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# On Some Properties of Dynamical Systems

PAVEL ŽAMPA

**The paper deals with some basic properties of dynamical systems, important from the mathematical simulating of controlled systems point of view. It specifies dynamical, causal, deterministic and stochastic systems. Attention is paid also to some basic properties of linear systems.**

## 1. INTRODUCTION

One of the basic tasks of automation is the determination of the control algorithm for a given, real-life physical system. This problem can be solved in two ways – experimentally and by simulating.

It is obvious, that the former will be useful for designing control algorithms to simple controlled systems, whereas by simulating, especially by using mathematical model, good results can be obtained even in constructing the control algorithms for more complicated controlled systems. Besides, at some controlled systems (nuclear reactor) a wrong control algorithm that could be used with the experimental method, must not be admitted in any case.

Simulating is therefore basal for theory and practice of automatic control. It has also many shortcomings which are given by the fact, that every model of a real-life system is a certain idealisation or approximation of this physical system. A control system that is designed for a model of controlled system can be the optimal control system of this model, but it need not be a suitable system for a control of a real-life object.

Good knowledge of properties of mathematical models or abstract systems that can be used for simulating of real-life systems is one of the conditions, necessary to overcome these difficulties.

The beginnings of the study of this branch may be dated from times of Isaac Newton. Only later the properties of dynamical systems begin to be studied more generally, especially in works of Poincaré [10], Birkhoff [2] and Nemytskii [9]. After the second world war the theory of dynamical systems becomes a part of general systems theory, the representatives of which are Bertalanffy [1] and Mesarović [8]. Also some works of Kalman [3], [4], [5] and Zadeh [12], [13] are devoted to dynamical systems from the needs of automatic control theory point of view. 

## 2. ABSTRACT SYSTEMS

Searching the nature around us we usually follow certain objects, that is, certain elements of objective reality that can exist in various places in the space or can appear in various qualities, quantities etc. Generally we shall say that these objects may appear in various forms.

Therefore each abstract system, that could serve as a model of a real-life system, should contain a set $Q$ of the abstract forms of the object O which could represent a set of all forms of existence of the real-life object so that one form of existence of an abstract object, that is one point of set $Q$ should be assigned to each form of existence of a real-life object.

Naturally a question arises, whether all forms of existence of a real-life object can be differed, a question of existence of set $Q$ and a corresponding mapping between real forms and elements $q \in Q$. This problems, however, will not be dealt here with [6].

Each real-life object exists however not only in different forms, but it exists also in time. Therefore an abstract system must contain a set $T$ which will represent a set of instants of time.

**Definition 1.** Let $T_1$ and $Q_1$ be given sets. If $t_1 \in T_1$, $q_1 \in Q_1$ the pair

$$w_1 = (t_1, q_1) \tag{1}$$

is called the *occurrence* of the object $O_1$. The set of all occurrences $w_1$ is then given by relation $W_1 = T_1 \times Q_1$ and thus $w_1 \in W_1$.

Occurrence $w_1 \in W_1$ expresses the form of existence of the given object $O_1$ and time, in which this form appeared. The occurrence is then fully defined by element $w_1 \in W_1$, without laying down any further conditions for sets $T_1$ and $Q_1$ or $W_1$.

For given object $O_1$ some occurrences are mutually exclusive, i.e. if there is an occurrence $w_1 \in W_1$ then there is such a set $W_1' \subseteq W_1$ that no further occurrence $w_1' \in W_1'$ can happen simultaneously with the occurrence $w_1$.

**Definition 2.** The set of all occurrences $w_1 \in W_1$ that can exist on given object $O_1$ simultaneously, and where no further occurrence can happen, shall be called the *elementary event* of the object $O_1$ on the set $W_1$, and we shall use symbols $\xi_{w_1}$ or $\xi_1$ for it.

**Definition 3.** Let $\xi_{w_i^j}$ be the elementary event of object $O_i$ on set $W_i^j$. Then the set

$$\xi_{w_i^j \cap w_i^k} = \xi_{w_i^j} \cap W_i^k \tag{2}$$

will be called *a part* of the elementary event $\xi_{w_i^j}$.

**Theorem 1.** *Each part $\xi_{w_i^j \cap w_i^k}$ of an elementary event $\xi_{w_i^j}$ on the set $W_i^j$ is an elementary event on the set $W_i^j \cap W_i^k$.*

Proof. It can be easily shown that

$$\xi_{w_i^j \cap w_i^k} \subseteq W_i^j \cap W_i^k \tag{3}$$

and that every occurrence of event $\xi_{w_i^j \cap w_i^k}$ belongs to set $W_i^j \cap W_i^k$.

Now we shall prove that set $\xi_{w_i^j \cap w_i^k}$, given by equation (2) and satisfying relation (3), contains the very occurences that can happen simultaneously, and that no further occurrence may happen. Let us suppose the opposite. Let such occurrence $w_i'$ exist,

$$w_i' \in W_i^j \cap W_i^k \tag{4}$$

$w_i' \notin \xi_{w_i^j \cap w_i^k}$, but it can happen simultaneously with event $\xi_{w_i^j \cap w_i^k}$. According to (4), $w_i' \in W_i^j$. As then according to supposition, occurrence $w_i'$ can happen simultaneously with event $\xi_{w_i^j \cap w_i^k}$, it can also happen simultaneously with event $\xi_{w_i^j}$. Then $w_i' \in \xi_{w_i^j}$ and according to (2) and (4), $w_i'$ belongs to set $\xi_{w_i^j \cap w_i^k}$. This is, however, a contradiction.

Let us suppose now that on the contrary there is $w_i' \in \xi_{w_i^j \cap w_i^k}$ and that occurrence $w_i'$ cannot happen simultaneously with event $\xi_{w_i^j \cap w_i^k}$. Then, however, this occurrence cannot happen simultaneously with event $\xi_{w_i^j}$ and therefore $w_i' \notin \xi_{w_i^j}$. According to (2) it means that $w_i' \notin \xi_{w_i^j \cap w_i^k}$. A part of the elementary event can therefore be only the elementary event.

**Theorem 2.** *Let sets $X_{w_i^j}$ and $X_{w_i^j \cap w_i^k}$ be sets of all elementary events $\xi_{w_i^j}$ and $\xi_{w_i^j \cap w_i^k}$ respectively. Then relation*

$$\xi_{w_i^j \cap w_i^k} = \xi_{w_i^j} \cap W_i^k \tag{5}$$

*defines the mapping $\psi_i^k$ of set $X_{w_i^j}$ onto set $X_{w_i^j \cap w_i^k}$.*

Proof. Relation (5) is the mapping of set $X_{w_i^j}$ into set $X_{w_i^j \cap w_i^k}$, because it is defined for all events $\xi_{w_i^j} \in X_{w_i^j}$ and according to Theorem 1 an element of set $X_{w_i^j \cap w_i^k}$ is an image of this event.

Now it is necessary to prove that every event $\xi_{w_i^j \cap w_i^k} \in X_{w_i^j \cap w_i^k}$ is an image of at least one event $\xi_{w_i^j}$. If there is no other event on set $W_i^j$ such that $\xi_{w_i^j \cap w_i^k} = \psi_i^k(\xi_{w_i^j})$, $\xi_{w_i^j \cap w_i^k}$ is the event on $W_i^j$ for which $\xi_{w_i^j \cap w_i^k} = \psi_i^j(\xi_{w_i^j \cap w_i^k})$.

**Theorem 3.** *Let set $\Psi_i$ be a set of all mappings $\psi_i^k$ defined in Theorem 2. Set $\Psi_i$ forms together with operation of composition $\circ$ of these mappings a commutative semigroup with unit element.*

Proof. From equation (5) it follows, that to every $W_i^k \in \mathfrak{W}_i$ where $\mathfrak{W}_i$ is the set of all subsets of set $W_i$, there exists only one mapping $\psi_i^k \in \Psi_i$ and therefore exists mapping $\lambda : \mathfrak{W}_i \to \Psi_i$.

Therefore we have following relations

$$\psi_i^k = \lambda(W_i^k)\,. \tag{6}$$

$$\psi_i^l = \lambda(W_i^l)\,, \tag{7}$$

$$\psi_i^{k,l} = \lambda(W_i^k \cap W_i^l)\,. \tag{8}$$

and therefore

$$\xi_{w_i{}^j \cap w_i{}^k} = \psi_i^k(\xi_{w_i{}^j})\,, \tag{9}$$

$$\xi_{w_i{}^j \cap w_i{}^k \cap w_i{}^l} = \psi_i^l(\xi_{w_i{}^j \cap w_i{}^k})\,, \tag{10}$$

$$\xi_{w_i{}^j \cap w_i{}^k \cap w_i{}^l} = \psi_i^{k,l}(\xi_{w_i{}^j})\,. \tag{11}$$

By composition of relations (9) and (10), by comparing with (11) and after substituting from (6), (7) and (8) we get

$$\lambda(W_i^k \cap W_i^i) = \lambda(W_i^k) \circ \lambda(W_i^l)\,. \tag{12}$$

According to this relation, set $\{\Psi_i, \circ\}$ is homomorphic with set $\{\mathfrak{W}_i, \cap\}$, which evidently forms a semigroup with unit element. The same structure has theregore even set $\{\Psi_i, \circ\}$ [7].

As we will show later, the semigroup properties of dynamical systems are the result of Theorem 3.

Occurrences that happen on the real-life object $O_1$ very often depend on occurrences or events of other objects. Therefore in the following we will watch $n$ objects $O_i$, $i \in N$ where $N = \{1, 2, \ldots, n\}$. For various objects, various sets $T_i$ of instants of time $t_i$ and various sets $Q_i$ of forms $q_i$ of object will be generally defined. Occurrence of $i$-th object will be then given by a pair $(t_i, q_i) = w_i$. Analogically, the subset $\xi_{w_i} \subseteq W_i$, which satisfies the terms of Definition 2, will be the event of $i$-th object. For the set of all events $\xi_{w_i}$ of object $O_i$ we will have symbol $X_{w_i}$. Therefore $\xi_{w_i} \in X_{w_i}$.

**Definition 4.** Let the sets $X_{w_1}, \ldots, X_{w_n}$ be the sets of all events $\xi_{w_i}$ of the objects $O_i$ on set $W_i$ for $i \in N = \{1, 2, \ldots, n\}$. Let

$$W = W_1 \times \ldots \times W_n\,. \tag{13}$$

Then the set $\mathbf{A}_w$, defined by relation

$$\mathbf{A}_w \subseteq X_{w_1} \times \ldots \times X_{w_n}\,. \tag{14}$$

will be called the *state of an abstract system on the set W* [8].

For objects $O_1, \ldots, O_n$ more states – e.g. ${}^1\mathbf{A}_w$, ${}^2\mathbf{A}_w, \ldots$ can be defined on the cartesian product. We will introduce therefore another concept.

**Definition 5.** Let $\{{}^i\mathbf{A}_w\}$, $i \in J$ be the set of states of abstract system on set $W$. Then the set

$$\mathbf{A}_w = \bigcup_{i \in J} {}^i\mathbf{A}_w \tag{15}$$

is called the *abstract system* on set $W$.

Comparing the fourth and fifth definition we can see, that every abstract system is also a state of the abstract system. The inverse statement, however, is not generally true.

As ${}^i\mathbf{A}_w \subseteq \mathbf{A}_w$, the abstract system always defines the weakest dependence among events of particular objects. If there is given a state of the abstract system, it means that relation among the elements of sets $X_{w_1}, \ldots, X_{w_n}$ is more close and that information or knowledge about the system behaviour is greater.

**Definition 6.** Let $\pi^k$ be such mapping of set $X_{w_1 J} \times \ldots \times X_{w_n J}$ onto set $X_{w_1 J \cap w_1 k} \times \times \ldots \times X_{w_n J \cap w_n k}$ that

$$(\xi_{w_1 J \cap w_1 k}, \ldots, \xi_{w_n J \cap w_n k}) = \pi^k(\xi_{w_1 J}, \ldots, \xi_{w_n J}) \tag{16}$$

if and only if

$$\xi_{w_i k \cap w_i J} = \psi_i^k(\xi_{w_i J}), \quad i = 1, 2, \ldots, n .$$

Then the set

$$\mathbf{A}_{wJ \cap wk} = \{\pi^k(\xi_{w_1 J}, \ldots, \xi_{w_n J}) \mid (\xi_{w_1 J}, \ldots, \xi_{w_n J}) \in \mathbf{A}_{wJ}\} \tag{17}$$

is called the *part of the state of abstract system* $\mathbf{A}_{wJ}$.

According to the fact that every abstract system is also the state of abstract system, we can speak, in similar sense, about the part of abstract system.

**Theorem 4.** *Set $\Pi$ of all mappings $\pi^k$ from Definition 6 forms together with operation of composition $\circ$ of these mappings a commutative semigroup with unit element.*

Proof. The theorem is the consequence of Theorem 3 and definition of mapping $\pi^k$.

## 3. ORIENTED ABSTRACT SYSTEMS

**Definition 7.** Let $\mathbf{A}_w$ be the state of abstract system, defined in Definition 4. Let us denote

$$X = X_{w_1} \times \ldots \times X_{w_r}, \tag{18}$$

$$Y = X_{w_{r+1}} \times \ldots \times X_{w_{r+s}}. \tag{19}$$

Then if from $\xi \in X$ follows that also $(\xi, \eta) \in \mathbf{A}_w$, $\eta \in Y$, we call the state of the system $\mathbf{A}_w$ the *state of oriented abstract system.* Set $X$ is called the set of input elementary events, set $Y$ is called the set of output elementary events [12].

The oriented system to the system given is not defined univocally. It is also obvious, that every system defined by relation (14) where $n > 1$ can always be defined as an oriented system. In cases where $n = 1$ we can always form a set $X$, which will contain for example one element and then define the oriented system with its help. In the following we will therefore deal only with oriented abstract systems.

## 4. DYNAMICAL SYSTEMS

**Definition 8.** Let $\mathbf{A}_w$ be the state of oriented abstract system

$$\mathbf{A}_w \subseteq X \times Y. \tag{20}$$

If the following two axioms are satisfied, the state of system is called the *state of oriented abstract dynamical system.*

**Axiom 1.** Every set $T_i$, $i = 1, 2, \ldots, n$ of dynamical system is a simply ordered set, its ordering being induced by a simply ordered set $\Theta_i$ so, that $T \subseteq \Theta_i$. Further the isomorphic mapping $\varkappa_{ij}$ of the simply ordered sets $\Theta_i$, $\Theta_j$, $i, j = 1, 2, \ldots, n$ is defined. For $\vartheta_i^1, \vartheta_i^2 \in \Theta_i$ and $\varkappa_{ij}$ is

$$\vartheta_j^1 \leqq \vartheta_j^2 \Leftrightarrow \varkappa_{ij}(\vartheta_i^1) \leqq \varkappa_{ij}(\vartheta_i^2) \tag{21}$$

Mapping $\varkappa_{ij}$ thus assigns instant of time $\vartheta_j \in \Theta_j$ to every instant of time $\vartheta_i \in \Theta_i$ so that instants of time $\vartheta_i$ and $\vartheta_j$ occur at the same time with respect to $\Theta_i$ as in general it is not necessary that $\varkappa_{ij} = \varkappa_{ji}^{-1}$. However, very often we will have $\Theta_1 = \ldots = \Theta_n$ and $\varkappa_{ij}$ will be supposed identical mappings.

**Axiom 2.** Element $\xi_{w_i}$ is the element of set $X_i$, $i = 1, 2, \ldots, n$ if and only if

$$\xi_{w_i} = \{(t_i, x(t_i)) \mid t_i \in T_i, x_i : T_i \to Q_i\}. \tag{22}$$

According to Axiom 1, in every set $T_i$ it can be said about two arbitrary elements, which of them is predecessor of the other. This axiom considers even such dependences of two objects for dynamical systems, where for instance description of every object is made in different time-space system. Otherwise sets can be chosen arbitrarily, e.g. set $T_j$ as a discrete set and set $T_k$ a continuous one.

According to Axiom 2 in dynamical systems there can be as elements of set $X_{w_i}$ only such events $\xi_{w_i}$ where in a given instant of time $t_i \in T_i$ only one form of existence $q_i \in Q_i$ of object $\mathrm{O}_i$ can occur.

Occurrences of dynamical system $\mathbf{A}_w$ can be defined on various sets $W_i = T_i \times Q_i$. However, it is most important to use such sets $T_i$, where $t_i \in T_i \Rightarrow t_i \geqq \vartheta_i$, $i = 1, 2, \ldots, n$, where $\vartheta_i$ are instants of time occurring simultaneously with respect to set $\Theta_j$.

**Definition 9.** Dynamical systems, whose occurrences are defined on sets $W_i = T_i \times Q_i$ so that for every $t_i \in T_i$, $i = 1, 2, \ldots, n$ is $t_i \geqq \vartheta_i$, are called ***right-hand side dynamical systems*** and denoted $\mathbf{A}_\vartheta$, where $\vartheta = (\vartheta_1, \ldots, \vartheta_n)$. $\vartheta_i$, $i = 1, 2, \ldots, n$ are instants which occur simultaneously with respect to set $\Theta_j$.

Very often we do not know the dynamical system behavior for all $t_i \geqq \vartheta_i$, but only for $t_i \in T'_i$ such that $t_i \in T'_i \Rightarrow \vartheta_i^1 \leqq t_i < \vartheta_i^2$ where $\vartheta_i^1, \vartheta_i^2$, $i = 1, 2, \ldots, n$ are instants of time that occur simultaneously with respect to some set $\Theta_j$. These systems will be denoted $\mathbf{A}_{\vartheta^1}^{\vartheta^2}$ or $\mathbf{A}_1^2$.

**Definition 10.** Let $\boldsymbol{A}_1^2$ be the part of the state of system $\boldsymbol{A}_1$, which is defined on set $T'$, for which we have

$$t_i \in T'_i \Rightarrow \vartheta_i^1 \leqq t_i < \vartheta_i^2 .$$

If to every $(\xi', \eta') \in \boldsymbol{A}_1^2$ is also $(\xi, \eta) \in \boldsymbol{A}_1$ and $\xi' \subseteq \xi$, $\eta' \subseteq \eta$, the state $\boldsymbol{A}_1^2$ is called the *causal state of dynamical system* $\boldsymbol{A}_1$ on set $T'$. If the state of system $\boldsymbol{A}_1$ is causal to every set $T'$, we call the state of system $\boldsymbol{A}_1$ the state of causal dynamical system.

The future behavior at causal systems has no influence on past behavior. These systems are very important in technical practice, as every physical system is a causal one.

## 6. DETERMINISTIC DYNAMICAL SYSTEMS

**Definition 11.** Let the state of right-hand side dynamical system $\boldsymbol{A}_\vartheta$ be given. If then the only one element $(\xi, \eta) \in \boldsymbol{A}_\vartheta$, $\eta \in \boldsymbol{Y}$ exists to every $\xi \in \boldsymbol{X}' \subseteq \boldsymbol{X}$ we call the state of system $\boldsymbol{A}_\vartheta$ the *state of deterministic dynamical system* on set $\boldsymbol{X}'$.

If $\boldsymbol{X}' = \boldsymbol{X}$, this state of system will be called the state of deterministic dynamical system.

**Theorem 5.** *Let the state of deterministic system $\boldsymbol{A}_{\vartheta 0} = \boldsymbol{A}_0$ be given. If then the event $\xi_0^1$, defined for all $t \in T^+$, $\vartheta_i^0 \leqq t_i < \vartheta_i^1$ is known, part $\boldsymbol{A}_1$ of the state of system $\boldsymbol{A}_0$ is also the state of deterministic system. We have*

$$\boldsymbol{A}_1 = \chi_1(\boldsymbol{A}_0, \xi_0^1) . \tag{23}$$

Proof. Let $\boldsymbol{X}_0^+$ be the set of all such events $\xi_0 \in \boldsymbol{X}_0$ where

$$\xi_0^1 = \psi_0^1(\xi_0) \tag{24}$$

where mapping $\psi_0^1$ is such mapping that assigns to event $\xi_0$ part $\xi_0^1$, which is defined on set $T^+ \times Q = W$. Then obviously $X_0^+ \subseteq X_0$. We define now a new state $\mathbf{A}_0^+$ by relation 

(25) $$\mathbf{A}_0^+ \subseteq X_0^+ \times Y_0$$

where is an implication

(26) $$(\xi_0, \eta_0) \in \mathbf{A}_0^+ \Rightarrow (\xi_0, \eta_0) \in \mathbf{A}_0 .$$

State $\mathbf{A}_0^+$ is obviously the state of deterministic system, as it follows from (26) that $\mathbf{A}_0^+ \subseteq \mathbf{A}_0$. Moreover, if state $\mathbf{A}_0^+$ is given, it is given also set $X_0^+$ for all elements of which is relation (24). Then for defining the element $\xi_0 \in X_0^+$ it is sufficient to know only element $\xi_1 \in X_1$, where $\xi_1 = \psi_1(\xi_0)$.

If there is given state $\mathbf{A}_0^+$, then to every $\xi_1 \in X_1$ exists the only element $(\xi_0, \eta_0) \in \mathbf{A}_0^+$ and therefore also the only element $(\xi_1, \eta_1) \in \mathbf{A}_1$ where $\mathbf{A}_1$ is the part of state $\mathbf{A}_0^+$.

**Definition 12.** Let $\mathbf{A}_1 = \mathbf{A}_1(\mathbf{A}_0, \xi_0^1)$ be the state of deterministic dynamical system. Then the set

(27) $$\mathbf{A}_1 = \bigcup_{\mathbf{A}_0, \xi_0^1} \mathbf{A}_1(\mathbf{A}_0, \xi_0^1)$$

is called the *deterministic dynamical system.*

According to (23), state $\mathbf{A}_1$ of deterministic dynamical system $\mathbf{A}_1$ is dependent on state $\mathbf{A}_0$ and on event $\xi_0^1$. It is therefore dependent only on the past of this system.

As the mapping (23) is not in general one-one mapping it is impossible to find the past of system univocally from the state given. It is possible to define only the equivalent classes of past behavior, where all elements of one class have the same influence on the future of the system. The state of deterministic system is therefore the minimal amount of information of the past, which is necessary for defining the future [5].

*Note 1.* Every state $\mathbf{A}_1(\mathbf{A}_0, \xi_0^1)$ of deterministic dynamical system can be considered, according to the proof of Theorem 5, as a part of state $\mathbf{A}_0^+$. If then the set of all mappings $\pi$, that assign the part of the state to the state, forms with respect to the operation of the composition a commutative semigroup with unit element, also the mappings $\chi$ have the same structure

(28) $$\mathbf{A}_0 = \chi_0(\mathbf{A}_0, \xi_0^0) ,$$

(29) $$\mathbf{A}_1 = \chi_1(\mathbf{A}_0, \xi_0^1) ,$$

(30) $$\mathbf{A}_2 = \chi_2(\chi_1(\mathbf{A}_0, \xi_0^1), \xi_1^2) .$$

*Note 2.* Deterministic systems, that satisfy the conditions of causality are called causal deterministic systems. If $\mathbf{A}_1$ is the state of deterministic system that it is also a causal system, there is only one $(\xi_1^2, \eta_1^2) \in \mathbf{A}_1^2$ to every $\xi_1^2$.

**Definition 13.** Let $A_0$ be a state of dynamical system $A_0 \subseteq X_0 \times Y_0$. Let $\mathfrak{X}_0$ and $\mathfrak{Y}_0$ be some Borel fields chosen on sets $X_0$ and $Y_0$ respectively. If there is to each $X_0' \in \mathfrak{X}_0$ such measure $P_0 = P_0(X_0')$ that $\{Y, \mathfrak{Y}_0, P_0\}$ forms a probability space, state $A_0$ is called the *state of stochastic dynamical system.*

**Theorem 6.** *Let $A_0$ be the state of stochastic dynamical system, $X_0^1$ $(Y_0^1)$ input (output) event that occurred on interval $\vartheta_i^0 \leqq t_i < \vartheta_i^1$. Then part $A_1$ of the state of stochastic dynamical system $A_0$ is also a state of stochastic dynamical system. We have*

$$A_1 = \varphi_1(A_0, X_0^1, Y_0^1)\,, \tag{31}$$

$$P_1 = \varrho_1(P_0, X_0^1, Y_0^1)\,. \tag{32}$$

Proof. We will prove that relation (32) is true. The rest can be proved as in the precedence paragraphs. Let

$$X_0^+ = \{\xi_0 \mid \xi_0 \in X_0, \psi_0^1(\xi_0) \in X_0^1\}\,, \tag{33}$$

$$Y_0^+ = \{\eta_0 \mid \eta_0 \in Y_0, \psi_0^1(\eta_0) \in Y_0^1\}\,. \tag{34}$$

Then the state of system $A_0^+$, that is given by relation

$$A_0^+ = \{(\xi_0, \eta_0) \mid \xi_0 \in X_0^+, \eta_0 \in Y_0^+, (\xi_0, \eta_0) \in A_0\} \tag{35}$$

is obviously also a state of stochastic dynamical system. It means that there is given a measure $P_0'$ to each set $X_0' \subseteq X_0^+$, $X_0' \in \mathfrak{X}_0$. We define set $X_0'$ by means of set $X_1' \subseteq X_1$, $X_1' \in \mathfrak{X}_1$ by relation

$$X_0' = \{\xi_0 \mid \xi_0 \in X_0^+, \psi_1(\xi_0) \in X_1'\}\,. \tag{36}$$

Measure $P_0'$ can be defined using sets $X_1'$ and $X_0^1$. The domain of measure $P_0'$ is set $\mathfrak{Y}_0$. But we must express measure $P_1$ with domain $\mathfrak{Y}_1$ which forms Borel field over set $Y_1$. Let us look for the probability $P_1(Y_1')$ of event $Y_1' \in \mathfrak{Y}_1$, supposing that events $Y_0^1$, $X_0^1$ and $X_1'$ occurred.

Let us define the event that may occur on interval $\vartheta_i^0 \leqq t_i < \infty$ supposing, that on $\vartheta_i^1 \leqq t_i < \infty$ occurs event $Y_1'$. For this event

$$Y_1^+ = \{\eta_0 \mid \eta_0 \in Y_0, \psi_1(\eta_0) \in Y_1'\}\,. \tag{37}$$

Its probability defines then probability of event $Y_1'$. If, however, event $Y_0^1$ is given, it is possible to find a conditional probability $P_0'(Y_1^+ \mid Y_0^+)$ for which is

$$P'(Y_1^+ \mid Y_0^+) = \frac{P_0'(Y_0^+ \cap Y_1^+)}{P(Y_0^+)} = P_1(Y_1')\,. \tag{38}$$

As $P'_0$ depends on $P_0$, $X_0^1$ and $X'_0$, $P_1$ depends on $P_0$, $X_0^1$, $Y_0^1$ and $X'_1$ and so really $P_1 = P_1(P_0, X_0^1, Y_0^1)$ exists to each $X'_1$ so that $P_1$ is a probability measure on set $\mathfrak{Y}_1$. 

## 8. LINEAR DYNAMICAL SYSTEMS

Up to now property of sets $Q_i$ were not specified in any way. In many cases the additive operation is supposed to be defined, and it forms a commutative group with this set.

If now an additive operation on sets $X_i$ will be defined with the help of this operation, even this set will form a commutative group with additive operation.

Let then set $R$ be the set of endomorphisms on set $X_i$. If $R$ is a field then if relations

(39) $$({}^j\xi_i + {}^k\xi_i)\,\mu = {}^j\xi_i\mu + {}^k\xi_i\mu\,,$$

(40) $${}^j\xi_i(\mu + \nu) = {}^j\xi_i\mu + {}^j\xi_i\nu\,,$$

(41) $${}^j\xi_i(\mu\nu) = ({}^j\xi_i\mu)\,\nu$$

are satisfied for arbitrary ${}^j\xi_i$, ${}^k\xi_i \in X_i$ and $\mu, \nu \in R$ and if $\varepsilon \in R$

(42) $${}^j\xi_i\varepsilon = {}^j\xi_i$$

set $X_i$ forms together with set of endomorphisms $R$ a linear space over field $R$. Element $\varepsilon \in R$ is obviously an identical endomorphism [7].

**Definition 14.** Let a deterministic system $\mathbf{S}_0$ be given. Let $\mathscr{S}_0$ be the set of all states $\mathbf{A}_0$, $\mathbf{B}_0$, $\mathbf{C}_0$, ... of this system. If set $\mathbf{S}_0$ forms linear space over field $K$ and if for arbitrary $\mathbf{A}_0$, $\mathbf{B}_0 \in \mathscr{S}_0$ and $\mu, \nu \in K$

(43) $$(\xi_a, \eta_a) \in \mathbf{A}_0\,,\quad (\xi_b, \eta_b) \in \mathbf{B}_0 \quad \text{or} \quad (\xi_a, \eta_a) \in \mathbf{B}_0\,,\quad (\xi_b, \eta_b) \in \mathbf{A}_0 \Leftrightarrow$$
$$\Leftrightarrow (\xi_a, \eta_a)\,\mu + (\xi_b, \eta_b)\,\nu \in \mathbf{C}_0\,,\quad C_0 \in \mathscr{S}_0$$

holds, system $\mathbf{S}_0$ is called a *linear deterministic dynamical system* over field $K$.

**Theorem 7.** *Elements $\mathbf{A}_0, \mathbf{B}_0, \ldots$ of set of states $\mathscr{S}_0$ of linear deterministic dynamical system $\mathbf{S}_0$ over field $K$ form disjunctive decomposition of set $\mathbf{S}_0$ in set of equivalent classes. So*

(44) $$\text{either} \quad \mathbf{A}_0 = \mathbf{B}_0 \quad \text{or} \quad \mathbf{A}_0 \cap \mathbf{B}_0 = \emptyset\,.$$

Proof. Theorem 7 is a direct result of Definition 14.

For simplicity of the record let us introduce symbols $(\xi_a, \eta_a) = a$, $(\xi_b, \eta_b) = b$.

**Theorem 8.** *Let set $\mathscr{S}_0$ be a set of all states of linear deterministic dynamical system $\mathbf{S}_0$ over field $K$. Then, if we define a sum $\mathbf{A}_0 + \mathbf{B}_0$ and a product $\mathbf{A}_0\mu$ by means*

$$\mathbf{A}_0 + \mathbf{B}_0 = \{(a + b) \mid a \in \mathbf{A}_0, b \in \mathbf{B}_0\}, \tag{45}$$

$$\mathbf{A}_0\mu = \{(a\mu) \mid a \in \mathbf{A}_0\}, \tag{46}$$

*where* $\mathbf{A}_0, \mathbf{B}_0 \in \mathcal{S}_0$, $\mu \in K$, *then set* $\mathcal{S}_0$ *forms linear space over field* $K$.

Proof. Let us define mapping $\lambda : \mathbf{S}_0 \to \mathcal{S}_0$ so, that

$$\mathbf{A}_0 = \lambda(a) \Leftrightarrow a \in \mathbf{A}_0 . \tag{47}$$

As every element $a \in \mathbf{S}_0$ is an element of some set $\mathbf{A}_0 \in \mathcal{S}_0$, $\lambda$ is defined on the whole set $\mathbf{S}_0$. According to Theorem 7 element $a \in \mathbf{S}_0$ may be the element of the only one set $\mathbf{A}_0 \in \mathcal{S}_0$ and so just one set $\mathbf{A}_0 \in \mathcal{S}_0$ is assigned to each $a \in \mathbf{S}_0$ by relation (47). $\lambda$ is therefore a mapping.

Let $a \in \mathbf{A}_0$, $b \in \mathbf{B}_0$, $\mathbf{A}_0, \mathbf{B}_0 \in \mathcal{S}_0$. Then according to (45) and (47)

$$\lambda(a + b) = \lambda(a) + \lambda(b) . \tag{48}$$

$\lambda$ is therefore a homomorphism of sets $\{\mathbf{S}_0, +\}$, $\{\mathcal{S}_0, +\}$ and as set $\{\mathbf{S}_0, +\}$ forms a commutative group, set $\{\mathcal{S}_0, +\}$ is a commutative group too.

Using (46) and (47) we have

$$\mathbf{A}_0\mu = \lambda(a\mu) . \tag{49}$$

Substituting into (49) from (47) we have

$$\lambda(a\mu) = \mu\, \lambda(a) \tag{50}$$

and therefore sets $\mathbf{S}_0$ and $\mathcal{S}_0$ are homomorphic with respect to set $K$. Thus the proof is accomplished. Set $\mathcal{S}_0$ forms additive commutative group and field $K$ is according to (50) also a field of endomorphisms on set $\mathcal{S}_0$. By means of equations (49) and (50) the validity od relations (39), (40), (41), (42) can be easily verified [7].

According to the definition of dynamical system, every event $\xi_0$ can be univocally defined by mapping $x : T \to Q$ and inversely every mapping $x$ is univocally defined by set $\xi_0$. Instead of $(\xi_0, \eta_0)$ we will introduce pair $(x, y)$. If $\mathbf{A}_0$ is the state of determined dynamical system, the only one $(\xi_0, \eta_0) \in \mathbf{A}_0$ exists to each $\xi_0 \in X_0$ and so the only $y$ exists to each $x$. This can be written in a multiplicative form as

$$x\mathbf{A} = y . \tag{51}$$

If $\mathbf{A} = {}^1\mathbf{A} + {}^2\mathbf{A}$ where

$${}^1x \,.\, {}^1\mathbf{A} = {}^1y , \tag{52}$$

$${}^2x \,.\, {}^2\mathbf{A} = {}^2y , \tag{53}$$

according to (45) 

$$(^1x + {}^2x)(^1\boldsymbol{A} + {}^2\boldsymbol{A}) = {}^1y + {}^2y \tag{54}$$

and using (52) and (53) we have

$$(^1x + {}^2x)(^1\boldsymbol{A} + {}^2\boldsymbol{A}) = {}^1x \,.\, {}^1\boldsymbol{A} + {}^2x \,.\, {}^2\boldsymbol{A} \,. \tag{55}$$

Product on the left side of the equation (55) is to be taken for a "scalar" product.
Analogically from equation (46) it follows

$$(^1x \,.\, \mu)(^1\boldsymbol{A}\mu) = (^1x \,.\, {}^1\boldsymbol{A})\,\mu \,. \tag{56}$$

Let us deal now with the change of state od linear dynamical system. Let two states of deterministic system $\boldsymbol{A}_0$, $\boldsymbol{B}_0 \in \mathscr{S}_0$ be given, and also state $\boldsymbol{A}_0 + \boldsymbol{B}_0 \in \mathscr{S}_0$. According to equation (23) state $\boldsymbol{A}_1 \in \mathscr{S}_1$ can be univocally assigned to every $\boldsymbol{A}_0 \in \mathscr{S}_0$ and to every $x_0^1$. We will record this univocal assignment in multiplicative form as follows

$$\boldsymbol{A}_0 \odot x_0^1 = \boldsymbol{A}_1 \,. \tag{57}$$

If now

$$^a x_0 \,.\, \boldsymbol{A}_0 = {}^a y_0 \,, \quad {}^b x_0 \,.\, \boldsymbol{B}_0 = {}^b y_0 \tag{58}$$

then analogically

$$\boldsymbol{A}_0 \odot {}^a x_0^1 = \boldsymbol{A}_1 \,, \quad \boldsymbol{B}_0 \odot {}^b x_0^1 = \boldsymbol{B}_1 \,. \tag{59}$$

According to (55) and (58) it can be written also

$$(^a x_0 + {}^b x_0)(\boldsymbol{A}_0 + \boldsymbol{B}_0) = {}^a y_0 + {}^b y_0 \tag{60}$$

and therefore we have also

$$(\boldsymbol{A}_0 + \boldsymbol{B}_0) \odot (^a x_0^1 + {}^b x_0^1) = \boldsymbol{A}_1 + \boldsymbol{B}_1 \,. \tag{61}$$

If we substitute into (61) for $\boldsymbol{A}_1$ and $\boldsymbol{B}_1$ from (59) we get

$$(\boldsymbol{A}_0 + \boldsymbol{B}_0) \odot (^a x_0^1 + {}^b x_0^1) = \boldsymbol{A}_0 \odot {}^a x_0^1 + \boldsymbol{B}_0 \odot {}^b x_0^1 \,, \tag{62}$$

product $\odot$ must be interpreted as "scalar" product.
Relation (57) can be obviously applied to $\boldsymbol{A}_1$. Then we get

$$\boldsymbol{A}_1 \odot x_1^2 = \boldsymbol{A}_2 \,. \tag{63}$$

Substituting (57) to (63) we get the multiplicative record of equation (30):

$$(\boldsymbol{A}_0 \odot x_0^1) \odot x_1^2 = \boldsymbol{A}_0 \odot (x_0^1 \odot x_1^2) \,. \tag{64}$$

We derived here some relations, which characterize in general the structure of linear systems, and which direct every concrete record of linear system. Moreover, some new concrete structures of linear dynamical systems that could be succesfully applied in theory and practice of automatic control, are supposed to be found in this way [5].

## 9. STRUCTURE

If $\mathbf{A}$ is the state of the system, then set $\mathbf{A} \subseteq \mathbf{X} \times \mathbf{Y}$ defines a certain relation on set $\mathbf{X} \times \mathbf{Y}$. This relation can be expressed for instance by tabulating all elements that belong to $\mathbf{A}$. This is, however, very often unrealizable, especially if set $\mathbf{A}$ contains a great or often infinite number of elements.

It will be therefore more convenient to find some other way, by means of which it could be possible to define relation $\mathbf{A}$, using only the finite number of some constants, that would gain various values for various relations. It is obvious, however, that these constants themselves cannot define a relation. It will be therefore necessary to define a general relation always by some suitable expression and the above mentioned constants will then more closely define the relation. Application of such means will be more general, although far from being general.

Relation $\mathbf{A}$ will be defined by choosing the set of basic relations, which will be called a structure of relation $\mathbf{A}$ and set of "constants", that will be called constituents of relation $\mathbf{A}$ at a given structure $Z$. The relation can be expressed then by means of structure $Z$ and constituent $\zeta$ of this relation at structure $Z$ as follows [8]

$$\mathbf{A} = \{Z, \zeta\}. \tag{65}$$

Let ${}^1\mathbf{A}$, ${}^2\mathbf{A}$ be two states of system $\mathbf{A}$. These relations may be expressed generally by using structures ${}^1Z$, ${}^2Z$ and constituents ${}^1\zeta$, ${}^2\zeta$. If, however, two states of one system are in question, structures ${}^1Z$ and ${}^2Z$ will be supposed to be equal and the states will differ only by constituents. Let

$${}^1\zeta = (z, {}^1s), \quad {}^2\zeta = (z, {}^2s) \tag{66}$$

where $z$ characterizes general properties of relation $\mathbf{A}$ and is the same for all states, whereas ${}^1s$ and ${}^2s$ specifies the specialities of states ${}^1\mathbf{A}$ and ${}^2\mathbf{A}$ respectively. The state of dynamical system can be therefore expressed by means of triple

$${}^1\mathbf{A} = \{Z, z, {}^1s\}. \tag{67}$$

## 10. EXAMPLE

Let us suppose, that we are to describe a dynamical system with time delay, transfer function of which is, as known, given by relation

$$F(p) = \frac{y(p)}{x(p)} = e^{-p\tau}.$$

According to Pade's approximation this transfer function can be expressed by 

$$F(p) = \frac{1 - ap + bp^2 - cp^3 + \ldots}{1 + ap + bp^2 + cp^3 + \ldots}$$

where $a, b, c, \ldots$ are certain real constants. In time area a differential equation accords with this transfer function

$$\ldots cy''' + by'' + ay' + y = x - ax' + bx'' - cx''' + \ldots$$

The state of the system is then often defined as vector $(y, y', y'')$.

A state defined in this way has, however, many lacks. At first according to Pade's approximation a state defined by vector $(y, y', y'')$ is just only the approximation of state of an actual system. Besides, there is a demand of differentiating of output (and input as well) function into the higher order, the more precise approximation is demanded. The input of real-life system naturally need not satisfy this demand. The state defined in this way does not give a good imagination about the physical essence of this concept.

Let us use the method given in previous paragraphs for description of this system. At the same time this example may illustrate concepts mentioned above.

Input and output of examined system will be denoted as object $O_1$ and $O_2$, respectively. In order to describe behavior of these objects, it is necessary to define sets $Q_1(Q_2)$ of all forms of objects $O_1$ $(O_2)$. Let therefore $Q_1 = Q_2 = Q = E_1$ where $E_1$ is the set of all real numbers. Analogically we define sets of instants of time $T_1 = T_2 = T = J$, where $J$ is interval $\langle 0, \infty)$.

The occurrence of the object $O_i$, $i = 1, 2$ is defined then by a pair

$$(t_i, q_i) = w_i\,, \quad t_i \in T_i\,, \quad q_i \in Q_i$$

it is

$$w_i \in W_i = T \times Q = E_1 \times J\,.$$

Further it is necessary to define the elementary event $\xi_i$ of object $O_i$. Let

$$\xi = \{(t_i, q_i) \mid t_i \in J, q_i = x_i(t_i)\}$$

where $x_i$ is the mapping of interval $J$ into $E_1$. Set of all $\xi$ will be denoted $\boldsymbol{X}_.$.

The state of the system will be defined by set $\boldsymbol{A}_w \subseteq \boldsymbol{X}_1 \times \boldsymbol{X}_2$

$$(68) \quad \boldsymbol{A}_w = \{(\xi_1, \xi_2) \mid x_2(t) = x_1(t - \tau) \text{ for } t \in \langle \tau, \infty),\ x_2(t) = s(t) \text{ for } t \in \langle 0, \tau)\}\,.$$

Let us denote $\boldsymbol{X}_1 = \boldsymbol{X}$, $\boldsymbol{X}_2 = \boldsymbol{Y}$. According to Definition 7 the state of system $\boldsymbol{A}_w$ can be taken for the state of oriented system, where $\boldsymbol{X}(\boldsymbol{Y})$ is a set of input (output) events.

We can easily make sure that axioms 1 and 2 are satisfied and therefore the state of system $\boldsymbol{A}_w$ can be considered as a state of dynamical system and according to

Definition 10 also as a state of right-hand side causal dynamical system. We will have symbol $\mathbf{A}_0$ for it.

As long as the expression $s(t)$ in relation (68) is a mapping of interval $\langle 0, \tau)$ into $Q$, function $x_2(t)$ is univocally assigned to each $x_1(t)$, $t \in T$ and event $\eta \in Y$ is univocally defined to each $\xi \in X$. Such state of system is then a state of determined dynamical system. Mapping $s$ is obviously given by the past of the system and can therefore gain various values e.g. from set $\Sigma$ of all admissible mapping on the interval $\langle 0, \tau)$. Set $\mathbf{A}_0 = \bigcup_{s \in \Sigma} \mathbf{A}_0$ is then called a deterministic dynamical system and set $\mathcal{S}_0 = \{\mathbf{A}_0(s) \mid s \in \Sigma\}$ the set of all states of this system.

Let $\mathbf{A}_0$, $\mathbf{B}_0 \in \mathcal{S}$, $\mu$, $\nu \in K$ where $K$ is the field of all real numbers. We can easily make sure that from $({}^a\xi, {}^a\eta) \in \mathbf{A}_0$, $({}^b\xi, {}^b\eta) \in \mathbf{B}_0$ follows that $({}^a\xi\mu + {}^b\xi\nu, {}^a\eta\mu + {}^b\eta\nu) \in \mathbf{C}_0 \in \mathcal{S}_0$. System $\mathbf{A}_0$ according to Definition 14 is a linear dynamical system.

Using the concept structure, state $\mathbf{A}_0$ of system $\mathbf{A}_0$ can be expressed by means of expression $\mathbf{A}_0 = \{Z, z, s\}$ where $Z$ represents all such systems, whose output is equal to time shifted input, $z = \tau$ is the value of this shifting and $s$ represents the state of the system, $s(t) = x_1(t - \tau)$, $t \in \langle 0, \tau)$.

The state of the system with time delay is given therefore univocally by the input value on interval $\langle -\tau, 0)$. It is obvious, that this definition of state is physically more precise, more general but also more descriptive.

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# O některých vlastnostech dynamických systémů

Pavel Žampa

Článek se zabývá studiem některých základních vlastností dynamických systémů, které jsou důležité zvláště z hlediska řízení a identifikace reálně existujících soustav. Na základě definice obecného systému, kterou uveřejnil M. D. Mesarović [8] jsou specifikovány systémy dynamické, kausální, deterministické a stochastické. Na dosti obecné úrovni je defimován stav a struktura systému. Pozornost je věnována i některým základním vlastnostem systémů lineárních. V závěru článku jsou pak na příkladu systému s dopravním zpožděním ilustrovány některé zavedené pojmy a dokumentována jejich užitečnost.

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