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RANK TESTS FOR SCALE: HÁJEK'S INFLUENCE AND RECENT DEVELOPMENTS

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In discussion are recent developments on rank tests for two-sample dispersion problems with continuous one-dimensional distribution functions F_1 and F_2. It turns out that the nonparametric theory strongly depends on whether the dispersion centers μ_1 and μ_2 are known (and equal) or unknown (and equal, or unequal). The linear rank tests are adjusted to the case "$\mu_1 = \mu_2$ unknown". Most of the literature following the book of Hájek–Šidák [12] tries to extend this theory to the case "$\mu_1 \neq \mu_2$ unknown". As is indicated in Section 1, these results are explicit or implicit within the framework of semiparametric models. Therefore, Section 2 of this paper starts with a nonparametric formulation of the dispersion problem as it is done for the location case in Lehmann [17]. This is followed by a dis ussion of the most important dispersion orderings. Section 3 gives a complete solution of the testing problem for the case "$\mu_1 = \mu_2$ known". In Sections 4 and 5 these results are extended to the cases "$\mu_1 = \mu_2$ unknown" and "$\mu_1 \neq \mu_2$ unknown".

Sections 2 – 5 survey a couple of partially still unpublished results that were received over the last 15 years. They reflect the influence which Hájek's ideas continue to have on the development of nonparametric statistics.

1. LINEAR RANK TESTS AND THEIR EXTENSION TO THE CASE OF TWO UNKNOWN AND UNEQUAL DISPERSION CENTERS

Let X_{ij}, $j = 1, \dots, n_i$, $i = 1, 2$, be independent random variables (r.v.) with one-dimensional continuous distribution functions (d.f.) F_i, not depending on j. We are interested in one-sided distribution-free tests for dispersion, i.e. whether F_1 is more dispersed than F_2, or not. In the framework of parametric or semiparametric statistics, these problems are formulated by means of a location-scale model

$$F_i(\cdot) = F\left(\frac{\cdot - \mu_i}{\sigma_i}\right), \quad \mu_i \in \mathbb{R}, \quad \sigma_i > 0, \quad i = 1, 2, \tag{1.1}$$

where F is a known or unknown continuous d.f. In such a model the hypotheses are

$$\boldsymbol{H} : \sigma_1^2 \leq \sigma_2^2, \quad \boldsymbol{K} : \sigma_1^2 > \sigma_2^2. \tag{1.2}$$

In classical statistics F is assumed to be the d.f. ϕ of the normal distribution $\mathcal{N}(0, 1)$. It is well known that in this case the F-tests are distribution-free on the boundary $\boldsymbol{J} : \sigma_1^2 = \sigma_2^2$ of the hypotheses, and this in the cases that the dispersion centers μ_1 and μ_2 are known or unknown.

If F is unknown (or known but different from ϕ) and μ_1 equals μ_2, it is standard to use linear rank tests: Let $R_{n1}, \ldots, R_{nn}, n := n_1 + n_2$, be the ranks of the r.v. $X_{11}, \ldots, X_{2n_2}$ w.r.t. the pooled sample and $b_{n1}, \ldots, b_{nn}$ be given scores. Then, the linear rank statistic

$$T_n = \sqrt{\frac{n_1 n_2}{n_1 + n_2}} \left(\frac{1}{n_1} \sum_{j=1}^{n_1} b_{nR_{nj}} - \frac{1}{n_2} \sum_{j=n_1+1}^{n} b_{nR_{nj}} \right) =: \sum_{\ell=1}^{n} c_{n\ell} b_{nR_{n\ell}} \tag{1.3}$$

is distribution-free and asymptotically normal $\mathcal{N}(0, \sigma_b^2)$ under $F_1 = F_2$, if the sequence of scores-generating step-functions

$$b_n(\cdot) = \sum_{\ell=1}^{n} b_{n\ell} \mathbb{1}_{(\frac{\ell-1}{n}, \frac{\ell}{n}]}(\cdot) \tag{1.4}$$

converges in $\mathbb{L}_2$ to a function $b \in \mathbb{L}_2$ with $\int b \mathrm{d}\boldsymbol{\lambda} = 0$ and $\sigma_b^2 := \int b^2 \mathrm{d}\boldsymbol{\lambda} > 0$. For this it is sufficient that the scores b_{nj} are either the *exact scores* [1] $Eb(U_{n\uparrow j})$, the *averaged scores* $n \int b \mathbb{1}_{(\frac{j-1}{n}, \frac{j}{n}]} \mathrm{d}\boldsymbol{\lambda}$ or – if, in addition, b is of locally bounded variation – the *approximate scores* $b(\frac{j}{n+1})$, for a given score-generating function $b \in \mathbb{L}_2$; cf. Hájek-Šidák [12], Ch. V.1.b.

In case of a dispersion problem it is reasonable to use U-shaped scores, i. e. scores b_{nj} fulfilling the condition

$$b_{n1} \geq \ldots \geq b_{n[\frac{n}{2}]} \leq \ldots \leq b_{nn}. \tag{1.5}$$

Well known examples are among others the *Capon-scores*

$$b_{nj} = E[\phi^{-1}(U_{n\uparrow j})]^2 - 1, \tag{1.6}$$

or the *Klotz-scores*

$$b_{nj} = \left[\phi^{-1} \left(\frac{j}{n+1} \right) \right]^2 - 1, \tag{1.7}$$

i. e. the exact and approximate scores for the score-function $b(\cdot) = [\phi^{-1}(\cdot)]^2 - 1$; cf. Hájek–Šidák [12], Ch. III.2.1. Cf. also Duran [9], which surveys also nonparametric tests for scale of different kinds.

Though these rank tests with U-shaped scores are very sensible in the case "$\mu_1 = \mu_2$ unknown", their application in the cases "$\mu_1 \neq \mu_2$ unknown" and "$\mu_1 = \mu_2$ known" is not without problems. In the first case the ranks of the X_{ij} have no meaning for testing the hypotheses (1.2); cf. Moses [18]. In order that this is the case – at least approximately –, first one has to estimate the dispersion centers μ_i by some statistics $\hat{\mu}_{in}$, $i = 1, 2$, and to determine the ranks $\hat{R}_{n1}, \ldots, \hat{R}_{nn}$ of the

[1] Here and in the following $U_1, \ldots, U_n$ denote independent uniform (rectangular) $\mathcal{R}(0,1)$-distributed r.v. and $U_{n\uparrow j}$ the j^{th} order statistic, $1 \leq j \leq n$. λ indicates the one-dimensional Lebesgue-measure.

centered r.v. $X_{11} - \hat{\mu}_{1n}, \ldots, X_{2n_2} - \hat{\mu}_{2n}$. Then the question is whether the modified rank statistic

$$\hat{T}_n = \sum_{\ell=1}^{n} c_{n\ell} b_{n\hat{R}_{n\ell}}$$

is at least asymptotically distribution-free. Hájek [11] – using the Jurečková-linearization of regression rank statistics, cf. [14] – has shown that this is the case for all d.f F, that permit a differentiable Lebesgue-density f satisfying

$$\int b b_F \mathrm{d}\boldsymbol{\lambda} = 0, \quad b_F := -\frac{f'}{f} \circ F^{-1}. \tag{1.8}$$

Since U-shaped score-functions b are quite often symmetric w.r.t. $1/2$, the condition (1.8) is met, if b_F is skew-symmetric w.r.t. $1/2$. This again is the case if the density f is symmetric w.r.t. 0. Symmetry conditions for handling this problem by means of Chernoff-Savage-type methods were already used by Raghavachari [22].

In the second case, i.e. in the case "$\mu_1 = \mu_2$ known", rank tests with U-shaped scores are biased, at least in a slightly more general nonparametric model.

Example 1.1. (Schäfer [23]) Let $n_1 = 2$, $n_2 = 1$, i.e. $n := n_1 + n_2 = 3$, $\mu = 1/10$. Define F_1, F_2 by their Lebesgue-densities $f_1 = 1\!\!1_{(0,1/10)} + \frac{9}{10} 1\!\!1_{(1,2)}, f_2 = 1\!\!1_{(0,1)}$. Then it holds true that

$$F_1(x) > F_2(x) \ \ \forall\, x < \mu, \quad F_1(x) < F_2(x) \ \ \forall\, x > \mu, \tag{1.9}$$

i.e. F_1 is more dispersed than F_2. Let φ be a linear rank test of level $\alpha = 1/3$ with U-shaped symmetric scores, i.e. with $b_{31} = b_{33} > b_{32}$. Then one easily verifies $E_{F_1 F_2}\varphi < 1/3$, i.e. all these tests (i.e. for instance the Klotz- and the Capon-test) are biased. Similarly, d.f. F_1, F_2 with

$$F_1(x) < F_2(x) \ \ \forall\, x < \mu, \quad F_1(x) > F_2(x) \ \ \forall\, x > \mu \tag{1.10}$$

can be given, for which F_1 is less dispersed than F_2 and for which the level α is not preserved, i.e. with $E_{F_1 F_2}\varphi > 1/3$.

The reason for the bias of linear rank tests is the fact that these tests are based on only one set of U-shaped scores. This implies that the scored ranks are minimal in a given point (typically in the median of the sample), whereas they should be minimal for those observations which are next to the known (common) dispersion center μ.

This intuitive argument suggests to use scores depending on the random number $v \in \{0, \ldots, n\}$ of observations, which are less or equal to μ. Using such *empirical scores* $b_{nj}(\frac{v}{n})$, $1 \le j \le n$, $0 \le v \le n$, we define *linear empirical rank statistics*

$$T_n^b = \sum_{\ell=1}^{n} c_{n\ell} b_{nR_{n\ell}} \left(\frac{V_n}{n}\right), \quad V_n = \#\{(i,j) : 1 \le j \le n_i, \ \ i = 1,2 : X_{ij} \le \mu\}. \tag{1.11}$$

As can be seen by means of Theorem 2.1, using the μ-ordering (2.13), the condition (1.5) has to be replaced by

$$b_{n1}\left(\frac{v}{n}\right) \geq \ldots \geq b_{nv}\left(\frac{v}{n}\right) \leq \ldots \leq b_{nn}\left(\frac{v}{n}\right) \quad \forall\, 0 \leq v \leq n. \tag{1.12}$$

Theorem 1.2. (Schäfer [23]) Let $n_1, n_2 \in \mathbb{N}$ be fixed, $n := n_1 + n_2$ and $b_{nj}(\frac{v}{n})$, $1 \leq j \leq n$, $0 \leq v \leq n$, be given empirical scores. Then it holds true:

a) Under $F_1 = F_2$ each empirical rank statistic (1.11) is conditionally distribution-free, given $V_n = v$.

b) If the condition (1.12) is fulfilled, then the empirical rank test with test statistic (1.11) is unbiased against all pairs of d.f. (F_1, F_2) with (1.9) and preserves its level under all pairs (F_1, F_2) with (1.10).

Example 1.3. (Burger [5]) Let $b : (0,1) \to \mathbb{R}$ be an integrable function and $\kappa \in (0,1)$. Then *exact empirical scores* are defined for $v \in \{0, \ldots, n\}$ by

$$b_{nj}^{\kappa}\left(\frac{v}{n}\right) = \begin{cases} Eb(\kappa U_{v\uparrow j}) & \text{for } j \leq v, \\ Eb(\kappa + (1-\kappa) U_{n-v\uparrow j-v}) & \text{for } j > v. \end{cases} \tag{1.13}$$

If b is *κ-isotonic*, i.e. antitonic in $[0,\kappa]$ and isotonic in $[\kappa, 1]$, then the condition (1.12) is fulfilled. The same holds true for the *approximate empirical scores*

$$b_{nj}^{\kappa}\left(\frac{v}{n}\right) = \begin{cases} b(\kappa \frac{j}{v+1}) & \text{for } j \leq v, \\ b(\kappa + (1-\kappa)\frac{j-v}{n-v+1}) & \text{for } j > v, \end{cases} \tag{1.14}$$

and the *averaged empirical scores*

$$b_{nj}^{\kappa}\left(\frac{v}{n}\right) = \begin{cases} \frac{v}{\kappa} \int_{(\kappa\frac{j-1}{v}, \kappa\frac{j}{v}]} b \,\mathrm{d}\boldsymbol{\lambda} & \text{for } j \leq v, \\ \frac{n-v}{1-\kappa} \int_{(\kappa+(1-\kappa)\frac{j-v-1}{n-v}, \kappa+(1-\kappa)\frac{j-v}{n-v}]} b \,\mathrm{d}\boldsymbol{\lambda} & \text{for } j > v. \end{cases}$$

For instance, if $\kappa = 1/2$ and $b = [\phi^{-1}]^2 - 1$, then (1.13) yields the *empirical Capon-scores*

$$b_{nj}^{1/2}\left(\frac{v}{n}\right) = \begin{cases} E[\phi^{-1}(\frac{1}{2} U_{v\uparrow j})]^2 - 1 & \text{for } j \leq v, \\ E[\phi^{-1}(\frac{1}{2} + \frac{1}{2} U_{n-v\uparrow j-v})]^2 - 1 & \text{for } j > v, \end{cases} \tag{1.15}$$

and (1.14) the *empirical Klotz-scores*

$$b_{nj}^{1/2}\left(\frac{v}{n}\right) = \begin{cases} [\phi^{-1}(\frac{2v+2-j}{2(v+1)})]^2 - 1 & \text{for } j \leq v, \\ [\phi^{-1}(\frac{n-2v+j+1}{2(n-v+1)})]^2 - 1 & \text{for } j > v. \end{cases} \tag{1.16}$$

Since $b = [\phi^{-1}]^2 - 1$ is 1/2-isotonic condition (1.12) is fulfilled. According to Theorem 1.2 the *empirical Capon-test* (1.11), (1.15) and the *empirical Klotz-test* (1.11), (1.16) are unbiased and level-preserving (in contrast to the usual Capon- and Klotz-test). The same holds true, for instance, for the analoguously defined empirical quartile-scores and the empirical Ansari–Bradley scores.

General remarks and notations. Whereas Raghavachari [22] and his followers – with the exception of Jurečková [15], where linear rank tests with estimated μ_1, μ_2 are discussed under contiguous alternatives – were only interested in achieving tests which are at least asymptotically distribution-free on the boundary of the hypotheses, Example 1.1 shows that it is necessary to take into account the alternative and the interior of the null hypothesis. This should be done in a nonparametric framework as it is indicated by the inequalities (1.9) and (1.10). Therefore, based on the ideas of Hoeffding [13] and Hájek [10], aspects of local optimality and a locally asymptotic approach have to be taken into account. For these discussions the concept of $\mathbb{L}_r$-differentiability, cf. [24], will be of importance. More precisely, $\mathbb{L}_1$-differentiability is the adequate tool for deriving locally optimal (invariant) tests, whereas $\mathbb{L}_2$- differentiability is needed for discussing asymptotic properties based on the approximating LAN-model.

It turns out that dispersion problems are of a different structure than location problems. This is because the dispersion centers enter as nuisance parameters, which cannot be eliminated completely by means of a reduction through invariance. In other words: Whereas the null distribution of a rank test for location is distribution-free, this is not the case for the null distribution of a rank test for scale. Instead, as indicated before, invariant tests for dispersion problems are conditionally distribution-free; cf. [7].

In Sections $2-5$, $\mathcal{F}_c$ denotes the set of all one-dimensional continuous d.f., $\mathbb{L}_r$ resp. $\mathbb{L}_r(F)$ the set of all measurable functions $b : (0,1) \to \mathbb{R}$ with $\int |b|^r \mathrm{d}\boldsymbol{\lambda} < \infty$ resp. $h : \mathbb{R} \to \mathbb{R}$ with $\int |h|^r \mathrm{d}F < \infty; r = 1, 2$. $\mathbb{L}_r^0$ resp. $\mathbb{L}_r^0(F)$ denote the subsets of all those $b \in \mathbb{L}_r$ resp. $h \in \mathbb{L}_r(F)$ with $\int b \mathrm{d}\boldsymbol{\lambda} = 0$ resp. $\int h \mathrm{d}F = 0$. Beyond that, $\mathbb{1}_B$ denotes the indicator-function of a set B, $\mathbb{1}_n$ the vector $(1, \ldots, 1)^\top \in \mathbb{R}^n$, $\hat{H}_n(\cdot) = \hat{H}_n(\cdot; x_{(n)})$ the empirical d.f. of the pooled sample $x_{(n)}$ and $\mathcal{M}^1(\mathcal{X}_0, \mathcal{B}_0)$ the class of all probability measures on the measurable space $(\mathcal{X}_0, \mathcal{B}_0)$. Finally, we abbreviate the "interior" of a null hypothesis $\boldsymbol{H}$, i.e. $\boldsymbol{H} \setminus \boldsymbol{J}$, by $\overset{\circ}{\boldsymbol{H}}$.

2. NONPARAMETRIC FORMULATION OF DISPERSION PROBLEMS

All results discussed in Section 1 were restricted to parametric or semiparametric model assumptions. From a practical point of view, though, it is only of interest whether F_1 is more dispersed than F_2, but not whether both distributions are of the same type. So the situation is similar to that in the location case. There it is only of interest, whether the observations X_{1j} are in some way larger than the observations X_{2j}, but not whether this is effected by shifting. Therefore, the dispersion problem should be formulated in a nonparametric way. A very general mode of doing this is to employ an appropriate ordering relation as Lehmann [17] did in the location case; cf. also Behnen [1]. This means that we need an appropriate dispersion ordering $\succeq$, such that the hypotheses can be formuled as

$$\boldsymbol{H} : F_1 \preceq F_2, \quad \boldsymbol{K} : F_1 \succeq F_2 \quad \text{with } F_1 \neq F_2. \tag{2.1}$$

The advantage of such a set-up is two-fold: On the one hand it is a very general way of describing the relevant aspects of the practical situation. On the other hand it allows a simple sufficient condition for unbiasedness of tests and for more general isotony statements of power functions; cf. Theorem 2.1.

The technique can also quite often be applied to other testing problems. It is only necessary that the ordering on the set $\mathcal{M}^1(\mathbb{R},\mathbb{B})$ of one-dimensional d.f. or, more general, on the set $\mathcal{M}^1(\mathcal{X}_0,\mathcal{B}_0)$ of probability measures on a Polish spare $\mathcal{X}_0$ with Borel sets $\mathcal{B}_0$ can be extended by a (closed) order relation $\succeq$ on the set $\mathbb{R}$ or $\mathcal{X}_0$. Then, for $\mathcal{H}$ being the set of all functions $h : \mathcal{X}_0 \to \mathbb{R}$ which are isotonic w.r.t. $\succeq$, according to a theorem of Kamae, Krengel, O'Brien [15], cf. also [25], Ch. 7.1.2, the ordering $\succeq$ on $\mathcal{M}^1(\mathcal{X}_0,\mathcal{B}_0)$ is defined by

$$F_1 \succeq F_2 \quad \Longleftrightarrow \quad \int h\mathrm{d}F_1 \geq \int h\mathrm{d}F_2 \quad \forall\, h \in \mathcal{H}. \tag{2.2}$$

Furthermore, according to that theorem, it holds true that for $(\mathcal{X}_0,\mathcal{B}_0)$-valued r.v. X_1, X_2 with $\mathcal{L}(X_1) \succeq \mathcal{L}(X_2)$ there exist $(\mathcal{X}_0,\mathcal{B}_0)$-valued r.v. Y_1, Y_2 (on a different probability space) with $\mathcal{L}(X_i) = \mathcal{L}(Y_i)$, $i = 1,2$, and $Y_1 \succeq Y_2$. It should be noticed that for instance for the stochastic ordering, but also for the dispersion ordering employed later on, this theorem is not really needed, since Y_i can be given in a constructive manner, namely as $Y_i = F_i^{-1}(U_i), i = 1,2$.

In all these cases the orderings on $\mathcal{X}_0$ can not only extended to $\mathcal{M}^1(\mathcal{X}_0,\mathcal{B}_0)$, but also to the class of joint distributions,

$$\mathcal{P} = \left\{ F_1^{(n_1)} \otimes F_2^{(n_2)} : F_i \in \mathcal{F}_c, \quad i = 1,2 \right\}.$$

For this, first define in a canonical way

$$x_i \preceq x_i' \quad :\Longleftrightarrow \quad x_{ij} \preceq x_{ij}' \quad \forall\, j = 1,\ldots,n_i, \quad i = 1,2. \tag{2.3}$$

By means of this abbreviation one gets an ordering on $(\mathcal{X},\mathcal{B}) = (\mathcal{X}_0^{(n_1+n_2)}, \mathcal{B}_0^{(n_1+n_2)})$ according to

$$(x_1, x_2) \preceq (x_1', x_2') \quad :\Longleftrightarrow \quad x_1 \preceq x_1', \;\; x_2 \succeq x_2' \tag{2.4}$$

and then an ordering on $\mathcal{P}$ by means of (2.2). This again can be described by a corresponding ordering on the parameter set $\Theta \subset \mathcal{F}_c \times \mathcal{F}_c$, defined by

$$(F_1, F_2) \preceq (F_1', F_2') \quad :\Longleftrightarrow \quad F_1 \preceq F_1', \quad F_2 \succeq F_2'. \tag{2.5}$$

Using this terminology and the above mentioned replacement in distribution also for the tupels of the $(n_1 + n_2)$ r.v. $X_{11}, \ldots, X_{2n_2}$ under (F_1, F_2) and (F_1', F_2'), i.e. by $F_1^{-1}(U_{11}), \ldots, F_2^{-1}(U_{2n_2})$ resp. $F_1'^{-1}(U_{11}), \ldots, F_2'^{-1}(U_{2n_2})$, one easily proves

Theorem 2.1. Let X_{ij}, $j = 1,\ldots,n_i$, $i = 1,2$, be independent real-valued r.v. with distributions F_i, and let φ be a test function on $\mathcal{X}_0^{(n_1+n_2)}$ which is isotonic w.r.t. (2.4). Then the following is true:

a) The power function $(F_1, F_2) \mapsto E_{F_1F_2}\varphi(X_1, X_2)$ is isotonic w.r.t. (2.5), i.e.:

$$(F_1', F_2') \succeq (F_1, F_2) \quad \Longrightarrow \quad E_{F_1'F_2'}\varphi(X_1, X_2) \geq E_{F_1F_2}\varphi(X_1, X_2). \tag{2.6}$$

b) If φ is α-similar on $\boldsymbol{J} : F_1 = F_2$, then it preserves its level on $\boldsymbol{H} : F_1 \preceq F_2$ and is unbiased against $\boldsymbol{K} : F_1 \succeq F_2$ with $F_1 \neq F_2$.

Now the question is which order relation should be used in the dispersion case. Whereas in the location case there is only one nonparametric ordering, appropriate for describing the hypotheses, namely the stochastic ordering, there are several orderings which could be used, at least in the first moment.

2.1. Spread ordering $\succeq_{\text{sp}}$ (cf. Bickel-Lehmann [3])

Here $F_1 \succeq_{\text{sp}} F_2$ is defined by the fact that the quantile distances under F_1 are always greater than or equal to the same distances under F_2, i.e.

$$F_1 \succeq_{\text{sp}} F_2 \quad :\Longleftrightarrow \quad F_1^{-1}(v) - F_1^{-1}(u) \geq F_2^{-1}(v) - F_2^{-1}(u) \quad \forall\, 0 < u < v < 1. \tag{2.7}$$

Though this has the advantage that no dispersion centers are needed, this ordering is not adequate for handling rank tests for dispersion. The main reason is that the largest group, which leaves the testing problem invariant, is the group of componentwise affine transformations, cf. [7], or [25], Ch. 7.1.2, and this does not reduce to ranks. Furthermore, the technical handling of (2.7) is very difficult.

2.2. μ-ordering $\succeq_\mu$ (cf. Schäfer [23] and Burger [5])

This is defined for fixed known $\mu \in \mathbb{R}$ by

$$F_1 \succeq_\mu F_2 \quad :\Longleftrightarrow \quad F_1(x) \gtreqless F_2(x) \quad \forall\, x \lesseqgtr \mu. \tag{2.8}$$

Since only continuous d.f. are admitted, the following holds true

$$F_1 \succeq_\mu F_2 \quad \Longrightarrow \quad F_1(\mu) = F_2(\mu) =: \nu. \tag{2.9}$$

Contrary to (2.7), the μ-ordering is very flexible. This is already indicated by the fact that the following statements are equivalent; cf. Burger [5] or [25].

a) $\quad F_1 \succeq_\mu F_2;$ (2.10)

b) $\quad F_1^{-1} \succeq_\nu F_2^{-1};$

c) $\quad |F_1^{-1}(u) - \mu| \geq |F_2^{-1}(u) - \mu| \quad \forall\, u \in (0,1);$

d) $\quad F_1^{-1}(v) - F_1^{-1}(u) \geq F_2^{-1}(v) - F_2^{-1}(u) \quad \forall\, 0 \leq u < \nu < v \leq 1;$ (2.11)

e) $\quad F_1(\cdot + \mu) \succeq_0 F_2(\cdot + \mu).$

Here in (2.10) $\succeq_\nu$ is defined formally as $\succeq_\mu$ in (2.8), i.e.

$$F_1^{-1} \succeq_\nu F_2^{-1} \quad :\Longleftrightarrow \quad F_1^{-1}(t) \lesseqgtr F_2^{-1}(t) \quad \forall\, t \lesseqgtr \nu. \tag{2.12}$$

Of course, in a) $\Longrightarrow$ b) one has $\nu := F_1(\mu) = F_2(\mu)$, whereas in b) $\Longrightarrow$ a) one needs $\mu := F_1^{-1}(\nu) = F_2^{-1}(\nu)$.

One more advantage of $\succeq_\mu$ compared with $\succeq_{\text{sp}}$ is the fact that the μ-ordering of one-dimensional d.f. can be generated by the *μ-ordering* on $\mathbb{R}$

$$x' \succeq_\mu x \quad :\iff \quad [x < \mu \Rightarrow x' \leq x, \quad x > \mu \Rightarrow x' \geq x]. \tag{2.13}$$

Using (2.2) one verifies that the induced ordering equals the ordering (2.8) and therefore, according to Theorem 2.1, unbiasedness and level-preservation can be ascertained. The property (2.11) indicates the connection with the spread ordering.

According to its definition the μ-ordering is adapted to the two-sample dispersion problem with known and equal dispersion centers. The case that μ_1 and μ_2 are known but unequal can trivially be reduced to the case $\mu_1 = \mu_2 =: \mu$ or equivalently be handled by the following extension of the μ-ordering (2.8)

$$F_1 \succeq_{\mu_1\mu_2} F_2 \quad :\iff \quad F_1(\cdot + \mu_1) \succeq_0 F_2(\cdot + \mu_2). \tag{2.14}$$

2.3. Free μ-orderings (cf. Burger [5])

The μ-ordering can also be modified in such a way that cases can be handled in which the dispersion centers are unknown (and equal or unequal). For these cases a location functional $\gamma : \mathcal{F}_c \to \mathbb{R}$ is needed which is equivariant for affine transformations, i.e.

$$\gamma\left(F\left(\frac{\cdot - u}{v}\right)\right) = v\gamma(F) + u \quad \forall\, u \in \mathbb{R} \quad \forall\, v > 0. \tag{2.15}$$

Well known examples are for instance the quantile functional $\gamma_\varrho(F) = F^{-1}(\varrho)$ for fixed $0 < \varrho < 1$ and the mean-value functional $\gamma(F) = \int x \mathrm{d}F(x)$, defined for all distributions F with finite first moments.

Using such a functional γ, the free μ-orderings $\succeq_\gamma^{\text{I}}$ and $\succeq_\gamma^{\text{II}}$, appropriate for the cases "$\mu_1 = \mu_2$ unknown" and "$\mu_1 \neq \mu_2$ unknown", are defined by

$$F_1 \succeq_\gamma^{\text{I}} F_2 \quad :\iff \quad F_1(\cdot + \mu) \succeq_0 F_2(\cdot + \mu), \quad \mu := \gamma(F_1) = \gamma(F_2), \tag{2.16}$$

$$F_1 \succeq_\gamma^{\text{II}} F_2 \quad :\iff \quad F_1(\cdot + \gamma(F_1)) \succeq_0 F_2(\cdot + \gamma(F_2)). \tag{2.17}$$

The adequacy of these two orderings for describing the two testing problems with equal resp. unequal dispersion centers is reflected by the implications

$$F_1 \succeq_\gamma^{\text{I}} F_2, \qquad F_i(\cdot) = F\left(\frac{\cdot - \mu}{\sigma_i}\right), \quad i = 1,2 \qquad \Longrightarrow \qquad \sigma_1 \geq \sigma_2,$$

$$F_1 \succeq_\gamma^{\text{II}} F_2, \qquad F_i(\cdot) = F\left(\frac{\cdot - \mu_i}{\sigma_i}\right), \quad i = 1,2 \qquad \Longrightarrow \qquad \sigma_1 \geq \sigma_2.$$

Finally, according to Burger [5], the following equivalence holds true

$$F_1 \succeq_{\text{sp}} F_1 \quad \iff \quad F_1 \succeq_{\gamma_\varrho}^{\text{II}} F_2 \quad \forall\, \varrho \in (0,1).$$

This makes clear why $\succeq_{\gamma_\varrho}^{\text{II}}$ allows a stronger reduction than $\succeq_{\text{sp}}$, namely to ranks, which is – as mentioned before – not possible for $\succeq_{\text{sp}}$.

3. EMPIRICAL RANK TESTS FOR THE CASE "$\mu_1 = \mu_2$ KNOWN"

As already mentioned, the nonparametric theory of dispersion problems strongly depends on whether the dispersion centers of the two samples are known (and equal) or unknown (and equal or unequal). Section 3 discusses the simplest case, namely that both centers are known and equal. The main goal is to justify the linear empirical rank test (1.11) as a locally optimal invariant test, and to derive its main asymptotic properties. Again, $X_{ij}, j = 1, \ldots, n_i, \; i = 1, 2$, are independent r.v., the distribution of which is independent of j. The hypotheses are

$$\boldsymbol{H}(\mu) : F_1 \preceq_\mu F_2, \quad \boldsymbol{K}(\mu) : F_1 \succeq_\mu F_2 \quad \text{with } F_1 \neq F_2, \tag{3.1}$$

where $\mu \in \mathbb{R}$ is given and for which the common boundary (for instance w.r.t. the totalvariation-norm) is $\boldsymbol{J}(\mu) : F_1 = F_2$, i.e. independent of μ the set $\boldsymbol{J} = \{(F, F) : F \in \mathcal{F}_c\}$. The theory of rank tests for scale is based on the fact that this testing problem is invariant against the group $\mathcal{G}(\mu)$ of all transformations on the sample space $\mathbb{R}^n$ of the form $(x_1, \ldots, x_n) \mapsto (\tau x_1, \ldots, \tau x_n)$, where $\tau : \mathbb{R} \to \mathbb{R}$ is surjective and strictly isotonic with $\tau\mu = \mu$. One easily verifies that $M_n = (R_n, V_n)$ is maximal invariant under $\mathcal{G}(\mu)$, where R_n is the vector of ranks and V_n is the number of observations, less than or equal to μ. Obviously, the null distribution of M_n is not distribution-free. This follows from the well known facts, that under $(F, F) \in \boldsymbol{J}$ the statistics R_n and V_n are independent, R_n has the discrete uniform distribution $\mathcal{L}_n$ on the group of permutations of $\{1, \ldots, n\}$ and V_n has the binomial-distribution $\mathcal{B}(n, \nu)$, $\nu := F(\mu)$, i.e.

$$\mathcal{L}_{FF}(M_n) = \mathcal{L}_n \otimes \mathcal{B}(n, \nu) \quad \forall (F, F) \in \boldsymbol{J}.$$

This implies the first important property: Each invariant test, i.e. each test of the form $\varphi_n = \psi_n(R_n, V_n)$, is only conditionally distribution-free, given $V_n = v$,

$$\mathcal{L}_{FF}(\psi_n(R_n, V_n) | V_n = v) \quad = \quad \mathcal{L}_{FF}(\psi_n(R_n, v)).$$

Example 3.1. The linear empirical rank test (1.11) only depends on R_n and V_n. Therefore it is a conditional rank test, the critical values of which can be fixed independently of $(F, F) \in \boldsymbol{J}$, conditionally given $V_n = v$. In particular it is true that T_n, given $V_n = v$, behaves like a linear rank statistic with the scores $b_{nj} = b_{nj}\left(\frac{v}{n}\right)$. If, in addition, condition (1.12) is fulfilled, the test (1.11) is unbiased against $\boldsymbol{K}(\mu)$ and preserves its level on $\boldsymbol{H}(\mu)$.

The second important property of empirical linear rank tests is that of local optimality among all invariant tests for finite sample size n. For this, as in the location case, we start from a score-function $b \in \mathbb{L}_1$, which enters the test statistic (1.11) only in a discretized form, namely in that of the empirical scores $b_{nj}\left(\frac{v}{n}\right)$, $1 \leq j \leq n$, $0 \leq v \leq n$, or equivalently in form of some generating function $b_n(\frac{v}{n}, \cdot)$. Correspondingly, for discussing the local asymptotic optimality among all tests we start from a score-function $b \in \mathbb{L}_2$. For proving these optimality properties and for asymptotic investigations of these tests one has to guarantee that the discretized versions of b converge to b in a sufficient manner. But contrary to the location case besides b another constant $\kappa \in (0,1)$ enters and, implied by this, the value $v \in \{0,\ldots,n\}$ of a corresponding statistic (namely of V_n). Therefore, as in the location case, it is useful to define a step-function b_n for $n \in \mathbb{N}$, which here also depends on κ and v. For technical reasons we first divide the intervals $[0,\kappa]$ and $[\kappa,1]$ into v resp. $n-v$ equally long subintervals and then define $b_n^\kappa(\frac{v}{n},\cdot)$ as the step-function which assumes the j^{th} empirical score $b_{nj}^\kappa\left(\frac{v}{n}\right)$ on the j^{th} of these subintervals, i.e.

$$b_n^\kappa\left(\frac{v}{n},\cdot\right)=\sum_{j=1}^{v} b_{nj}^\kappa\left(\frac{v}{n}\right)\mathbb{1}_{(\kappa\frac{j-1}{v},\kappa\frac{j}{v}]}(\cdot)+\sum_{j=v+1}^{n} b_{nj}^\kappa\left(\frac{v}{n}\right)\mathbb{1}_{(\kappa+(1-\kappa)\frac{j-v-1}{n-v},\kappa+(1-\kappa)\frac{j-v}{n-v}]}(\cdot). \tag{3.2}$$

Then it can be shown, cf. Burger [6], that the systems of exact, averaged and – if b is of locally bounded variation – approximate empirical scores $b_{nj}^\kappa\left(\frac{v}{n}\right)$ are *stable in* $\mathbb{L}_r$, i.e. it holds true that

$$\textbf{a)}\qquad b_n^\kappa\left(\frac{v_n}{n},\cdot\right)\to b(\cdot)\ \text{ in } \mathbb{L}_r \quad \text{for } \min\{v_n,n-v_n\}\to\infty; \tag{3.3}$$

$$\textbf{b)}\qquad \sup_{n\in\mathbb{N}}\ \sup_{0\le v\le n}\left\|b_n^\kappa\left(\frac{v}{n},\cdot\right)\right\|_{\mathbb{L}_r}<\infty. \tag{3.4}$$

Now it is possible to define in each point $(F_0,F_0)\in \boldsymbol{J}$ with $F_0(\mu)=\kappa$ one-parameter classes $\mathcal{F}(F_0;b,\kappa)$ of alternatives along which the linear empirical rank test with scores $b_{nj}^\kappa\left(\frac{v}{n}\right)$ turns out to be locally optimal invariant (for $r=1$ in case of exact empirical scores) resp. asymptotically optimal (for $r=2$ in case of $\mathbb{L}_2$ -stable empirical scores). These classes $\{P_\eta : \eta\in\mathbb{R}\}$ have to be *compatible with the hypotheses* (3.1) in the sense that it holds true that

$$P_0\in\boldsymbol{J},\quad P_\eta\in\overset{\circ}{\boldsymbol{H}}(\mu)\ \text{ for } \eta<0 \ \text{ and } \ P_\eta\in\boldsymbol{K}(\mu)\ \text{ for } \eta>0.$$

For defining these classes one first fixes a distribution $F_0\in\mathcal{F}_c$, i.e. a point $(F_0,F_0)\in\boldsymbol{J}$. Then, for given $b\in\mathbb{L}_r, \kappa:=F_0(\mu)\in(0,1), v\in\{0,\ldots,n\}$ and the corresponding averaged empirical scores $b_{nj}^\kappa\left(\frac{v}{n}\right)$, i.e. for given step-functions $b_n(\frac{v}{n},\cdot)$, one-dimensional d.f. F_Δ for $\Delta\in\mathbb{R}$ are defined by

$$\frac{\mathrm{d}F_\Delta}{\mathrm{d}F_0}(\cdot)=1+\Delta b_k^\kappa\left(\frac{v_k}{n},F_0(\cdot)\right),\quad k=k(\Delta):=\left[\frac{\kappa\wedge(1-\kappa)}{2|\Delta|\|b\|_{\mathbb{L}_r}}\right],\quad v_k:=[\kappa k(\Delta)]. \tag{3.5}$$

Finally, (n_1+n_2)-dimensional d.f. P_η, $\eta\in\mathbb{R}$, are defined by

$$P_\eta=F^{(n_1)}_{\frac{\eta}{n_1}\sqrt{\frac{n_1n_2}{n_1+n_2}}}\otimes F^{(n_2)}_{-\frac{\eta}{n_2}\sqrt{\frac{n_1n_2}{n_1+n_2}}}. \tag{3.6}$$

Using the basic results on $\mathbb{L}_r$-differentiability, cf. [24], and the terminology introduced before, the assertions a) – c) of the next theorem follow immediately. Part d) guarantees the compatibility of the class $\mathcal{P}=\{P_\eta : \eta\in\mathbb{R}\}$ with the hypotheses (3.1).

Theorem 3.2. Let $\kappa \in (0,1)$, $b \in \mathbb{L}_2^0$ and $F_0 \in \mathcal{F}_c$ with $F_0(\mu) = \kappa$. Then it holds that

a) the one-parameter class $\mathcal{F}(F_0; b, \kappa) := \{F_\Delta : \Delta \in \mathbb{R}\}$ is $\mathbb{L}_r(0)$-differentiable, with derivative $b \circ F_0$;

b) the one-parameter class $\mathcal{P}(F_0; b, \kappa) := \{P_\eta : \eta \in \mathbb{R}\}$ is $\mathbb{L}_r(0)$-differentiable, with derivative $S_n^{b\kappa F_0}(x_1, \ldots, x_n) := \sum_{\ell=1}^n c_{n\ell} b(F_0(x_\ell))$;

c) If $M_n = (R_n, V_n)$ denotes the maximal invariant statistic, then the one-parameter class of their distributions is $\mathbb{L}_r(0)$-differentiable, with derivative

$$E_{F_0F_0}(S_n^{b\kappa F_0}|M_n) = \sum_{\ell=1}^{n} c_{n\ell} b_{nR_{n\ell}}\left(\frac{V_n}{n}\right) = T_n^{b\kappa}. \tag{3.7}$$

In particular, the linear empirical rank test (1.11), (1.13), is locally optimal at the point $(F_0, F_0) \in \boldsymbol{J}$ along the family $\mathcal{P}(F_0; b, \kappa)$ among all invariant tests.

d) $\mathcal{P}(F_0; b, \kappa)$ is compatible with the hypotheses (3.1) provided $\int b\mathrm{d}\boldsymbol{\lambda} \leq_\kappa 0$, i.e. when

$$\int_{(0,u)} b\mathrm{d}\boldsymbol{\lambda} \leq 0 \quad \text{for } u \leq \kappa, \qquad \int_{(u,1)} b\mathrm{d}\boldsymbol{\lambda} \geq 0 \quad \text{for } u > \kappa. \tag{3.8}$$

It is sufficient for (3.8) that the following two conditions are fulfilled:

$$b \;\; \kappa\text{-isotonic}; \qquad \int_0^\kappa b\mathrm{d}\boldsymbol{\lambda} = \int_\kappa^1 b\mathrm{d}\boldsymbol{\lambda} = 0. \tag{3.9}$$

e) If $b_{nj}^\kappa\left(\frac{v}{n}\right)$, $1 \leq j \leq n$, $0 \leq v \leq n$, are the exact or approximate empirical scores for b and κ, the test (1.11) is unbiased against $\boldsymbol{K}(\mu)$ and preserves its level on $\boldsymbol{H}(\mu)$.

Example 3.3. Let $\kappa \in (0,1)$, $b \in \mathbb{L}_r^0$ with (3.9) and $F_0 \in \mathcal{F}_c$ with $F_0(\mu) = \kappa$. Then, according to Theorem 3.2 d, the class $\mathcal{P}(F_0; b, \kappa)$ is compatible with (3.1) and therefore the linear empirical rank test $\varphi_n^{b\kappa}$ with test statistic (1.11) and exact empirical scores is a locally optimal invariant test. Since because of (3.9) the condition (1.12) is also fulfilled, according to Theorem 2.1 $\varphi_n^{b\kappa}$ is an unbiased level preserving test.

Obviously, according to Theorem 3.2c, the optimality property of the test $\varphi_n^{b\kappa}$ is independent of the special point (F_0, F_0) with $F_0(\mu) = \kappa$. It is now the question whether – for given b and κ with (3.8) – also in all other points $(F, F) \in \boldsymbol{J}$, i.e. those with $F(\mu) \neq \kappa$, a one-parameter class exists, along which the test $\varphi_n^{b\kappa}$ is locally optimal invariant (for $r = 1$) resp. locally asymptotically optimal (for $r = 2$). This is indeed true. But, contrary to the location problem, the score-function b which defines the alternatives according to (3.5) and (3.6), has to be "adapted" to the point $(F, F) \in \boldsymbol{J}$ or, more precisely, to the value $\nu = F(\mu)$. To be concrete, let $b \in \mathbb{L}_r$ and $F_0, F \in \mathcal{F}_c$ with $\kappa = F_0(\mu)$, $\nu = F(\mu)$. Then the *adapted score-function for b and κ* is defined by

$$b_\nu(u) = \begin{cases} b(\frac{\kappa}{\nu}u) & \text{for} \quad u \leq \nu, \\ b\left(\kappa + \frac{1-\kappa}{1-\nu}(u-\nu)\right) & \text{for} \quad u > \nu. \end{cases} \tag{3.10}$$

One easily verifies the following properties which motivate Theorem 3.5:

a) $\nu = \kappa \quad \Longrightarrow \quad b_\nu = b;$

b) b (strictly) κ-isotonic $\quad \Longrightarrow \quad b_\nu$ (strictly) ν-isotonic; (3.11)

c) $\displaystyle\int b \mathrm{d}\boldsymbol{\lambda} \leq_\kappa 0 \quad \Longrightarrow \quad \int b_\nu \mathrm{d}\boldsymbol{\lambda} \leq_\nu 0;$ (3.12)

d) $\mathcal{P}(F_0; b, \kappa)$ is compatible with the hypotheses (3.1) in $(F_0, F_0) \quad \Longrightarrow$

$\mathcal{P}(F; b_\nu, \nu)$ is compatible with the hypotheses (3.1) in (F, F);

e) The exact, approximate and averaged empirical scores for b, κ and b_ν, ν coincide, i.e.

$$b_{nj}^\nu\left(\frac{v}{n}\right) = b_{nj}^\kappa\left(\frac{v}{n}\right), \quad 1 \leq j \leq n, \quad 0 \leq v \leq n, \quad 0 < \kappa < 1. \tag{3.13}$$

Example 3.4. Let $b_0 : (0,1) \to \mathbb{R}$. Then the following holds true:

$$b(u) = \begin{cases} b_0(\frac{\kappa-u}{\kappa}) & \text{for} \quad u \leq \kappa \\ b_0(\frac{u-\kappa}{1-\kappa}) & \text{for} \quad u > \kappa \end{cases} \quad \Longrightarrow \quad b_\nu(u) = \begin{cases} b_0(\frac{\nu-u}{\nu}) & \text{for} \quad u \leq \nu \\ b_0(\frac{u-\nu}{1-\nu}) & \text{for} \quad u > \nu. \end{cases} \tag{3.14}$$

Beyond this implication we have

$$b_0 \text{ isotonic} \quad \Longleftrightarrow \quad b\ \kappa\text{-isotonic} \quad \Longleftrightarrow \quad b_\nu\ \nu\text{-isotonic}.$$

The exact, approximate and averaged empirical scores for $j \leq v$ resp. $j > v$ simplify in this case to

$$\begin{aligned} b_{nj}\left(\tfrac{v}{n}\right) &= Eb_0(U_{v\uparrow v-j+1}) && \text{resp.} \quad Eb_0(U_{n-v\uparrow j-v}), \\ b_{nj}\left(\tfrac{v}{n}\right) &= b_0\left(\tfrac{v-j+1}{v+1}\right) && \text{resp.} \quad b_0\left(\tfrac{j-v}{n-v+1}\right), \\ b_{nj}\left(\tfrac{v}{n}\right) &= v\int_{(\frac{v-j}{v}, \frac{v-j+1}{v})} b_0 \mathrm{d}\boldsymbol{\lambda} && \text{resp.} \quad (n-v)\int_{(\frac{j-v-1}{n-v}, \frac{j-v}{n-v})} b_0 \mathrm{d}\boldsymbol{\lambda}. \end{aligned}$$

Now it is clear, what has to be done. Replace $(F_0, F_0) \in \boldsymbol{J}$ with $F_0(\mu) = \kappa$ by an arbitrary point $(F, F) \in \boldsymbol{J}$ with $F(\mu) = \nu$ and b by b_ν. Define classes $\mathcal{F}(F; b_\nu, \nu)$ and $\mathcal{P}(F; b_\nu, \nu)$ according to (3.5) resp. (3.6). Because of (3.12) $\mathcal{P}(F; b_\nu, \nu)$ is compatible with the hypotheses (3.1) in the neighborhood of (F, F). Then, in generalization of Theorem 3.2c, the following holds true:

Theorem 3.5. If $M_n = (R_n, V_n)$ denotes the maximal invariant statistic and if the one-parameter class $\mathcal{P}(F; b_\nu, \nu)$ is $\mathbb{L}_r(0)$-differentiable with derivative $S_n^{b_\nu \kappa F}$, then the class $\mathcal{P}^{M_n}(F; b_\nu, \nu)$ of distributions of M_n is $\mathbb{L}_r(0)$ differentiable, with derivative

$$E_{FF}(S_n^{b_\nu \kappa F}|M_n) = \sum_{\ell=1}^{n} c_{n\ell} b_{nR_{n\ell}} \left(\frac{V_n}{n}\right) = T_n^{b\kappa}. \tag{3.15}$$

In particular, the linear empirical rank test (1.11) is locally optimal in each point $(F, F) \in \boldsymbol{J}$ along $\mathcal{P}(F; b_\nu, \nu)$ among all invariant tests.

The special set-up (3.5) leaves open the question whether there are further one-parameter $\mathbb{L}_r(0)$-differentiable classes, which are also compatible with the hypotheses (3.1) and which would lead to a different type of locally optimal tests. It is therefore important, to determine the totality of all those one-parameter classes or at least of their $\mathbb{L}_r(0)$-derivatives. One easily verifies that this *tangent-set*, cf. Pfanzagl [21] or [26], consists of all functions

$$\sum_{\ell=1}^{n_1} b_1 \circ F_0(x_\ell) + \sum_{\ell=n_1+1}^{n} b_2 \circ F_0(x_\ell) \ : \ b_1, b_2 \in \mathbb{L}_r^0, \quad \int (b_1 - b_2) \mathrm{d}\boldsymbol{\lambda} \leq_\kappa 0, \tag{3.16}$$

where $\int (b_1 - b_2)\mathrm{d}\boldsymbol{\lambda} \leq_\kappa 0$ means $\int_{(0,u)} (b_1 - b_2)\mathrm{d}\boldsymbol{\lambda} \gtreqless 0$ for $u \lesseqgtr \kappa$.

For practical purposes it is sufficient to determine the subset of all those $\mathbb{L}_r(0)$-derivatives, which are "orthogonal" w.r.t. the boundary. This so-called *co-set* consists of all functions

$$\frac{1}{n_1} \sum_{\ell=1}^{n_1} b \circ F_0(x_\ell) - \frac{1}{n_2} \sum_{\ell=n_1+1}^{n} b \circ F_0(x_\ell) : \quad b \in \mathbb{L}_r^0, \quad \int b \mathrm{d}\boldsymbol{\lambda} \leq_\kappa 0. \tag{3.17}$$

The third important property of linear empirical rank statistics is that of asymptotic normality. This can be proved by means of Theorem 3.2 for $r = 2$, using a Pythagoras-kind of argument; cf. [26]. On the one hand $S_n^{b\kappa F_0}$ is a sum of independent r.v., standardized in a form appropriate for applying the central limit theorem. On the other hand – because of (3.15) and properties of conditional expectations – $T_n^{b\kappa}$ is orthogonal to $S_n^{b\kappa F_0} - T_n^{b\kappa}$ w.r.t. $\mathbb{L}_2(F_0)$, i. e. it holds

$$\mathrm{Var}_{F_0 F_0}(S_n^{b\kappa F_0} - T_n^{b\kappa}) = \mathrm{Var}_{F_0 F_0} S_n^{b\kappa F_0} - \mathrm{Var}_{F_0 F_0} T_n^{b\kappa}. \tag{3.18}$$

Now $\mathrm{Var}_{F_0 F_0} S_n^{b\kappa F_0} = \sum_{\ell=1}^{n} c_{n\ell}^2 \int b^2 \mathrm{d}\boldsymbol{\lambda} \to \int b^2 \mathrm{d}\boldsymbol{\lambda}$ according to the Markov-inequality and Slutsky's theorem it is sufficient to verify

$$\mathrm{Var}_{F_0 F_0} T_n^{b\kappa} \quad \longrightarrow \quad \int b^2 \mathrm{d}\boldsymbol{\lambda}. \tag{3.19}$$

But this is achieved as in the location case; cf. [7].

By means of Theorem 3.2 asymptotic normality can be established under any other distribution $(F, F) \in \boldsymbol{J}$. For instance one verifies

$$\operatorname{Var}_{FF} T_n^{b\kappa} \quad \longrightarrow \quad \sigma_{b_\nu\kappa}^2 := \frac{\nu}{\kappa} \int_{(0,\kappa)} b^2 \mathrm{d}\boldsymbol{\lambda} + \frac{1-\nu}{1-\kappa} \int_{(\kappa,1)} b^2 \mathrm{d}\boldsymbol{\lambda}, \tag{3.20}$$

if $F(\mu) = \nu$. In particular, if b is of the form (3.14), then it holds

$$\sigma_{b_\nu\kappa}^2 = \int b_0^2 \mathrm{d}\boldsymbol{\lambda}.$$

Also, using Le Cam's third lemma, cf. Hájek-Šidák [11], Lemma VI.1.4, or [25], corollary 6.139, the limit of the power function under contiguous alternatives (F_n^a, F_n^a) of the form (3.5), (3.6) with some function $a \in \mathbb{L}_2^0$ instead of the score-function $b \in \mathbb{L}_2^0$ can be determined. Using these tools and the terminology introduced above we come to

Theorem 3.6. Let $\kappa \in (0,1)$, $b \in \mathbb{L}_2^0$ κ-isotonic and $F \in \mathcal{F}_c$ with $F(\mu) = \nu$. Then it holds true that

a) the one-parameter class $\mathcal{P}(F; b_\nu, \kappa)$ is $\mathbb{L}_r(0)$ -differentiable, with derivative $S_n^{b_\nu\kappa F}$;

b) $\mathcal{L}_{FF}(S_n^{b_\nu\nu F}) \xrightarrow[\mathcal{L}]{} \mathcal{N}(0, \sigma_{b_\nu\kappa}^2)$;

c) $\mathcal{L}_{FF}(T_n^{b\kappa}) \xrightarrow[\mathcal{L}]{} \mathcal{N}(0, \sigma_{b_\nu\kappa}^2)$; (3.21)

d) $\mathcal{L}_{F_n^a F_n^a}(T_n^{b\kappa}) \xrightarrow[\mathcal{L}]{} \mathcal{N}\left(\eta \int a b_\nu \mathrm{d}\boldsymbol{\lambda}, \sigma_{b_\nu\kappa}^2\right)$;

e) $\left.\begin{array}{l} \int b_\nu \mathrm{d}\boldsymbol{\lambda} \leq_\nu 0 \\ a \ \ \nu\text{-isotonic} \end{array}\right\} \Longrightarrow \int a b_\nu \mathrm{d}\boldsymbol{\lambda} \geq 0.$

In particular the linear empirical rank tests $\varphi_n^{b\kappa}$ are asymptotically unbiased and level-preserving in the vicinity of each point $(F, F) \in \boldsymbol{J}$ along all classes $\mathcal{P}(F; a, \nu)$ with ν-isotonic a, $\int_{(0,\nu)} a \mathrm{d}\boldsymbol{\lambda} = \int_{(\nu,1)} a \mathrm{d}\boldsymbol{\lambda} = 0$.

Besides (3.21) the limit of the conditional distribution of $T_n^{b\kappa}$, given $V_n = v$, can be determined and the asymptotic equivalence of the corresponding asymptotic test $\hat{\varphi}_n^{b\kappa} = \mathbb{1}(T_n^{b\kappa} > u_\alpha \sigma_{b_\nu\kappa})$ and the conditional level α-test can be verified.

Theorem 3.7. Under the assumptions of Theorem 3.6 it holds true that

a) $\mathcal{L}_{FF}(T_n^{b\kappa} | V_n = v_n) \xrightarrow[\mathcal{L}]{} \mathcal{N}(0, \sigma_{b_\nu\kappa}^2)$ for almost all sequences(w_n);

b) $\varphi_n^{b\kappa}$ and $\hat{\varphi}_n^{b\kappa}$ are asymptotically equivalent under all pairs $(F, F) \in \boldsymbol{J}$.

If b is strictly κ-isotonic, then the asymptotic equivalence holds under each pair $(F_1, F_2) \in \boldsymbol{H}(\mu) \cup \boldsymbol{K}(\mu)$.

We omit efficiency results since these are immediate consequences of the supplement in Theorem 3.6. But we wish to mention that also consistency can be proved under very general assumptions. Similar to the location case this is based on a law of large numbers for standardized linear empirical rank statistics.

Theorem 3.8. Let $\kappa \in (0,1)$, $b \in \mathbb{L}_1^0$ κ-isotonic and $T_n^{b\kappa}$ a linear empirical rank statistic (3.7) with $\mathbb{L}_1$-stable scores. Let $\varphi_n^{b\kappa}$ be a test with test statistic $T_n^{b\kappa}, \lambda \in (0,1)$, $H_\lambda = \lambda F_1 + (1-\lambda)F_2$ and $\gamma_\lambda^{b\kappa}(F_1, F_2; \mu) = \int \dot{\upsilon}_\nu(H_\lambda)(\mathrm{d}F_1 - \mathrm{d}F_2)$. Then for $n \to \infty$ with $n_1/n \to \lambda$ it holds for all $(F_1, F_2) \in \boldsymbol{H}(\mu) \cup \boldsymbol{K}(\mu)$:

a) $T_n^{b\kappa} \to \gamma_\lambda^{b\kappa}(F_1, F_2; \mu) \quad (F_1, F_2)$-a.e.;

b) $$E_{F_1F_2}\varphi_n^{b\kappa} \to \begin{cases} 1, & \text{if } \gamma_\lambda^{b\kappa}(F_1,F_2;\mu) > 0, \\ 0, & \text{if } \gamma_\lambda^{b\kappa}(F_1,F_2;\mu) < 0. \end{cases}$$

In particular, if b is $\boldsymbol{\lambda}$-a.e. strictly κ-isotonic, then $\varphi_n^{b\kappa}$ is consistent for the hypotheses (3.1).

The supplement follows from the fact that for strictly κ-isotonic b the functional $\gamma_\lambda^{b\kappa}(F_1, F_2; \mu)$ characterizes the hypotheses according to

$$(F_1, F_2) \in \overset{\circ}{\boldsymbol{H}}(\mu) \iff \gamma_\lambda^{b\kappa}(F_1,F_2;\mu) < 0, \tag{3.22}$$

$$(F_1, F_2) \in \boldsymbol{J} \iff \gamma_\lambda^{b\kappa}(F_1,F_2;\mu) = 0, \tag{3.23}$$

$$(F_1, F_2) \in \boldsymbol{K}(\mu) \iff \gamma_\lambda^{b\kappa}(F_1,F_2;\mu) > 0. \tag{3.24}$$

4. EMPIRICAL RANK TESTS FOR THE CASE "$\mu_1 = \mu_2$ UNKNOWN"

For making the testing problem "$\mu_1 = \mu_2$ unknown" precise it is necessary to have a location functional $\gamma : \mathcal{F}_c \to \mathbb{R}$ for fixing the unknown common dispersion center μ. Then, by means of the free μ-ordering $\preceq_\gamma^{\mathrm{I}}$ defined in Section 2.3, the hypotheses can be formulated as

$$\boldsymbol{H} : F_1 \preceq_\gamma^{\mathrm{I}} F_2, \quad \boldsymbol{K} : F_1 \succeq_\gamma^{\mathrm{I}} F_2 \text{ with } F_1 \neq F_2. \tag{4.1}$$

In this situtation it suggests itself to use the test statistic $T_n =: T_n(\mu)$ from Section 3 with μ replaced by an estimator $\hat{\mu}_n$. If this depends only on the order statistic of the pooled sample – which for instance is the case for the canonical estimator $\gamma(\hat{H}_n)$ – the same is true for $\hat{H}_n(\hat{\mu}_n)$. Therefore, since in this case R_n and $\hat{H}_n(\hat{\mu}_n)$ are independent under $\boldsymbol{J} : F_1 = F_2$, the resulting test statistic

$$T_n^{b\kappa}(\hat{\mu}_n) = \sum_{\ell=1}^{n} c_{n\ell} b_{nR_{n\ell}}(\hat{H}_n(\hat{\mu}_n)) \tag{4.2}$$

is conditionally distribution-free, given $V_n = v$.

This again implies that the critical value can conditionally be determined independently of the special point $(F, F) \in \boldsymbol{J}$. More precisely, if $V_n = v$ implies $\hat{H}_n(\hat{\mu}_n) = w/n$, then $T_n^{b\kappa}(\hat{\mu}_n)$ behaves conditionally, given $\hat{H}_n(\hat{\mu}_n) = w/n$, as a linear rank statistic with the scores $b_{nj} = b_{nj}(\frac{w}{n})$, $1 \leq j \leq n$.

If $\hat{\mu}_n$ is an equivariant estimator for μ the substitution of μ by $\hat{\mu}_n$ can be justified by invariance against the translation group $\mathcal{G}_t$ of all transformations $(x_1, \ldots, x_n) \mapsto$

$(x_1+u,\ldots,x_n+u), u \in \mathbb{R}$, which in the case "$\mu_1 = \mu_2$ unknown" leaves the testing problem invariant. Since $(x_1 - \hat{\mu}_n(x), \ldots, x_n - \hat{\mu}_n(x))$ is a maximal invariant statistic, the justification follows from

$$R_n(x - \hat{\mu}_n(x)\mathbb{1}_n) = R_n(x), \quad \hat{H}_n(0; x - \hat{\mu}_n(x)\mathbb{1}_n) = \hat{H}_n(\hat{\mu}_n(x); x). \tag{4.3}$$

Using this method it is near at hand to ask whether the results about local optimality, asymptotic normality and asymptotic unbiasedness, formulated in Section 3 for the case "$\mu_1 = \mu_2$ known", can be extended to this more general case. This is indeed possible, if $\hat{\mu}_n$ is $\sqrt{n}$-consistent and depends only on $\hat{H}_n$, i.e. only on the order statistic. All these results are based on the asymptotic equivalence of $T_n^{b\kappa}(\hat{\mu}_n)$ and $T_n^{b\kappa}(\mu)$ under fixed distributions $(F,F) \in \boldsymbol{J}$, which, of course, extends to contiguous alternatives.

Theorem 4.1. Let $\kappa \in (0,1)$, $b \in \mathbb{L}_2$ be κ-isotonic and $T_n^{b\kappa}(\mu)$ for fixed $\mu \in \mathbb{R}$ a linear empirical rank statistic (3.7) with $\mathbb{L}_2$-stable scores. If $\hat{\mu}_n$ is a $\sqrt{n}$-consistent estimator for μ depending only on $\hat{H}_n$, $\varphi_n^{b\kappa}$ a test with test statistic $T_n^{b\kappa}(\hat{\mu}_n)$ and F_n^a a d.f. defined by (3.5) with some function $a \in \mathbb{L}_2^0$ instead of $b \in \mathbb{L}_2^0$, then for all $F \in \mathcal{F}_c$ the following holds true:

a) $T_n^{b\kappa}(\hat{\mu}_n) - T_n^{b\kappa}(\mu) \to 0 \quad$ in (F,F)-probability;

b) $\mathcal{L}_{FF}(T_n^{b\kappa}(\hat{\mu}_n)) \xrightarrow[\mathcal{L}]{} \mathcal{N}(0, \sigma^2_{b_\nu\kappa})$;

c) $\mathcal{L}_{F_n^a F_n^a}(T_n^{b\kappa}(\hat{\mu}_n)) \xrightarrow[\mathcal{L}]{} \mathcal{N}(\eta \int a b_\nu \mathrm{d}\boldsymbol{\lambda}, \sigma^2_{b_\nu\kappa})$;

d) $\varphi_n^{b\kappa}$ is asymptotically unbiased and level-preserving;

e) $\varphi_n^{b\kappa}$ is a locally asymptotically optimal test.

Besides the limit of the (unconditional) distribution of the test statistic $T_n^{b\kappa}(\hat{\mu}_n)$ we can also find the limit of the conditional distribution, given $\hat{H}_n(\hat{\mu}_n) = w_n/n$, for almost all sequences (w_n/n). More precisely, these limit distributions are the same and the tests turn out to be asymptotically equivalent.

Theorem 4.2. Under the assumptions of Theorem 4.1 it holds for all $F \in \mathcal{F}_c$ and (F,F)-almost all sequences (w_n/n) that

a) $\mathcal{L}_{FF}\left(T_n^{b\kappa}(\hat{\mu}_n) | \hat{H}_n(\hat{\mu}_n) = \frac{w_n}{n}\right) \xrightarrow[\mathcal{L}]{} \mathcal{N}(0, \sigma^2_{b_\nu\kappa})$.

b) The tests with the test statistic $T_n^{b\kappa}(\hat{\mu}_n)$ and the conditional tests of the same level with the same test statistic, given $\hat{H}_n(\hat{\mu}_n) = w_n/n$, are asymptotically equivalent under all $(F,F) \in \boldsymbol{J}$. If b is strictly κ-isotonic, then both tests are asymptotically equivalent under all $(F_1,F_2) \in \boldsymbol{H} \cup \boldsymbol{K}$.

Consistency can also be proved. In analogy to Theorem 3.8 this is based on the validity of a law of large numbers for standardized linear rank statistics.

Theorem 4.3. Let $\kappa \in (0,1)$, $b \in \mathbb{L}_1^0$ be κ-isotonic and $T_n^{b\kappa}(\mu)$ for fixed $\mu \in \mathbb{R}$ a linear empirical rank statistic (3.7) with $\mathbb{L}_1$-stable scores. Let $\hat{\mu}_n$ be a $\sqrt{n}$-consistent estimator for μ depending only on $\hat{H}_n$, $\varphi_n^{b\kappa}$ a test with test statistic $T_n^{b\kappa}(\hat{\mu}_n)$, $H_\lambda := \lambda F_1 + (1-\lambda)F_2$ and $\gamma_\lambda^{b\kappa}(F_1, F_2; \mu) = \int b_\nu(H_\lambda)(\mathrm{d}F_1 - \mathrm{d}F_2)$. Then for $n \to \infty$, $n_1/n \to \lambda \in (0,1)$, it holds for all $(F_1, F_2) \in \boldsymbol{H} \cup \boldsymbol{K}$ that

a) $T_n^{b\kappa}(\hat{\mu}_n) \to \gamma_\lambda^{b\kappa}(F_1, F_2; \mu) \quad (F_1, F_2) - \text{a. e.};$

b) $E_{F_1F_2}\varphi_n^{b\kappa} \to \begin{cases} 1, & \text{if } \gamma_\lambda^{b\kappa}(F_1, F_2; \mu) > 0, \\ 0, & \text{if } \gamma_\lambda^{b\kappa}(F_1, F_2; \mu) < 0. \end{cases}$

In particular, if b is $\boldsymbol{\lambda}$-a.e. strictly κ -isotonic, then $\varphi_n^{b\kappa}$ is consistent for the hypotheses (4.1).

We now concentrate on the special case that in (4.1) $\gamma = \gamma_\varrho$ is the ϱ-quantile functional[2]. Then it is near at hand to use the estimator $\hat{\mu}_n = \gamma_\varrho(\hat{H}_n) = \hat{H}_n^{-1}(\varrho)$, such that $\hat{H}_n(\hat{\mu}_n) = \hat{H}_n(\hat{H}_n^{-1}(\varrho))$ is at least asymptotically equal to ϱ. This implies

Theorem 4.4. Let $\varrho \in (0,1)$ be fixed and $\gamma = \gamma_\varrho$ the ϱ-quantile functional for fixing the hypotheses (4.1) and estimating the unknown dispersion center. Then for sufficiently large n it holds true that

a) $\hat{H}_n(\hat{\mu}_n) = \hat{H}_n(\hat{H}_n^{-1}(\varrho)) \approx \varrho \quad$ (*which is not random*);

b) The empirical scores $b_{nj}(\hat{H}_n(\hat{\mu}_n))$ can be approximated by the usual scores b_{nj}, defined by the ϱ-isotonic function b;

c) The linear empirical rank statistic (4.2) with estimator $\hat{\mu}_n = \hat{H}_n^{-1}(\varrho)$ can be approximated by the usual linear rank statistic for the ϱ-isotonic score-function b.

Theorem 4.4c gives an asymptotic justification of applying linear rank tests in the case "$\mu_1 = \mu_2$ unknown", as it was mentioned in Section 1 as a standard technique. It can be extended to a justification for finite sample sizes by invariance considerations. This is based on the fact that the functional γ_ϱ is equivariant for the group $\mathcal{G}$ of all transformations on the sample space of the form $(x_1, \dots, x_n) \mapsto (\tau x_1, \dots, \tau x_n)$, where $\tau : \mathbb{R} \to \mathbb{R}$ is surjective and strictly isotonic. This implies that the testing problem (4.1) is invariant against $\mathcal{G}$. For this group the rank vector $R_n(x)$ is maximal invariant, cf. Lehmann [17], Ch. 6.7, and linear rank tests with exact scores are locally optimal invariant tests.

Theorem 4.5. Let $\varrho \in (0,1)$ be fixed and the hypotheses (4.1) be defined by the quantile functional γ_ϱ. Then linear rank tests with exact scores for ϱ-isotonic score-functions b are locally optimal invariant tests.

We mention that for the hypotheses (4.1) unbiasedness (and level preservation) cannot be proved by means of Theorem 2.1, since the ordering $\succeq_\gamma^{\mathrm{I}}$ cannot be transfered to the sample space. We also point out that invariance properties strongly

[2] In contrast to Section 3 now ϱ (rather than μ) is fixed.

depend on the special choice of the functional γ. Furthermore note that the use of linear rank tests for dispersion problems in the case "$\mu_1 = \mu_2$ unknown" is motivated in a slightly different way in Behnen-Neuhaus [2]. Our approach goes back to Schäfer [23] and was extended and supplemented by Burger [4].

5. EMPIRICAL RANK TESTS FOR THE CASE "$\mu_1 \neq \mu_2$ UNKNOWN"

According to (4.1) the hypotheses are formulated as

$$\boldsymbol{H} : F_1 \preceq_\gamma^{\mathrm{II}} F_2, \quad \boldsymbol{K} : F_1 \succeq_\gamma^{\mathrm{II}} F_2 \quad \text{with } F_1 \neq F_2. \tag{5.1}$$

Here again γ is a location functional which fixes the testing problem and the dispersion centers μ_1 and μ_2. In analogy to the question whether linear rank tests can be extended to the case "$\mu_1 \neq \mu_2$ unknown" discussed in Section 1 it suggests itself first to estimate the dispersion centers μ_i by some estimators $\hat{\mu}_{in}$ and then to use the ranks $\hat{R}_{n1}, \ldots, \hat{R}_{nn}$ of the r.v. $Y_{11} = X_{11} - \hat{\mu}_{1n}, \ldots, Y_{2n_2} = X_{2n_2} - \hat{\mu}_{2n}$ resp. the corresponding empirical rank statistic (1.11), i.e.

$$\tilde{T}_n = \sum_{\ell=1}^{n} c_{n\ell} b_{n\hat{R}_{n\ell}}(\hat{H}_n(0; Y_{(n)})). \tag{5.2}$$

Here $\hat{H}_n(\cdot; Y_{(n)})$ denotes the empirical d.f. of the r.v. $Y_{11}, \ldots, Y_{2n_2}$. But contrary to the test statistic (4.2) this statistic is not conditionally distribution-free, given $\hat{H}_n(0; Y_{(n)})$. Therefore, we have to ask whether $\tilde{T}_n$ under appropriate conditions is at least asymptotically distribution-free. Following Hájek [11] this shall be done by linearization. For this the Jurečková-linearization has to be generalized to the case of linear empirical rank statistics.

Lemma 5.1. (Burger [4]) Let $T_n^{b\kappa}$ be a linear empirical rank statistic and $T_n^{\Delta b\kappa}$ the same statistic, evaluated for the r.v. $X_{ij} - \Delta_i, j = 1, \ldots, n_i, i = 1, 2$. Let b_ν be the adapted score-function (3.10) and b_F be defined as in (1.8). Then, for $n \to \infty$, it holds true that

$$T_n^{\Delta b\kappa} - T_n^{b\kappa} - \Delta \int b_\nu b_F \mathrm{d}\boldsymbol{\lambda} \to 0 \quad \text{in } (F, F)\text{-probability}. \tag{5.3}$$

According to this lemma, the statistics $\tilde{T}_n$ and

$$\hat{T}_n = \sum_{\ell=1}^{n} c_{n\ell} b_{nR_{n\ell}}(\hat{H}_n(0)) \tag{5.4}$$

have the same limit distribution in a point $(F, F) \in \boldsymbol{J}$, if

$$\int b_\nu b_F \mathrm{d}\boldsymbol{\lambda} = 0. \tag{5.5}$$

According to the discussion of (1.8) this condition is fulfilled, if b_ν is symmetric and b_F skew-symmetric w.r.t. 1/2. This again is typically the case if $\nu = 1/2$ and F has a Lebesgue-density f which is symmetric w.r.t. 0 and for which we have $\int |b_\nu b_F| \mathrm{d}\boldsymbol{\lambda} < \infty$.

But also for $\nu \neq 1/2$ there are situations of practical interest in which (5.5) is met. For this we have to generalize the concepts of symmetry for score-functions and d.f.[3] Let $\kappa \in (0,1)$ be fixed. Then a score-function $b : (0,1) \to \mathbb{R}$ is called *quasi-symmetric* w.r.t. κ iff

$$b(\kappa - \kappa u) = b(\kappa + (1-\kappa)u) \quad \forall\, u \in (0,1). \tag{5.6}$$

This is equivalent to the existence of a function $b_0 : (0,1) \to \mathbb{R}$ with

$$b(u) = b_0\left(\frac{\kappa - u}{\kappa}\right) \quad \text{for} \quad u \leq \kappa, \quad b(u) = b_0\left(\frac{u-\kappa}{1-\kappa}\right) \quad \text{for} \quad u > \kappa; \tag{5.7}$$

cf. Example 3.4. A d.f. $F : \mathbb{R} \to \mathbb{R}$ is called *κ-symmetric* w.r.t. $\mu \in \mathbb{R}$ iff

$$\frac{F(\mu) - F(\mu - x)}{\kappa} = \frac{F(\mu + x) - F(\mu)}{1-\kappa} \quad \forall\, x > 0. \tag{5.8}$$

If F has a Lebesgue-density f, this is equivalent to

$$\frac{1}{\kappa} f(\mu - x) = \frac{1}{1-\kappa} f(\mu + x) \quad \boldsymbol{\lambda}\text{-a.e.}, \tag{5.9}$$

which again implies that $b_F \circ F$ is skew-symmetric w.r.t. μ. If in addition b is quasi-symmetric w.r.t. κ, then it holds

$$\int \tilde{b} b_F \mathrm{d}\boldsymbol{\lambda} = 0. \tag{5.10}$$

Here $\tilde{b}$ is a slight modification of b, namely

$$\tilde{b} = \sqrt{\frac{1-\nu}{\nu}}\, b \mathbb{1}_{(0,\kappa]} + \sqrt{\frac{\nu}{1-\nu}}\, b \mathbb{1}_{(\kappa,1)}. \tag{5.11}$$

Obviously among others we get $\tilde{b} \in \mathbb{L}_2$, $(\tilde{b}_\nu) = (\tilde{b})_\nu$ and $\tilde{b} = b$ for $\nu = 1/2$.

Under condition (5.5) resp. (5.10) the statistics (5.4) and (5.2) can be compared. This is also possible with regard to the validity of other asymptotic properties as local optimality or unbiasedness. For the formulation of these assertions it is useful to extend – according to (2.7) – the results of Section 3 to the case "$\mu_1 \neq \mu_2$ known". Denoting the statistic thus maintained by $T_n^{b\kappa}(\mu_1, \mu_2)$, (5.2) is of the form $T_n^{\tilde{b}\kappa}(\hat{\mu}_{1n}, \hat{\mu}_{2n})$. Before using this as a test statistic, ν has to be known (as in the case that γ is a quantile functional) or has to be estimated by some consistent estimator $\hat{\nu}_n$.

[3] Examples of dispersion problems in practice, in which the dispersion center of a one-dimensional distribution is not its median, are given in Deshpandé-Kusum [8].

Theorem 5.2. (Burger [4]) Let $\kappa \in (0,1), b : (0,1) \to \mathbb{R}$ be κ-isotonic and F_1, F_2 is such that (5.5) is fulfilled, $\tilde{b}$ defined by (5.1) and $T_n^{\tilde{b}\kappa}(\mu_1,\mu_2)$ for fixed $(\mu_1,\mu_2) \in \mathbb{R}^2$ a linear empirical rank statistic with $\mathbb{L}_2$-stable scores for testing the hypotheses

$$\boldsymbol{H} : F_1 \preceq_{\mu_1\mu_2} F_2, \quad \boldsymbol{K} : F_1 \succeq_{\mu_1,\mu_2} F_2 \quad \text{with } F_1 \neq F_2,$$

where $\preceq_{\mu_1\mu_2}$ is defined by (2.14). If $(\hat{\mu}_{1n},\hat{\mu}_{2n}) = (\gamma(\hat{F}_{1n}),\gamma(\hat{F}_{2n}))$ is a $\sqrt{n}$-consistent estimator for (μ_1,μ_2) and φ_n a test with $T_n^{\tilde{b}\kappa}(\hat{\mu}_{1n},\hat{\mu}_{2n})$ as test statistic, the following holds true

a) $T_n^{\tilde{b}\kappa}(\hat{\mu}_{1n},\hat{\mu}_{2n}) - T_n^{\tilde{b}\kappa}(\mu_1,\mu_2) \to 0$ in (F_1,F_2)-probability;

b) $\mathcal{L}_{FF}(T_n^{\tilde{b}\kappa}(\hat{\mu}_{1n},\hat{\mu}_{2n})) \xrightarrow[\mathcal{L}]{} \mathcal{N}(0,\sigma^2_{\tilde{b}_\nu\kappa})$;

c) φ_n is asymptotically unbiased and level-preserving;

d) φ_n is a locally optimal test.

In the special case that γ is the ϱ-quantile-functional and $\tilde{b}$ fulfils (5.11), then a) – d) are true for all κ-symmetric F_1 (w.r.t. μ_1) and F_2 (w.r.t. μ_2). Beyond that φ_n is a linear rank test with a ϱ-isotonic score-function.

Finally, we discuss a completely different method for handling hypotheses (5.1). Whereas the asymptotic justification of tests considered up to now was based on the $\mathbb{L}_r$-convergence of the step-functions b_n against some function b, now an LAN-approximation is used; cf. [25], Ch. 6.3.2. More precisely, by means of an appropriate localization one comes to a limit problem governed by a multidimensional normal distribution, which makes possible an exact solution.

The starting point is the fact that after fixing the one-parameter class of distributions $\mathcal{F}(F;b,\kappa)$ we have a three-parametric testing problem with the one-dimensional main parameter Δ and the two-dimensional nuisance parameter (μ_1,μ_2). Assuming some regularity conditions it is possible to simplify the testing problem by localization according to

$$\Delta = \eta/\sqrt{n}, \quad \mu_1 = \xi_1/\sqrt{n}, \quad \mu_2 = \xi_2/\sqrt{n}.$$

Assuming that F is $\boldsymbol{\lambda}$-continuous with an absolute continuous density f and finite Fisher-information $I(f) = \int b_F^2 \mathrm{d}\boldsymbol{\lambda}, b_F = -\frac{f'}{f} \circ F^{-1}$, and using the abbreviations

$$\mathsf{J}_{11} = I(f) \in (0,\infty), \quad \mathsf{J}_{12} = \mathsf{J}_{21} = \int b b_F \mathrm{d}\boldsymbol{\lambda} \left(\sqrt{1-\lambda}, -\sqrt{\lambda}\right) \in \mathbb{R}^{1\times 2},$$

$$\mathsf{J}_{22} = I(f)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in \mathbb{R}^{2\times 2},$$

the limit problem of the LAN-approximation is

$$\mathcal{L}_{\eta\xi}\begin{pmatrix} U \\ V \end{pmatrix} = \mathcal{N}\left(\begin{pmatrix} \mathsf{J}_{11}\eta + \mathsf{J}_{12}\xi \\ \mathsf{J}_{21}\eta + \mathsf{J}_{22}\xi \end{pmatrix}, \begin{pmatrix} \mathsf{J}_{11} & \mathsf{J}_{12} \\ \mathsf{J}_{21} & \mathsf{J}_{22} \end{pmatrix}\right), \tag{5.12}$$

$$\hat{\boldsymbol{H}} : \eta \leq 0, \quad \xi \in \mathbb{R}^2, \quad \hat{\boldsymbol{K}} : \eta > 0, \quad \xi \in \mathbb{R}^2. \tag{5.13}$$

Here the nuisance parameter ξ can be eliminated by conditioning, given $V = v$; cf. Lehmann [17] and [24], Ch. 3.3. Since $S = U - \mathrm{J}_{12}\mathrm{J}_{22}^{-1}V$ is stochastically independent of V, it is possible to transform the conditional test into an unconditional one. This corresponds to transforming the given score-functions b into

$$b^1 = b - \frac{\int bb_F \mathrm{d}\boldsymbol{\lambda}}{I(f)} b_F, \tag{5.14}$$

for which the orthogonality conditions (5.10) is fulfilled. Of course, b^1 can be standardized according to $b^0 = b^1/||b^1||_{\mathbb{L}_2}$. This technique for handling parametric testing problems with nuisance parameters goes back to Neyman [20]. The corresponding tests are known in asymptotic statistic as $C(\alpha)$-tests; cf. [25], Ch. 6.4.3.

As is seen from (5.14) the resulting test depends on the unknown $F \in \mathcal{F}_c$. Besides the factors $\int bb_F \mathrm{d}\boldsymbol{\lambda}$ and $I(f)$, the score-function b_F has to be estimated consistently, as it was done for the first time in Hájek–Šidák [12], Ch. VII.1.5.

Using b^0 as the score-function we end up with a linear rank statistic or a linear empirical rank statistic, depending on whether the location functional γ, which was used for making the hypotheses (5.1) precise, is a quantile functional or not. For details we refer the reader to Burger [4].

6. TWO-SIDED TESTS AND CONCLUDING REMARKS

Most of the results can be extended to the two-sided case just as in the location problem; cf. [19]. Since in the nonparametric approach the alternative comprises those pairs (F_1, F_2) for which F_1 is either less or more dispersed than F_2, the boundary of the hypotheses is typically the same as in the one-sided case, i.e. $\boldsymbol{J} : F_1 = F_2$. If the one-parameter subclass $\mathcal{P}_1 = \{P_\Delta : \Delta \in \mathbb{R}\}$ is sufficiently smooth at $\Delta = 0$, i.e. if the power function of any test is twice differentiable at $\Delta = 0$ and if the differentiation can be done under the expectation sign, the locally optimal test for $\boldsymbol{H} : \Delta = 0$, $\boldsymbol{K} : \Delta \neq 0$ can be defined as a test maximizing the curvature of the power function among all locally unbiased level α-tests. If, in addition, $\mathcal{P}_1$ is twice $\mathbb{L}_1(0)$ -differentiable with derivatives $\dot{L}_0$ and $\ddot{L}_0$, the test can be made explicit by the generalized fundamental lemma as a solution of

$$\begin{aligned} \nabla\nabla E_\Delta \varphi|_{\Delta=0} &= E_0(\varphi \ddot{L}_0) = \sup\nolimits_\varphi, \\ \nabla E_\Delta \varphi|_{\Delta=0} &= E_0(\varphi \dot{L}_0) = 0, \\ E_0 \varphi &= \alpha. \end{aligned}$$

In the same way this can be done for two-sided locally optimal rank tests. In all these cases it turns out that the optimal test statistic is asymptotically quadratic, i.e. it is the square of the optimal one-sided test statistic plus a term which is typically asymptotically negligible. Among others this implies that the two-sided test asymptotically depends only on the first derivative $\dot{L}_0$ resp. $E_{F_0F_0}(\dot{L}_0|M_n)$, which is typical for asymptotic statistics.

As is shown in Burger [4], an analoguous theory can be developed for permutation tests. Finally, it should be mentioned that several results of Sections 3 – 5 hold also for other orderings, for norms distinct from the total variation norm and for other nonparametric testing problems, e. g. for location problems with censored data or correlation problems.

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