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ON GENERIC PROPERTIES OF LINEAR SYSTEMS: AN OVERVIEW

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The topological space of linear, time-invariant systems is considered. A property of linear systems is called generic if the systems equipped with this property occupy an open and dense subspace of the space of systems. Several generic properties of linear systems are reviewed, including controllability, observability, invertibility, structural stability, as well as some topological properties of orbits of the feedback group. A new estimate for the number of orbits of the feedback group is produced, and a differential geometric proof of the existence of generic controllability indices is given.

1. INTRODUCTION

Consider a linear system

$$\dot{x} = Ax + Bu\,, \quad y = Cx\,,$$

with m-inputs, n-states, and p-outputs, $m \leqq p \leqq n$. Clearly, each such a system, denoted by σ, can be viewed as a point of the Euclidean space $\mathbb{R}^{mn+n^2+np}$ via the identification $\sigma = (A, B, C)$. Thus the set of linear systems becomes a topological space $\Sigma = \mathbb{R}^{mn+n^2+np}$.

If $\mathscr{P}$ denote a property of linear systems, we define the extension of $\mathscr{P}$ as $\Sigma_{\mathscr{P}} = \{\sigma \in \Sigma : \sigma \text{ has the property } \mathscr{P}\}$. We want to describe $\Sigma_{\mathscr{P}}$, as a subspace of Σ, in topological terms. Throughout this paper we will be interested in those properties $\mathscr{P}$ for which $\Sigma_{\mathscr{P}}$ is an open and dense subset of Σ. These properties of linear systems will be called generic.

Let us observe that generic properties combine two important elements. The first is that generic means open, i.e. some appropriately small perturbations of the matrices of the system having a generic property do not destroy this property. The second element is that generic implies dense, hence any system which does not have a generic property can be changed into a system having this property just by arbitrarily small perturbations. One may say that generic properties are jointly robust and universal.

The concept of generic properties comes from the global analysis of dynamical systems and catastrophe theory [9], [12]. The relevance of generic properties to the theory of linear systems has been recognized by Kalman [7] and Wonham [16]. The fact that structural stability of linear systems is generic has been proved by Willems [14]. Some general aspects of genericity have been considered by Olbrot [8] and Tchoń [11].

2. CONTROLLABILITY, OBSERVABILITY AND INVERTIBILITY

The properties of controllability, observability and invertibility of linear systems can be seen as the properties which allow one to calculate some aspects of systems behaviour given some other ones. In particular controllability means a possibility of reconstruction of inputs given states, observability implies that it is possible to reconstruct states of the system given its outputs, while invertibility, called sometimes functional reproducibility, means that input functions can be reconstructed when output functions are known [2], [10].

Let us remind the well-known criteria for controllability, observability and invertibility of linear systems [2], [10], [16]. A system $\sigma = (\mathbf{A}, \mathbf{B}, \mathbf{C})$ is controllable if and only if rank $[\mathbf{B}\ \mathbf{AB}\ \dots\ \mathbf{A}^{n-1}\mathbf{B}] = n$, observable if and only if rank $[\mathbf{C}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}\mathbf{C}^{\mathrm{T}} \dots$ $\dots (\mathbf{A}^{n-1})^{\mathrm{T}}\, \mathbf{C}^{\mathrm{T}}] = n$, invertible if and only if

$$\operatorname{rank}\begin{bmatrix} \mathbf{CB} & \mathbf{0} & \mathbf{0\ 0\ 0\ 0} & \\ \mathbf{CAB} & \mathbf{CB} & \mathbf{0\ 0\ 0\ 0} & \\ \dots & \dots & \dots & \dots \\ \mathbf{CA}^{n-1}\mathbf{B} & \dots & \mathbf{CB} & \\ \dots & \dots & \dots & \dots \\ \mathbf{CA}^{2n-1}\mathbf{B} & \dots & \mathbf{CA}^{n-1}\mathbf{B} & \end{bmatrix}_{2np \times nm} = nm\,.$$

It can be easily observed that in all the cases the criterion states that a matrix whose entries depend polynomially on the entries of $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ should have its maximal rank. Let $\mathscr{P}$ denote any of these properties. Then those systems for which $\mathscr{P}$ does not hold (i.e. certain determinant functions vanish) are given by a set of polynomial equations in the elements of $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$. Such a set, defined by vanishing of a finite number of polynomials, is called algebraic. The basic property of an algebraic subset S of Σ is that if $S \neq \Sigma$ then the complement $\Sigma - S$ is open and dense in Σ [6]. So, to prove that $\mathscr{P}$ is generic it is enough to show that there exist some systems which have the property $\mathscr{P}$. Since controllable, observable and invertible linear systems clearly exist, we have proved the following.

Proposition 2.1. Controllability, observability and invertibility are generic properties of linear systems.

The Proposition states that the sets consisting of controllable, observable and invertible systems are open and dense in Σ. Clearly, their intersection is also open and dense yielding that the property "to be jointly controllable, observable and invertible" is generic. This last property is evidently "smaller" than each one stated in Proposition 2.1 since, in general, the properties of controllability, observability and invertibility do not imply each other. Also observe that Proposition 2.1 yields the conclusion that there exists a system being at the same time controllable, observable and invertible (moreover, there is an open and dense set of such systems). The last conclusion illustrates and application of the so-called category argument to prove the existence of some objects.

3. FEEDBACK GROUP

Assume from now on that the system $\sigma = (A, B, C)$ has $p = n$ and $C = I_n$ (the n by n identity matrix). Then $\sigma = (A, B)$ and $\Sigma = \mathbb{R}^{mn+n^2}$. Consider another system $\sigma' = (A', B') \in \Sigma$. These systems can be described as follows:

$$\sigma : \dot{x} = Ax + Bu\,, \quad \sigma' : \dot{y} = A'y + B'v\,, \tag{3.1}$$

where (x, u), (y, v) denote some coordinate systems in $\mathbb{R}^n \times \mathbb{R}^m$. We want to find a diffeomorphism $\varphi : (x, u) \to (y, v)$ which would establish an equivalence between σ and σ'. Assume that we are looking for a diffeomorphism φ of the form $\varphi = (\varphi_1, \varphi_2)$, where $\varphi_1(x) = y$ and $\varphi_2(x, u) = v$. Substituting y and v into the equation for σ' gives us the following expression:

$$A'\,\varphi_1(x) + B'\,\varphi_2(x, u) = \varphi_{1*}Ax + \varphi_{1*}Bu\,. \tag{3.2}$$

Here φ_{1*} denotes the Jacobi matrix of φ_1, $\varphi_{1*} = \left[\frac{\partial \varphi_{1i}}{\partial x_j}\right]$.

It is natural to assume that φ_1 is linear, $\varphi_1 = Px$ with P non-singular n by n, and that $\varphi_2(x, u) = Kx + Qu$ with Q non-singular m by m and K an arbitrary m by n matrix. Thus our equivalence requires

$$A'Px + B'Kx + B'Qu = PAx + PBu$$

or, equivalently, that under the equivalence defined by $\varphi = (\varphi_1, \varphi_2)$ the system $\sigma' = (A', B')$ transforms according to the rule

$$A' \to P^{-1}A'P + P^{-1}B'K = A\,, \quad B' \to P^{-1}B'Q = B\,. \tag{3.3}$$

If (3.3) holds, the systems σ, σ' will be called feedback equivalent.

It can be easily proved that the matrices $g = \begin{bmatrix} P & 0 \\ K & Q \end{bmatrix}$ which define the transformation (3.3) form a subgroup $F(m, n)$ of the general linear group $GL(m + n, \mathbb{R})$, acting on the space of systems Σ according to (3.3). The group $F(m, n)$ is called the feedback group [1], [6], [16]. It can be checked easily that the multiplication in $F(m, n)$ is the ordinary matrix multiplication, that

$$\begin{bmatrix} P & 0 \\ K & Q \end{bmatrix}^{-1} = \begin{bmatrix} P^{-1} & 0 \\ -Q^{-1}KP^{-1} & Q^{-1} \end{bmatrix},$$

and that the identity element in $F(m, n)$ is equal to $\begin{bmatrix} I_n & 0 \\ 0 & I_m \end{bmatrix}$.

It is well known that the action $F(m, n)$ does not influence controllability, i.e. controllable systems are invariant with respect to this action. Therefore the following set is invariant:

$$C(m, n) = \{\sigma = (A, B) \in \Sigma : \text{rank}\, B = m \text{ and } \sigma \text{ controllable}\} . \tag{3.4}$$

Clearly, the complement to $C(m, n)$ in Σ is algebraic, hence $C(m, n)$ is open and dense in Σ since there exist controllable systems with the matrix B of the maximal rank. Denote the action of $g \in F(m, n)$ on σ as $g\sigma$. Thus the orbit of σ under feedback can be defined as

$$F(m, n)\, \sigma = \{g\sigma : g \in F(m, n)\} . \tag{3.5}$$

Let $\Omega(m, n)$ denote the set of all orbits of $F(m, n)$ in $C(m, n)$. Then the following result about the classification of the orbits $\omega \in \Omega(m, n)$ is due to Brunovsky [3].

Theorem 3.1. There exists a bijection between orbits $\omega \in \Omega(m, n)$ and lists of integers $\varkappa(\omega) = (\varkappa_1, \varkappa_2, \ldots, \varkappa_m)$ with $\varkappa_1 \geqq \varkappa_2 \geqq \ldots \geqq \varkappa_m$ and $\sum_{i=1}^{m} \varkappa_i = n$. If a system σ belongs to the orbit described by $\varkappa = (\varkappa_1, \ldots, \varkappa_m)$ then σ can be reduced under $F(m, n)$ to the Brunovsky canonical form $\sigma(\varkappa) = (A', B')$, where

$$A' = \text{diag}\,\{J_1, \ldots, J_m\} , \quad B' = \text{diag}\,\{e_1, \ldots, e_m\}$$

and

$$J_i = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & \ldots & 0 & 1 \\ 0 & \ldots & \ldots & 0 & 0 \end{bmatrix}_{\varkappa_i \times \varkappa_i} , \quad e_i = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix}_{\varkappa_i \times 1} , \quad i = 1, 2, \ldots, m .$$

The integers $\varkappa_1, \ldots, \varkappa_m$ are called controllability indices and can be calculated according to the following formulas. Let $S_0 = \text{Im}\, B$, $S_j = \text{Im}\, B + AS_{j-1}$, $j = 1, 2, \ldots, n - 1$, and let $\varrho_j = \dim S_j - \dim S_{j-1}$. Then $\varkappa_i$ = number of $\varrho_0, \varrho_1, \ldots, \varrho_{n-1}$ which are greater than or equal to i, $i = 1, 2, \ldots, m$.

Theorem 3.1 allows one to calculate the number of orbits in $\Omega(m, n)$ by calculating the number of possible sequences of controllability indices. This, however, appears to be rather complicated. To obtain an estimate of this number we can use the concept of partition of a natural number r [4]. By the partition of r we mean any representation of r as a sum of natural numbers (i.e. partitions of 4 are $1 + 1 + 1 + 1$ $1 + 1 + 2$, $2 + 2$, $1 + 3$, 4). There is no explicit formula for the number of partitions $p(r)$ of a natural number r, however, there are known some recurrent and asymptotic expressions for $p(r)$. The most useful for our purposes seems to be the following:

$$p(r) = \frac{1}{2\pi\sqrt{2}}\frac{\mathrm{d}}{\mathrm{d}r}\frac{\exp\left(\pi\sqrt{(2(r-1/24)/3)}\right)}{\sqrt{(r-1/24)}} + O(\exp k\sqrt{r}),$$

where $|O(x)/x| \leqq M$, M fixed for x big enough, and $k < \pi/\sqrt{6}$ [4]. It can be derived from this general formula that $p(r)$ for big r obeys the asymptotic formula: $p(r) \sim \sim (1/4r\sqrt{3}) \exp(\pi\sqrt{(2r/3)})$. There exists a table of values of $p(r)$ for $r \leqq 200$ [5]. Below we quote the first 10 elements of this table.

r	1	2	3	4	5	6	7	8	8	10	...	200
$p(r)$	1	2	3	5	7	11	15	22	30	42	...	3972999029388.

We will use $p(r)$ to estimate the number of orbits in $\boldsymbol{\Omega}$ as in the following.

Proposition 3.2. Let $n = km + r$, $0 \leqq r < m$. Then the number $N(\boldsymbol{\Omega})$ of orbits in $\boldsymbol{\Omega}(m, n)$ satisfies the following inequalities: $p(r) \leqq N(\boldsymbol{\Omega}) \leqq p(n - m)$, where $p(r)$ denotes the number of partitions of r. In particular, if $k = 1$ then $N(\boldsymbol{\Omega}) = p(r) = = p(n - m)$.

Proof. Let s be an arbitrary number. Fix a natural $q \geqq s$. Then we can represent each partition of s as a sequence of non-increasing numbers of length q. For instance, if $s = 4$, $q = 5$ we get $(1, 1, 1, 1, 0)$, $(2, 2, 0, 0, 0)$, $(3, 1, 0, 0, 0)$, $(2, 1, 1, 0, 0)$, $(4, 0, 0, 0)$. Now the lower bound $p(r)$ can be produced as follows. Set $k_i = m + i$th element in a partition of r of length m, $i = 1, 2, \ldots, m$. Clearly, the number of sequences $(k_1, k_2, \ldots, k_m)$ obtained in this way equals $p(r)$. On the other hand, if $k > 1$ then for each $(k_1, \ldots, k_m)$ there exists a sequence of controllability indices of the form $\varkappa_1 = k_1 + k_m - 1$, $\varkappa_2 = k_2, \ldots$, $\varkappa_{m-1} = k_{m-1}$, $\varkappa_m = 1$. But this sequence is not among previously produced, so $p(r) \leqq N(\boldsymbol{\Omega})$. As concerns the upper bound, consider $K_i = 1 + i$th element in a partition of $n - m$ of length n, $i = 1, 2, \ldots, m$. It is easy to see that the number of different $(K_1, \ldots, K_m)$ is equal to $p(n - m)$ when $n - m \leqq m$ and is less than $p(n - m)$ otherwise. Therefore $N(\boldsymbol{\Omega}) \leqq p(n - m)$. □

The feedback group $\boldsymbol{F}(m, n)$ is a Lie group and an algebraic group, hence its orbits are regular submanifolds of $\boldsymbol{C}(m, n)$. Pick a sequence of controllability indices $\boldsymbol{\varkappa} = (\varkappa_1, \ldots, \varkappa_m)$ and let $\boldsymbol{\sigma}(\boldsymbol{\varkappa})$ denote the appropriate Brunovsky canonical form. Then we can define the isotropy subgroup of $\boldsymbol{F}(m, n)$, stabilizing $\boldsymbol{\sigma}(\varkappa)$, as

$$\boldsymbol{H}_{\sigma(\varkappa)} = \{\boldsymbol{g} \in \boldsymbol{F}(m, n) : \boldsymbol{g}\, \boldsymbol{\sigma}(\boldsymbol{\varkappa}) = \boldsymbol{\sigma}(\boldsymbol{\varkappa})\}.$$

For each $\boldsymbol{\varkappa}$ there is a diffeomorphism between the orbit $\boldsymbol{F}(m, n)\, \boldsymbol{\sigma}(\boldsymbol{\varkappa})$ and $\boldsymbol{F}(m, n) / / \boldsymbol{H}_{\sigma(\varkappa)}$ given as $\boldsymbol{g}\boldsymbol{H}_{\sigma(\varkappa)} \to \boldsymbol{g}\, \boldsymbol{\sigma}(\boldsymbol{\varkappa})$, [13]. This diffeomorphism enables us to calculate the dimension of the orbit passing through $\boldsymbol{\sigma}(\boldsymbol{\varkappa})$ (as a submanifold) via calculating the dimension of $H_{\sigma(\varkappa)}$. The last problem was solved by Brockett [4] whose result can be formulated as the following.

Proposition 3.3. Let $\varkappa = (\varkappa_1, \ldots, \varkappa_m)$ be such that there is in $\varkappa$ k_i-integers equal to n_i, $i = 1, 2, \ldots, s$, $m \geqq n_1 > n_2 > \ldots > n_s \geqq 1$, s a natural number. Clearly $\sum_{i=1}^{s} k_i n_i = n$, $\sum_{i=1}^{s} k_i = m$. Then

$$\dim \omega(\varkappa) = n^2 + mn - \sum_{i=1}^{s-1} k_i \sum_{j=i+1}^{s} k_j(n_i - n_j - 1).$$

It is easy to see that $\dim \omega(\varkappa) \leqq n^2 + mn$ and that $\dim \omega(\varkappa)$ assumes its maximum if and only if $n_i = n_j + 1$ for each $j > i$. Let $n = km + r$. Then the last conditions can be satisfied only in the case when $s = 2$, $k_1 = r$, $k_2 = m - r$, $n_1 = k + 1$, $n_2 = k$ (if $r = 0$, we get $k_1 = 0$, $k_2 = m$, $n_2 = k$). Therefore, if $\varkappa' = (\underbrace{k+1, \ldots, k+1}_{r \text{ times}}, \underbrace{k, \ldots, k}_{m-r \text{ times}})$ then $\dim \omega(\varkappa') = n^2 + mn = \dim C(m, n) = \dim \Sigma$. But this implies that $\omega(\sigma')$ is open in $C(m, n)$, and in Σ. Moreover, since all the other orbits have dimensions less than $n^2 + mn$, they must have empty interiors in $C(m, n)$. Now the finite number of orbits with empty interiors form the complement of $\omega(\sigma')$ to $C(m, n)$ and, in consequence, $\omega(\varkappa')$ is actually dense in $C(m, n)$. Finally, since $\omega(\varkappa')$ is open-dense in $C(m, n)$ and $C(m, n)$ is open-dense in Σ, the Reduction lemma [11] yields the following.

Proposition 3.4. Let $n = km + r$, $0 \leqq r < m$. Then the property "to have controllability indices of the form $\varkappa' = (\underbrace{k + 1, \ldots, k + 1}_{r\times}, \underbrace{k, \ldots, k}_{m-r\times})$" is a generic property of systems $\sigma = (\mathbf{A}, \mathbf{B}) \in \Sigma$.

We have given a Lie group theoretic proof of the above Proposition. A linear algebraic proof can be found e.g. in [16]. However, the Lie-theoretic approach provides us with some more complete topological characterization of the maximal orbit $\omega(\varkappa')$, i.e. of almost all linear systems $\sigma = (\mathbf{A}, \mathbf{B})$. Namely, from the results obtained by Brockett [1] one can derive the following.

Proposition 3.5. The maximal orbit $\omega(\varkappa')$ of the feedback group is diffeomorphic to the quotient group $F(m, n)/H_{\sigma(\varkappa')}$, where $H_{\sigma(\varkappa')}$ consists of the matrices

$$g = \begin{bmatrix} \mathbf{P}' & \mathbf{0} \\ \mathbf{K}' & \mathbf{Q}' \end{bmatrix}$$

with

$$\mathbf{P}' = \begin{bmatrix} \mathbf{P}_{11} & \mathbf{0} \\ \mathbf{P}_{21} & \mathbf{P}_{22} \end{bmatrix}, \quad \mathbf{Q}' = \begin{bmatrix} \mathbf{Q}_{11} & \mathbf{0} \\ \mathbf{Q}_{21} & \mathbf{Q}_{22} \end{bmatrix}, \quad \mathbf{K}' = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{K}_{21} & \mathbf{0} \end{bmatrix}$$

being respectively of dimensions n by n, m by m, and m by n. Moreover, we have

$$\mathbf{P}_{11} = \mathbf{D}_{rr} \otimes \mathbf{I}_{k+1}, \quad \mathbf{P}_{21} = \mathbf{D}_{mr} \otimes \mathbf{J}_{k,k+1} + \mathbf{E}_{mr} \otimes \mathbf{J}^*_{k,k+1},$$
$$\mathbf{P}_{22} = \mathbf{D}_{mm} \otimes \mathbf{I}_k, \quad \mathbf{Q}_{11} = \mathbf{D}_{rr}, \quad \mathbf{Q}_{21} = \mathbf{E}_{mr}, \quad \mathbf{Q}_{22} = \mathbf{D}_{mm},$$
$$\mathbf{K}_{21} = \mathbf{D}_{mr} \otimes \mathbf{e}^*_{k+1}.$$

Here $\mathbf{D}_{rr}$, $\mathbf{D}_{mm}$ are non-singular matrices of dimensions r by r and $(m - r)$ by $(m - r)$, $\mathbf{D}_{mr}$, $\mathbf{E}_{mr}$ are arbitrary matrices $(m - r)$ by r, $\mathbf{e}^*_{k+1} = [0, \ldots, 0, 1]_{1 \times (k+1)}$,

$$\mathbf{J}_{k,k+1} = \begin{bmatrix} 1 & 0 & \ldots & . & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 1 & 0 \end{bmatrix}_{k \times (k+1)}, \quad \mathbf{J}^*_{k,k+1} = \begin{bmatrix} 0 & 1 & \ldots & . & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 0 & 1 \end{bmatrix}_{k \times (k+1)},$$

and $\mathbf{A} \otimes \mathbf{B}$ denotes the Kronecker product of matrices $\mathbf{A}$, $\mathbf{B}$, i.e.

$$\underset{n \times m}{\mathbf{A}} \otimes \underset{p \times r}{\mathbf{B}} = \begin{bmatrix} a_{11}\mathbf{B} & \ldots & a_{1m}\mathbf{B} \\ \ldots & \ldots & \ldots \\ a_{n1}\mathbf{B} & \ldots & a_{nm}\mathbf{B} \end{bmatrix}_{np \times mr}.$$

Additionally, if $n = km + r$ and $r \neq 0$, the maximal orbit is connected and pathwise connected (any two systems from this orbit can be joined by a continuous path lying entirely inside the orbit).

Having defined the feedback equivalence of systems we can introduce a concept of structural stability. We will call a system $\sigma \in \Sigma$ to be structurally stable if σ has a neighbourhood consisting of systems equivalent to σ. This means that σ is structurally stable if appropriately small perturbations of the system's matrices can be compensated by feedback. Clearly, the structural stability of σ is equivalent to the openness of the $F(m, n)$ orbit of σ. And since the maximal orbit $\omega(\varkappa')$ is open, all systems belonging to $\omega(\varkappa')$ are structurally stable. We can formulate this as the following.

Proposition 3.6. Structural stability is a generic property of linear systems.

There are some other equivalence relations which can be imposed on linear systems, see Willems [14]. It appears, however, that the concepts of structural stability based on these equivalence relations coincide with that one introduced above, and the structural stability in any of these senses is generic.

4. FINAL REMARKS

The purpose of this paper was to present a sample of generic properties of linear systems, and describe some methods with which the properties can be handled. We paid particular attention to algebraic geometric and differential geometric methods. The properties whose genericity has been proved might suggest that linear systems have very regular structure, and all their properties would be generic. However this is not always the case. There is an important group of properties of linear systems related to the existence of solutions to some design problems (disturbance decoupling, regulation, pole placement, etc.) which are generic provided that some extra conditions are satisfied or which hold on some very "thin" sets of linear systems, see [14], [15].

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