of calculating a specific solution to equations (3). This solution is dictated by the problem at hand. For example, to construct the optimal regulator discussed in the Introduction we have to find a solution X_0, Y_0, and Z_0 of equations (2) such that

(25) $$\langle Z_0 \rangle = 0$$

This particular solution is termed here "optimal" and it enjoys the following properties.

Theorem 3. There exists a unique optimal solution to equations (3).

Proof. To prove the existence, let X, Y, and Z be any solution of (3). Then the optimal solution X_0, Y_0, and Z_0 is seen to be

(26) $$\begin{aligned} X_0 &= X - WB \\ Y_0 &= Y + WA \\ Z_{0*} &= Z_* - C_* W \end{aligned}$$

where

(27) $$W = \langle C_* \rangle^{-1} \langle Z_* \rangle$$

Note that the indicated inverse exists because C is a spectral factor. The uniqueness then follows from (27). □

Comparing the highest-degree coefficients of each row in (3b) we can immediately see that the optimal solution X_0, Y_0, Z_0 has the following alternative characterization: the degree of any column of Y_0 is less than the degree of the corresponding column of A.

Since any solution X, Y and Z of equations (3) can also be obtained as a solution X, Y and $\bar{Z}$ of the premultiplied equation (14), it is of interest to know how the optimal solution is characterized among the solutions of (14). A glance at the transformation relationship (15) reveals that the condition (25) is equivalent to the requirement that the degree of the i-th row of $\bar{Z}$ be less than the degree of the i-th row of $\bar{C}$ for all $i = 1, 2, \ldots, m$.

7. RELATION TO UNILATERAL EQUATIONS

The special structure of the bilateral matrix equations (3) renders it possible to relate their solutions with the solutions of a unilateral equation.

Theorem 4. Let X, Y and Z be an arbitrary solution of equations (3). Then

$$XA_1 + YB_1 = C \tag{28}$$

Proof. Adding (3a) postmultiplied by A_1 to (3b) postmultiplied by B_1 gives

$$C_*(XA_1 + YB_1) + Z_*(AB_1 - BA_2) = A_{1*}A_1 + B_{1*}B_1$$

Now using (4) and (5) we arrive to (28). □

Thus any X and Y satisfying (3) also satisfies (28). Unfortunately the converse is not true in general. To see this consider two solutions X, Y and X_0, Y_0 of (28) related by

$$X = X_0 - TB$$

$$Y = Y_0 + TA$$

where T is an $m \times l$ polynomial matrix. Comparing this with Theorem 2 where W is a real, not polynomial matrix, the claim becomes evident.

A case of special interest occurs when the matrix A is proper (this is always true for scalar polynomial A). Then equation (28) possesses a unique "minimal" solution X_0, Y_0 such that the degree of each column of Y_0 is less than the degree of the corresponding column of A. This property, however, is also shared by the optimal solution X_0, Y_0, and Z_0 of equation (3) and hence the two must coincide. Thus a proper A entails that the optimal solution of (3) can be found as the minimal solution of (28). The computational advantages resulting from this identification are discussed below.

8. COMPUTATIONAL ALGORITHMS

To complete the picture, let us summarize some techniques of calculating the solutions to equations (3) and (28). For a detailed discussion including the computational algorithms the reader is referred to Kučera [5].

The unilateral equation (28) can be solved as follows. Form the matrix $\begin{pmatrix} A_1 & I_m & 0 \\ B_1 & 0 & I_l \end{pmatrix}$ and, using elementary (unimodular) row operations, carry out the transformation

$$\begin{bmatrix} A_1 & I_m & 0 \\ B_1 & 0 & I_l \end{bmatrix} \to \begin{bmatrix} D_{11} & D_{12} & D_{13} \\ 0 & D_{22} & D_{23} \end{bmatrix}$$

where D_{11} is $m \times m$ and upper triangular and D_{22} is $l \times l$. Further solve a triangular system of linear equations to express C as

$$C = C_1 D_{11}$$

Then $C_1 D_{12}$, $C_1 D_{13}$ is a solution of (28) and any solution X, Y is generated by

$$X = C_1 D_{12} + T D_{22}$$

$$Y = C_1 D_{13} + T D_{23}$$

where T is an $m \times l$ polynomial matrix.

If D_{23} is a proper polynomial matrix, the minimal solution X_0, Y_0 of (28) can be obtained by applying the division algorithm for polynomial matrices. If M is the quotient and N is the remainder after dividing D_{23} into $C_1 D_{13}$, then simply

$$X_0 = C_1 D_{12} - M D_{22}$$

$$Y_0 = N$$

The solution of the bilateral equations (3) is more complicated. The method recommended here is as follows. Transform (3) into (14) using (13) and calculate Smith forms for $\bar{C}$ and $[-B \;\; A]$:

$$\mathscr{S}(\bar{C}) = U_1 \bar{C} U_2 = \begin{bmatrix} e_1 & & \\ & \ddots & \\ & & e_m \end{bmatrix}$$

$$\mathscr{S}[-B \;\; A] = U_3 [-B \;\; A] \, U_4 = \begin{bmatrix} f_1 & & & \\ & \ddots & & \\ & & f_l & 0 \dots \end{bmatrix}$$

Write g_{ij} for the elements of the matrix $U_1 [\bar{A}_1 \;\; \bar{B}_1] \, U_4$ and calculate any polynomials p_{ij} and q_{ij} satisfying the (decoupled) equations

$$e_i p_{ij} + q_{ij} f_j = g_{ij}$$

for $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, l$ and

$$e_i p_{ij} = g_{ij}$$

for $i = 1, 2, \dots, m$ and $j = l + 1, \dots, l + m$.

Form the matrices

$$\tilde{X} = [p_{ij}], \quad i = 1, 2, \dots, m \quad \text{and} \quad j = 1, 2, \dots, m$$

$$\tilde{Y} = [p_{ij}], \quad i = 1, 2, \dots, m \quad \text{and} \quad j = m + 1, \dots, m + l$$

$$\tilde{Z} = [q_{ij}], \quad i = 1, 2, \dots, m \quad \text{and} \quad j = 1, 2, \dots, l$$

and write

$$U_4^{-1} = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix}$$

where U_{11} is $m \times m$. Then $U_2(\tilde{X}U_{11} + \tilde{Y}U_{21})$, $U_2(\tilde{X}U_{12} + \tilde{Y}U_{22})$, and $U_1^{-1}\tilde{Z}U_3$ is a solution of equation (14). To obtain the optimal solution divide $\bar{C}$ into $U_1^{-1}\tilde{Z}U_3$ from the left. If M is the quotient and N is the remainder, then

$$\begin{aligned} X_0 &= U_2(\tilde{X}U_{11} + \tilde{Y}U_{21}) - MB \\ Y_0 &= U_2(\tilde{X}U_{12} + \tilde{Y}U_{22}) + MA \\ Z_0 &= N_* R \end{aligned}$$

is the optimal solution of equations (3).

All solutions to equations (3), if they are of interest, can be generated from the particular solution X_0, Y_0, and Z_0 according to Theorem 2.

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