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The Characteristic Polynomial of the Feedback Connection of Dynamical Systems

MILOSLAV HÁJEK

The paper proves by means of the state space theory the theorem in which the characteristic polynomial of the determinate feedback connection of two controllable and observable dynamical systems is stated. The characteristic polynomial is expressed by the use of the transfer function matrices of each dynamical system.

1. INTRODUCTION

Let C^n and R^q be n and q dimensional complex and real Euclidean spaces, respectively, and let C^{nn} and R^{qp} be sets of $n \times n$ and $q \times p$ matrices with complex and real elements, respectively.

Definition 1. A dynamical system $\mathscr{S} \equiv \{\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}\}$ is the following mathematical structure:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\,\mathbf{x}(t) + \mathbf{B}\,\mathbf{u}(t)\,, \tag{1}$$

$$\mathbf{y}(t) = \mathbf{C}\,\mathbf{x}(t) + \mathbf{D}\,\mathbf{u}(t)\,, \tag{2}$$

where $\mathbf{x} \in C^n$, $\mathbf{y} \in R^q$, $\mathbf{u} \in R^p$, $\mathbf{A} \in C^{nn}$, $\mathbf{B} \in C^{np}$, $\mathbf{C} \in C^{qn}$, $\mathbf{D} \in R^{qp}$; $q, p \leqq n$; $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ are constant matrices.

Euclidean space C^n is called the *state space* of the dynamical system and is generally denoted by Σ; Euclidean spaces R^q and R^p are called the *output space* and the *input space* of the dynamical system, respectively. A vector function $\mathbf{x}(t) \in \Sigma$ is called the *state* of the dynamical system; vector functions $\mathbf{y}(t) \in R^q$ and $\mathbf{u}(t) \in R^p$ are called the *output* and the *input* of the dynamical system, respectively. It is supposed, that $\mathbf{u}(t)$ is a piecewise continuous function with simple (or jump) discontinuities.

Remark 1. More accurately, the dynamical system in definition 1 is a finite dimensional, continuous time, linear, time-invariant, differential dynamical sys em and will be shortly called the dynamical system.

Let $\mathbf{x}(t)$, $\mathbf{y}(t)$, $\mathbf{u}(t)$ be Laplace transformable functions. Let us denote $\hat{\mathbf{x}}(s)$, $\hat{\mathbf{y}}(s)$, $\hat{\mathbf{u}}(s)$ the Laplace transforms of the $\mathbf{x}(t)$, $\mathbf{y}(t)$, $\mathbf{u}(t)$. Let us express a dependence of the output $\mathbf{y}$ on the input $\mathbf{u}$ from equations (1) and (2) by means of the Laplace transform provided the zero initial conditions, i.e. $\mathbf{x}(0) = \mathbf{0}$:

$$\hat{\mathbf{y}}(s) = [\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1} \mathbf{B} + \mathbf{D}] \hat{\mathbf{u}}(s) . \tag{3}$$

A matrix $\mathbf{G}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1} \mathbf{B} + \mathbf{D}$, $\mathbf{G} \in C^{qp}$ is called the *transfer function matrix* of the dynamical system $\mathscr{S} \equiv \{\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}\}$. The dynamical system $\mathscr{S}$ is *completely characterized* by its transfer function matrix $\mathbf{G}(s)$ if and only if this dynamical system is controllable and observable. Elements of the transfer function matrix are relatively prime rational functions and the degree of the numberator is at most equal to the degree of the denominator.

Let us denote $\Delta = \det(s\mathbf{I} - \mathbf{A})$ a characteristic polynomial of the dynamical system $\mathscr{S} \equiv \{\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}\}$. Let us define a characteristic polynomial of the transfer function matrix $\mathbf{G}(s)$ of the dynamical system $\mathscr{S}$.

Definition 2. A characteristic polynomial $\hat{\Delta}$ of the transfer function matrix $\mathbf{G}(s)$ is the least common denominator of all the minors of $\mathbf{G}(s)$.

For so defined characteristic polynomial of the transfer function matrix the following theorem holds:

Theorem 1. $\Delta = \hat{\Delta}$ *if and only if the dynamical system* $\mathscr{S}$ *is completely described by its transfer function matrix* $\mathbf{G}$.

The proof is given in [4].

Let two dynamical systems $\mathscr{S}_i \equiv \{{}_i\mathbf{A}, {}_i\mathbf{B}, {}_i\mathbf{C}, {}_i\mathbf{D}\}$, $i = 1, 2$, be given. A state space of the i-th dynamical system is denoted by Σ_i. A transfer function matrix ${}_i\mathbf{G}(s)$ of the dynamical system $\mathscr{S}_i$ is

$${}_i\mathbf{G}(s) = {}_i\mathbf{C}(s\mathbf{I} - {}_i\mathbf{A})^{-1}\, {}_i\mathbf{B} + {}_i\mathbf{D} . \tag{4}$$

Let us define

$${}_i\hat{\mathbf{G}}(s) = {}_i\mathbf{C}(s\mathbf{I} - {}_i\mathbf{A})^{-1}\, {}_i\mathbf{B} . \tag{5}$$

Then

$${}_i\mathbf{G}(s) = {}_i\hat{\mathbf{G}}(s) + {}_i\mathbf{D} . \tag{6}$$

Let us denote ${}_i\Delta = \det(s\mathbf{I} - {}_i\mathbf{A})$ the characteristic polynomial of the matrix ${}_i\mathbf{A}$ (or the characteristic polynomial of the dynamical system $\mathscr{S}_i$) and ${}_i\hat{\Delta}$ the characteristic polynomial of the transfer function matrix ${}_i\boldsymbol{G}$ of the dynamical system $\mathcal{S}_i$. It is supposed that the dynamical systems $\mathcal{S}_1$ and $\mathcal{S}_2$ are in the beginning at the zero states, i.e. ${}_1\boldsymbol{x}(0) = \boldsymbol{0}$ and ${}_2\boldsymbol{x}(0) = \boldsymbol{0}$, and their mutual connection is performed at time $t = 0$. Let $\boldsymbol{y} \in R^q$ and $\boldsymbol{u} \in R^p$ be the output and the input of a composed dynamical system.

Let $\mathcal{S} \equiv \{\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}\}$ be a dynamical system which arose by arbitrary connection of two dynamical systems $\mathcal{S}_1$ and $\mathcal{S}_2$. We mutualy connect only inputs and outputs of the individual dynamical systems and the states ${}_1\boldsymbol{x}$, ${}_2\boldsymbol{x}$ of the dynamical systems $\mathcal{S}_1$, $\mathcal{S}_2$ are kept. A state $\boldsymbol{x}$ of the composed dynamical system $\mathcal{S}$ is determined by states of the dynamical systems $\mathcal{S}_1$ and $\mathcal{S}_2$. So

$$\boldsymbol{x} = \begin{bmatrix} {}_1\boldsymbol{x} \\ {}_2\boldsymbol{x} \end{bmatrix} \tag{7}$$

and a state space Σ of the composed dynamical system $\mathcal{S}$ is the direct sum of the state spaces of the dynamical systems $\mathcal{S}_1$ and $\mathcal{S}_2$:

$$\Sigma = \Sigma_1 \oplus \Sigma_2 \,. \tag{8}$$

We shall suppose below that the dynamical system $\mathcal{S}_1$ and $\mathcal{S}_2$ are controllable and observable or that they are completely described by their transfer function matrices ${}_1\boldsymbol{G}$ and ${}_2\boldsymbol{G}$.

2. FEEDBACK CONNECTION OF DYNAMICAL SYSTEMS

Let us have two controllable and observable dynamical systems $\mathcal{S}_i \equiv \equiv \{{}_i\boldsymbol{A}, {}_i\boldsymbol{B}, {}_i\boldsymbol{C}, \boldsymbol{D}\}$, $i = 1, 2$. A feedback connection of the dynamical systems

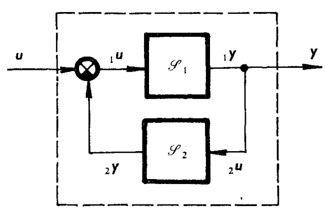

Fig. 1. The feedback connection of dynamical systems $\mathcal{S}_1$ and $\mathcal{S}_2$.

$\mathcal{S}_1$, $\mathcal{S}_2$ with dynamical system $\mathcal{S}_2$ in the feedback is schematically pictured in Fig. 1. It is clear from Fig. 1 that ${}_1\boldsymbol{u} = \boldsymbol{u} - {}_2\boldsymbol{y}$, ${}_2\boldsymbol{u} = {}_1\boldsymbol{y} = \boldsymbol{y}$, so $p = {}_1p = {}_2q$, ${}_2p = {}_1q = q$. A dynamical system $\mathcal{S}$ which arose by feedback connection of the dynamical systems

$\mathscr{S}_1$, $\mathscr{S}_2$ is described by the following equations:

$$(9)\quad \begin{bmatrix} {}_1\dot{\boldsymbol{x}} \\ {}_2\dot{\boldsymbol{x}} \end{bmatrix} = \begin{bmatrix} {}_1\boldsymbol{A} - {}_1\boldsymbol{B}_2\boldsymbol{D}(\boldsymbol{I} + {}_1\boldsymbol{D}_2\boldsymbol{D})^{-1}\,{}_1\boldsymbol{C} & -{}_1\boldsymbol{B}_2\boldsymbol{C} + {}_1\boldsymbol{B}_2\boldsymbol{D}(\boldsymbol{I} + {}_1\boldsymbol{D}_2\boldsymbol{D})^{-1}\,{}_1\boldsymbol{D}_2\boldsymbol{C} \\ {}_2\boldsymbol{B}(\boldsymbol{I} + {}_1\boldsymbol{D}_2\boldsymbol{D})^{-1}\,{}_1\boldsymbol{C} & {}_2\boldsymbol{A} - {}_2\boldsymbol{B}(\boldsymbol{I} + {}_1\boldsymbol{D}_2\boldsymbol{D})^{-1}\,{}_1\boldsymbol{D}_2\boldsymbol{C} \end{bmatrix} \begin{bmatrix} {}_1\boldsymbol{x} \\ {}_2\boldsymbol{x} \end{bmatrix} +$$

$$+ \begin{bmatrix} {}_1\boldsymbol{B} - {}_1\boldsymbol{B}_2\boldsymbol{D}(\boldsymbol{I} + {}_1\boldsymbol{D}_2\boldsymbol{D})^{-1}\,{}_1\boldsymbol{D} \\ {}_2\boldsymbol{B}(\boldsymbol{I} + {}_1\boldsymbol{D}_2\boldsymbol{D})^{-1}\,{}_1\boldsymbol{D} \end{bmatrix} \boldsymbol{u},$$

$$(10)\qquad \boldsymbol{y} = (\boldsymbol{I} + {}_1\boldsymbol{D}_2\boldsymbol{D})^{-1} \left\{ \begin{bmatrix} {}_1\boldsymbol{C} & -{}_1\boldsymbol{D}_2\boldsymbol{C} \end{bmatrix} \begin{bmatrix} {}_1\boldsymbol{x} \\ {}_2\boldsymbol{x} \end{bmatrix} + {}_1\boldsymbol{D}\boldsymbol{u} \right\}$$

where

$$(11)\qquad \det(\boldsymbol{I} + {}_1\boldsymbol{D}_2\boldsymbol{D}) \neq 0\,.$$

Let us express a dependence of $\hat{\boldsymbol{y}}(s)$ on $\hat{\boldsymbol{u}}(s)$ by means of the Laplace transform under assumption that initial conditions are zero, i.e. $\begin{bmatrix} {}_1\boldsymbol{x}(0) \\ {}_2\boldsymbol{x}(0) \end{bmatrix} = \boldsymbol{0}$:

$$(12)\qquad \hat{\boldsymbol{y}}(s) = (\boldsymbol{I} + {}_1\boldsymbol{G}_2\boldsymbol{G})^{-1}\,{}_1\boldsymbol{G}\,\hat{\boldsymbol{u}}(s)\,,$$

where

$$(13)\qquad \det(\boldsymbol{I} + {}_1\boldsymbol{G}_2\boldsymbol{G}) \neq 0$$

for all $s \in C$ with the exception of the finite number of points. The condition (13) guarantees that the output $\boldsymbol{y}$ and the state $\boldsymbol{x}$ of the composed dynamical system $\mathscr{S}$ are by the input $\boldsymbol{u}$ and the initial conditions $\begin{bmatrix} {}_1\boldsymbol{x}(0) \\ {}_2\boldsymbol{x}(0) \end{bmatrix}$ determined uniquely. In such case we say that the feedback connection of the dynamical systems $\mathscr{S}_1$, $\mathscr{S}_2$ is *determinate* (see [1]).

We can easily persuade that

$$(14)\qquad \boldsymbol{I} + {}_1\boldsymbol{D}_2\boldsymbol{D} = \lim_{s\to\infty}(\boldsymbol{I} + {}_1\boldsymbol{G}_2\boldsymbol{G})$$

and $\det(\boldsymbol{I} + {}_1\boldsymbol{D}_2\boldsymbol{D}) \neq 0 \Rightarrow \det(\boldsymbol{I} + {}_1\boldsymbol{G}_2\boldsymbol{G}) \neq 0$. So the condition (11) is stronger than the condition (13). We shall further deal with only the determinate feedback connections of the dynamical systems $\mathscr{S}_1$ and $\mathscr{S}_2$ and in addition we shall suppose that the relationship (11) holds.

Let us denote $\boldsymbol{G} = (\boldsymbol{I} + {}_1\boldsymbol{G}_2\boldsymbol{G})^{-1}\,{}_1\boldsymbol{G}$ a transfer function matrix of the feedback connection of the dynamical systems $\mathscr{S}_1$, $\mathscr{S}_2$. For determinants of matrices $(\boldsymbol{I} + {}_1\boldsymbol{G}_2\boldsymbol{G})$ and $(\boldsymbol{I} + {}_2\boldsymbol{G}_1\boldsymbol{G})$ the following theorem holds: (See [4, 5]).

Theorem 2. $\det(\boldsymbol{I}_p + {}_2\boldsymbol{G}_1\boldsymbol{G}) = \det(\boldsymbol{I}_q + {}_1\boldsymbol{G}_2\boldsymbol{G})$, *where* $\boldsymbol{I}_p$ *and* $\boldsymbol{I}_q$ *are the identity matrices of the p-th and q-th order, respectively.*

Proof. Let us have the following determinant

$$\det \begin{bmatrix} \mathbf{I}_q & {}_1\mathbf{G} \\ -{}_2\mathbf{G} & \mathbf{I}_p \end{bmatrix}. \tag{15}$$

By using the known formulas for calculation of the determinant of a block matrix (see [2]) we obtain:

$$\det(\mathbf{I}_q)\det(\mathbf{I}_p + {}_2\mathbf{G}_1\mathbf{G}) = \det(\mathbf{I}_q + {}_1\mathbf{G}_2\mathbf{G})\det(\mathbf{I}_p)\,. \quad \square \tag{16}$$

For the transfer function matrix $\mathbf{G}$ the following theorem holds (see [3]):

Theorem 3. $\mathbf{G} = (\mathbf{I} + {}_1\mathbf{G}_2\mathbf{G})^{-1}\,{}_1\mathbf{G} = {}_1\mathbf{G}(\mathbf{I} + {}_2\mathbf{G}_1\mathbf{G})^{-1}$.

Proof. If matrix $(\mathbf{I} + {}_1\mathbf{G}_2\mathbf{G})^{-1}$ exists then, according to theorem 2, the matrix $(\mathbf{I} + {}_2\mathbf{G}_1\mathbf{G})^{-1}$ also exists. The next part of the proof is performed by multiplying the relationship in theorem 3 from right by matrix $(\mathbf{I} + {}_2\mathbf{G}_1\mathbf{G})$ and from left by matrix $(\mathbf{I} + {}_1\mathbf{G}_2\mathbf{G})$. $\square$

Theorems 2 and 3 will be useful in the next chapter.

3. THE CHARACTERISTIC POLYNOMIAL OF THE FEEDBACK CONNECTION OF DYNAMICAL SYSTEMS

Let us have a determinate feedback connection of two controllable and observable dynamical systems $\mathscr{S}_1$ and $\mathscr{S}_2$. Such a composed dynamical system $\mathscr{S}$ is described by equations (9) and (10). Though dynamical systems $\mathscr{S}_1$ and $\mathscr{S}_2$ are controllable and observable the composed dynamical system $\mathscr{S}$ need not be controllable and observable and so need not be completely described by its transfer function matrix $\mathbf{G}$. It is true that the characteristic polynomial of a feedback connection $\mathscr{S}$ can be determined on the basis of the equation (9), but in many cases it will be very useful to have a possibility to determine this polynomial from the transfer function matrices ${}_1\mathbf{G}$ and ${}_2\mathbf{G}$ directly regardless of the fact whether the composed dynamical system is controllable and observable or not. The solution of the insinuated problem is given in the following theorem, which is the main result of this paper.

Theorem 4. *Let $\mathscr{S}$ is a determinate feedback connection of controllable and observable dynamical systems $\mathscr{S}_1$ and $\mathscr{S}_2$ (see* Fig. 1*) which is described by the equations* (9) *and* (10). *Let Δ is the characteristic polynomial of the feedback connection $\mathscr{S}$; let ${}_i\hat{\Delta}$ is the characteristic polynomial of the transfer function matrix ${}_i\mathbf{G}$ of the dynamical system $\mathscr{S}_i$, $i = 1, 2$. (It is evident that, in accordance with the given assumption, ${}_i\hat{\Delta} = {}_i\Delta$.) Let*

$$\det(\mathbf{I} + {}_1\mathbf{G}_2\mathbf{G}) = \frac{M(s)}{N(s)}, \tag{17}$$

where M and N are relatively prime polynomials and let

$$\overline{M} = \frac{{}_1\hat{\Delta}\,{}_2\hat{\Delta}}{N}. \tag{18}$$

Then

$$\Delta = M\overline{M} \quad [\text{mod const.}]. \tag{19}$$

Proof. First of all we show that $\overline{M}$ is a polynomial. Let us repeat that ${}_i\hat{\Delta}$ is, in accordance with definition 2, the least common denominator of all the minors of the transfer function matrix ${}_i\boldsymbol{G}$. Using pure algebraic operations it is possible to show that $N(s)$ divides ${}_1\hat{\Delta}_2\hat{\Delta}$ and therefore the expression on the right hand side of the equation (18) is really a polynomial. The next part of the proof is performed by direct calculation and by suitable arrangement of the characteristic polynomial $\Delta = \det(s\boldsymbol{I} - \boldsymbol{A})$, where $\boldsymbol{A}$ is the respective matrix in the equation (9)

$$\Delta = \det\begin{bmatrix} s\boldsymbol{I} - {}_1\boldsymbol{A} + {}_1\boldsymbol{B}_2\boldsymbol{D}(\boldsymbol{I} + {}_1\boldsymbol{D}_2\boldsymbol{D})^{-1}\,{}_1\boldsymbol{C} & {}_1\boldsymbol{B}_2\boldsymbol{C} - {}_1\boldsymbol{B}_2\boldsymbol{D}(\boldsymbol{I} + {}_1\boldsymbol{D}_2\boldsymbol{D})^{-1}\,{}_1\boldsymbol{D}_2\boldsymbol{C} \\ -{}_2\boldsymbol{B}(\boldsymbol{I} + {}_1\boldsymbol{D}_2\boldsymbol{D})^{-1}\,{}_1\boldsymbol{C} & s\boldsymbol{I} - {}_2\boldsymbol{A} + {}_2\boldsymbol{B}(\boldsymbol{I} + {}_1\boldsymbol{D}_2\boldsymbol{D})^{-1}\,{}_1\boldsymbol{D}_2\boldsymbol{C} \end{bmatrix}. \tag{20}$$

Using a relationship for the calculation of the determinant of a block matrix we obtain:

$$\begin{aligned} \boldsymbol{A} = {} & \det[s\boldsymbol{I} - {}_1\boldsymbol{A} + {}_1\boldsymbol{B}_2\boldsymbol{D}(\boldsymbol{I} + {}_1\boldsymbol{D}_2\boldsymbol{D})^{-1}\,{}_1\boldsymbol{C}]\det\{s\boldsymbol{I} - {}_2\boldsymbol{A} + {}_2\boldsymbol{B}(\boldsymbol{I} + {}_1\boldsymbol{D}_2\boldsymbol{D})^{-1}\,{}_1\boldsymbol{D}_2\boldsymbol{C} + \\ & + {}_2\boldsymbol{B}(\boldsymbol{I} + {}_1\boldsymbol{D}_2\boldsymbol{D})^{-1}\,{}_1\boldsymbol{C}[s\boldsymbol{I} - {}_1\boldsymbol{A} + {}_1\boldsymbol{B}_2\boldsymbol{D}(\boldsymbol{I} + {}_1\boldsymbol{D}_2\boldsymbol{D})^{-1}\,{}_1\boldsymbol{C}]^{-1} \,. \\ & .[{}_1\boldsymbol{B}_2\boldsymbol{C} - {}_1\boldsymbol{B}_2\boldsymbol{D}(\boldsymbol{I} + {}_1\boldsymbol{D}_2\boldsymbol{D})^{-1}\,{}_1\boldsymbol{D}_2\boldsymbol{C}]\}, \end{aligned} \tag{21}$$

where the following unequality must hold:

$$\det[s\boldsymbol{I} - {}_1\boldsymbol{A} + {}_1\boldsymbol{B}_2\boldsymbol{D}(\boldsymbol{I} + {}_1\boldsymbol{D}_2\boldsymbol{D})^{-1}\,{}_1\boldsymbol{C}] \neq 0. \tag{22}$$

It is possible to arrange the determinant in the relationship (22) as follows, if $(s\boldsymbol{I} - {}_1\boldsymbol{A})$ is factored out, theorem 2 is used, $(\boldsymbol{I} + {}_1\boldsymbol{D}_2\boldsymbol{D})^{-1}$ is factored out and the relationships (5), (6) are used:

$$\begin{aligned} & \det[s\boldsymbol{I} - {}_1\boldsymbol{A} + {}_1\boldsymbol{B}_2\boldsymbol{D}(\boldsymbol{I} + {}_1\boldsymbol{D}_2\boldsymbol{D})^{-1}\,{}_1\boldsymbol{C}] = \\ & \quad = {}_1\Delta\det(\boldsymbol{I} + {}_1\boldsymbol{G}_2\boldsymbol{D})\det(\boldsymbol{I} + {}_1\boldsymbol{D}_2\boldsymbol{D})^{-1}. \end{aligned} \tag{23}$$

If we reason on the assumptions in theorem 4 and the next details from chapter 2, the condition (22) is evidently fulfilled.

By substitution of (23) into (21) and by several simple arragements we obtain

$$\begin{aligned} \Delta = {} & {}_1\Delta_2\Delta\det(\boldsymbol{I} + {}_1\boldsymbol{G}_2\boldsymbol{D})\det(\boldsymbol{I} + {}_1\boldsymbol{D}_2\boldsymbol{D})^{-1}\det\{\boldsymbol{I} + (s\boldsymbol{I} - {}_2\boldsymbol{A})^{-1} \,. \\ & .\,{}_2\boldsymbol{B}(\boldsymbol{I} + {}_1\boldsymbol{D}_2\boldsymbol{D})^{-1}[{}_1\boldsymbol{D} + {}_1\boldsymbol{C}[\boldsymbol{I} + (s\boldsymbol{I} - {}_1\boldsymbol{A})^{-1}\,{}_1\boldsymbol{B}_2\boldsymbol{D}(\boldsymbol{I} + {}_1\boldsymbol{D}_2\boldsymbol{D})^{-1}\,{}_1\boldsymbol{C}]^{-1} \,. \\ & .\,(s\boldsymbol{I} - {}_1\boldsymbol{A})^{-1}\,{}_1\boldsymbol{B}[\boldsymbol{I} - {}_2\boldsymbol{D}(\boldsymbol{I} + {}_1\boldsymbol{D}_2\boldsymbol{D})^{-1}\,{}_1\boldsymbol{D}]]\,{}_2\boldsymbol{C}\}. \end{aligned} \tag{24}$$

Using theorems 2, 3 and relationships (5), (6), expression (24) can be arranged like this:

(25) $$\Delta = {}_1\Delta_2\Delta \det(\boldsymbol{I} + {}_1\boldsymbol{G}_2\boldsymbol{D}) \det(\boldsymbol{I} + {}_1\boldsymbol{D}_2\boldsymbol{D})^{-1} \det\{\boldsymbol{I} + {}_2\hat{\boldsymbol{G}}(\boldsymbol{I} + {}_1\boldsymbol{D}_2\boldsymbol{D})^{-1} \cdot \\ \cdot [{}_1\boldsymbol{D} + [\boldsymbol{I} + {}_1\hat{\boldsymbol{G}}_2\boldsymbol{D}(\boldsymbol{I} + {}_1\boldsymbol{D}_2\boldsymbol{D})^{-1}]^{-1} {}_1\hat{\boldsymbol{G}}[\boldsymbol{I} - {}_2\boldsymbol{D}(\boldsymbol{I} + {}_1\boldsymbol{D}_2\boldsymbol{D})^{-1} {}_1\boldsymbol{D}]]\} .$$

When $[\boldsymbol{I} + {}_1\hat{\boldsymbol{G}}_2\boldsymbol{D}(\boldsymbol{I} + {}_1\boldsymbol{D}_2\boldsymbol{D})^{-1}]^{-1}$ is factored out, the obtained expression is arranged, theorem 2 is used, $[\boldsymbol{I} + {}_1\hat{\boldsymbol{G}}_2\boldsymbol{D}(\boldsymbol{I} + {}_1\boldsymbol{D}_2\boldsymbol{D})^{-1}]^{-1}$ is again factored out and the same arragement as between the expressions (22) and (23) is performed, equation (25) is changed on

(26) $$\Delta = {}_1\Delta_2\Delta \det[\boldsymbol{I} + ({}_1\hat{\boldsymbol{G}}_2\boldsymbol{D} + {}_1\boldsymbol{G}\hat{\boldsymbol{G}}_2)(\boldsymbol{I} + {}_1\boldsymbol{D}_2\boldsymbol{D})^{-1}] .$$

When $(\boldsymbol{I} + {}_1\boldsymbol{D}_2\boldsymbol{D})^{-1}$ is factored out, relationship (6) is used and a small rearragement is performed, the relationship

(27) $$\Delta = {}_1\Delta_2\Delta \det(\boldsymbol{I} + {}_1\boldsymbol{G}_2\boldsymbol{G}) \det(\boldsymbol{I} + {}_1\boldsymbol{D}_2\boldsymbol{D})^{-1}$$

is obtained. By this the proof is in fact completed because $\det(\boldsymbol{I} + {}_1\boldsymbol{D}_2\boldsymbol{D})$ is, in accordance with the assumption, a nonzero constant and ${}_i\Delta = {}_i\hat{\Delta}$, $i = 1, 2$. When (17) and (18) are substituted into (27) the last relationship (19) is obtained. □

Remark 2. There is a conjecture in [3] which is not identical with the content of theorem 4. but which was the initial impulse for the formulation of theorem 4. Theorem 4 has not been found in literature so far by the author.

4. SOME PRACTICAL CONSEQUENCES OF THEOREM 4

Let us have a feedback connection of one-parameter dynamical systems $\mathscr{S}_1$ and $\mathscr{S}_2$, i.e. dynamical systems with one input and one output. In this case a transfer function matrix ${}_i\boldsymbol{G}$ of the dynamical system $\mathscr{S}_i$ is reduced on the point matrix and if seen scalarly, we briefly speak about the transfer function ${}_iG$ of the one-parameter dynamical system $\mathscr{S}_i$. Each transfer function is a rational function

(28) $${}_iG(s) = \frac{{}_iP(s)}{{}_iQ(s)},$$

where ${}_iP$, ${}_iQ$ are relatively prime polynomials and the degree of ${}_iP$ is less or equal to the degree of ${}_iQ$.

A characteristic polynomial of the transfer function ${}_iG$ is ${}_i\hat{\Delta} = {}_iQ$. Using theorem 4 we obtain a characteristic polynomial of the one-parameter feedback control system (feedback connection) in a form

(29) $$\Delta = {}_1Q_2Q + {}_1P_2P .$$

In literature is currently stated that

$$(30) \qquad 1 + {}_1G_2G = 0$$

is the characteristic equation of the one-parameter feedback control system. This is, of course, true only under the assumption that the polynomials $({}_1Q_2Q + {}_1P_2P)$ and ${}_1Q_2Q$ are relatively prime. The situation is ilustrated in the following example.

Example 1. Let us have a one-parameter feedback control system (see Fig. 1), where ${}_1G = 1/(s - 1)$, ${}_2G = (s - 1)/s$ and let us examine the stability of this feedback control system. In accordance with (29) the characteristic equation is

$$(31) \qquad (s + 1)(s - 1) = 0\,,$$

so the feedback control system is unstable. It is clear that by using equation (30) we get an incorrect result. It is known, of course, that it is not permitted to compensate poles and zeros lying in the right hand side of the complex plane. In our case we should choose ${}_2G = [s - (1 + \varepsilon)]/s$, where ε is a small parameter expressing the fact that a zero point of the transfer function ${}_2G$ is not physically realized with absolute exactness. By means of the small parameter ε we arrive to the correct conclusion also by using equation (30). So theorem 4 is not in contradiction with the present procedure for the examination of the stability of a one-parameter feedback control system. The characteristic polynomial of a one-parameter feedback contol system is more precisely formulated only by theorem 4.

The practical usefulness of theorem 4 may be seen more precisely in multidimensional control systems where the situation is rather more complicated. The equation

$$(32) \qquad \det(\boldsymbol{I} + {}_1\boldsymbol{G}_2\boldsymbol{G}) = 0$$

is called in literature the characteristic equation of a feedback control system. We are going to show on the basis of an example that this equation does not often characterize the feedback control system not even from the point of view of stability.

Example 2. Let us consider a feedback control system (see Fig. 1), where

$${}_1\boldsymbol{G} = \begin{bmatrix} \dfrac{1}{s-1} & \dfrac{1}{s-1} \\ 0 & \dfrac{1}{s-1} \end{bmatrix}, \quad {}_2\boldsymbol{G} = \begin{bmatrix} \dfrac{2}{(s+1)^2} & -\dfrac{s+2}{s+1} \\ \dfrac{-2}{(s+1)^2} & \dfrac{s+2}{s+1} \end{bmatrix}.$$

Substituing ${}_1\boldsymbol{G}$ and ${}_2\boldsymbol{G}$ into equation (32) we obtain

$$(33) \qquad \frac{s^2 + s + 1}{(s + 1)(s - 1)} = 0\,,$$

so the feedback control system should be stable. By using theorem 4 we find out, however, that the characteristic polynomial

$$\Delta = (s - 1)(s + 1)(s^2 + s + 1), \tag{34}$$

for

$$M = s^2 + s + 1, \quad N = (s + 1)(s - 1), \quad {}_1\hat{\Delta} = (s - 1)^2, \quad {}_2\hat{\Delta} = (s + 1)^2.$$

So the feedback connection is unstable. The same result should be obtained by suitably introducing small parameters, similarly as at example 1, which, however, to a certain extent depens, on the technical feeling. On the other hand theorem 4 is entirely exact and its using is more comfortable.

We have supposed in examples 1 and 2 that at least one of the dynamical systems $\mathcal{S}_1$ and $\mathcal{S}_2$ is unstable. The following corollary results from theorem 4 for the feedback connections of the stable dynamical systems.

Corollary 1. *Let $\mathcal{S}$ is a determinate feedback connection of the controllable, observable and stable dynamical systems $\mathcal{S}_1$ and $\mathcal{S}_2$ (see* Fig. 1*). Then a stability of the feedback connection $\mathcal{S}$ is determined by the roots of equation*

$$\det(\boldsymbol{I} + {}_1\boldsymbol{G}_2\boldsymbol{G}) = 0. \tag{32}$$

5. CONCLUSION

Theorem 4 which has a considerable practical importance for it enables to determine a characteristic polynomial of the feedback connection of dynamical systems on the basis of the transfer function matrices of individual dynamical systems is proven in the paper by means of the state space theory. It is shown that if the dynamical systems $\mathcal{S}_1$ and $\mathcal{S}_2$ are stable then the stability of feedback connection is uniquely determined by the roots of so called characteristic equation $\det(\boldsymbol{I} + {}_1\boldsymbol{G}_2\boldsymbol{G}) = 0$; in the opposite case the given equation may lead to incorrect results.

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Charakteristický polynom zpětnovazebného spojení dynamických systémů

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V práci je na základě teorie stavového prostoru dokázána věta, v níž je formulován charakteristický polynom determinovaného zpětnovazebného spojení dvou řiditelných a pozorovatelných dynamických systémů. Charakteristický polynom je vyjádřen pomocí přenosových matic jednotlivých dynamických systémů.

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