Ivan Kramosil
A selection-based formal description of an environment

A Selection-Based Formal Description of an Environment

IVAN KRAMOSIL

Some former results ([3]) show that a non-restricted joining of a new formula with the formal representation of the environment is not justifiable because of a danger of inconsistency. In this paper a criterion is proposed which measures the quality of a formal representation of the environment. This criterion combines the properties of several more special criteria and its value can be statistically estimated on the basis of a random sample. A decision procedure is proposed which joins a new formula with the formal representation of the environment only in case when, with at least some probability given a priori, the enriching does not make the quality of the formal representation worse.

1. INTRODUCTION AND PROBLEM FORMULATION

Among the papers dealing with automaton-environment systems the deterministic approach prevails no matter whether the used models or investigation methods are considered. Papers [3] and [4] can serve as an illustration of our attempts to suggest another, statistically based point of view to this field of mathematics and its methods. The results of these papers show that probabilistic way of reasoning covers a number of aspects which the deterministic approach must ommit or eliminate by appropriate a priori assumptions. However, at the same time a certain number of new questions and problems have arisen, problems connected with the process of formal representation of the environment inside the automaton. Our aim is to answer, at least partially, some of these questions in this paper.

Let us start with a very short description of the model investigated in [3] and [4] which will play the fundamental role also here; the knowledge of [3] and [4] is necessary for a more detailed study of this paper.

The notion "automaton" is not used here in its theoretical sense (like in the automaton theory) but in an intuitive and rather rough sense; automaton is a device which is able to move in the medium and to fulfil some tasks in it. The automaton

is supposed to be able to perceive, at least in certain measure, the environment and to be able to form and use a formal image of the medium and its activity is not goalless, the automaton tends to modify the medium in such a way that certain properties, given usually a priori, were satisfied by the newly reached configuration of the environment. Perhaps the expression "cognitive robot" would be appropriate and we would be tempted to use it if we did not realize some undesirable anthropomorfic associations connected with it.

Environment will be understood here as the set of all pairs of integers, i.e. points in the Euclidean plain the both co-ordinates of which are integers; the notation $a = \langle a_1, a_2 \rangle$ for points and I^2 for the set will be used. They are many properties which are or are not possessed by particular points and this may be subjected to changes, as the time passes, due to the automaton activity as well as due to some random influences. Only properties of points are considered, i.e. unary predicates, relations will be mentioned below. Using an appropriate formal language (e.g. first order predicate calculus) we are allowed to profit of Gödel procedure, enumerate the properties and, after all, introduce a unique predicate V (ternary) ascribing to any formula $V(a, i, s)$, $a \in I^2$, the meaning: "the point a possesses the i-th property in the situation s".

The properties of points in I^2 may satisfy some relations which, supposing they are known to the subject or designer, may be implemented into the automaton. Here we shall do it using the notion of dependence axioms. It is possible that if a point a_1 possesses the i_1-th property in the situation s_1, a_2 possesses the i_2-th property in the situation s_2, ..., a_n possesses the i_n-th property in the situation s_n, then a point a_0 possesses i_0-th property in s_0 and, moreover, a_0, i_0, s_0 can be effectively computed from $a_1, ..., a_n, i_1, ..., i_n, s_1, ..., s_n$. This knowledge can be formally written as

$$\left(\bigwedge_{j=1}^{n} V(a_j, i_j, s_j) \right) \rightarrow V\big(f(a_1, ..., a_n), g(i_1, ..., i_n), h(s_1, ..., s_n)\big),$$

this formula is called *dependence axiom* and may be implemented into the automaton storage.

In every situation the automaton is situated just in one point $R(s) \in I^2$. It is supposed to be able to perform the following actions:

1) moves: eight possible moves to any point neighbour to the actual position.

2) operations: express the automaton ability to change the properties of the point where it is situated. An operation is described by a pair $\langle j, k \rangle$ of naturals the meaning of which is: if the point $R(s)$, where the automaton is situated now, possesses the j-th property, then operation is applicable and, if applied, the point will possess the k-th property.

3) observations: any of them is defined by a pair $\langle a, j \rangle$, $a \in I^2$, $j = 1, 2, ...$ and means: to find whether the point a possesses the j-th property or not.

4) deductions: consist in using a dependence axiom in order to prove $V(a_0, i_0, s_0)$ under the condition that the validity of premises is known to the automaton.

5) actualizations: can be seen as a special case of deductions; the validity of a fact in a past situation is extended also to the present one supposing no later information coming from the environment prevents us to do so.

During any action excluding the moves the automaton is supposed not to change its position in I^2. The notion of situation is formalized in our model by the sequence of all the actions performed by the automaton since a starting situation s_0. A formal representation $H(s)$ is connected with any situation s, $H(s)$ is a finite set of formulas of the form $V(a, i, s')$ (s' may vary, not necessarily $s = s'$). A formula representing the automaton position in the present situation is always supposed to be in $H(s)$. With any among the actions given above a transformation of $H(s)$ is associated; roughly speaking, $H(s)$ is joined with the new formula the validity of which has been verified (observations, deductions), supposed (actualization), or involved (moves, operations,), for more details see [3].

The basic idea of our probabilistic modification of this model consists in the assumption that the automaton actions involve, besides the expected and formally described consequences also some other consequences which are supposed to be of random nature. The automaton has no systematic information about these random events, it may learn some of them a posteriori, by an observation, however, in general, its formal representation $H(s)$ not necessarily corresponds completely to the actual state. $H(s)$ may contain even non-valid formulas, which it has obtained using the actualization and not having known that the fact valid in past had changed at random before it used the actualization.

We shall not develope this model into more details and close this introductory section with mentioning some of its properties (precise formulations and proofs in [3]).

Not only the possibility that $H(s)$ contains a non-valid formula, but even the possibility that $H(s)$ is inconsistent is not excluded. Probabilities for both these events are studied in [3] and are proved to increase, under some rather general conditions, to one when the number of actions performed by the automaton increases.

During the automaton activity the formal representation $H(s)$ plays the basic role, namely in the stage of decision making and planning. Consequently, the inadequacy or even inconsistency of $H(s)$ threatens the proper sense of constituting such formal representation. These results do not admit their simple accepting or constating, they force us to modify the process of formal representation of the environment, if possible, in order to avoid or minimize the undesirable consequences. To judge the possibilities of such a modification will be our aim in this paper.

2. PRELIMINARY REQUESTS TO REPRESENTATION PROCEDURES

Instead of to suggest a procedure and then to prove its convenability we shall follow the opposite pattern and it is why we start with some preliminary demands try-ing, later, to deduce the desired procedure almost immediately from this pattern.

No substantial changes concerning the way of obtaining an information from the environment or the automaton operations will be considered. It is caused by the fact that these parts of the automaton activity stand in a close connection with its physical and technical features and abilities and any new assumptions concerning these matters would involve immediately doubts on the possibilities of their realization at these non-mathematical levels.

However, an other possibility remains. In the foregoing papers the representation $H(s)$ was formed by joining to $H(s)$, in an implicit or explicit way, any formula the validity of which followed from the performed action. In other words, if the action assured the validity of a formula the formula has been implemented to $H(s)$ without any further testing. The basic part of this paper is based on the idea that a formula obtained from the environment is nothing more than a candidate of being joined with $H(s)$; a further decision procedure must solve the question of its actual joining with $H(s)$.

Generally spoken, there are at least two types of such tests. First of them can be called global test; in this case a number of formulas entering the automaton is joined with $H(s)$ without any testing, then the automaton interrupts, for a while, its outer activity and tests and transforms its formal representation; having reached an appro-priate modification of $H(s)$ the automaton goes on in his interaction with the en-vironment.

The other type of test can be called individual and consists in facing every new formula to the set $H(s)$ (the facing being of the form of a test) and the formula is joined definitely with $H(s)$ supposing it satisfies the test. Clearly, both the types of tests can be combined, in this paper we shall concern our attention to the indivi-dual tests because of the fact that under certain more conditions any global test can be expressed as an individual one.

Only such tests will be taken into consideration which satisfy the two following criteria or principles. Both of them express the necessity of some storage and com-plexity limitations connected with a possibility of a practical implementation and use of the suggested testing procedures.

Local principle: Any decision concerning the set $H(s)$ is made only using a subset $H_0(s)$ of $H(s)$ $(H_0(s)$ may be, e.g., a random sample), the cardinality of $H_0(s)$ is supposed to be given a priori and this cardinality will be a free parameter of the procedure. The manipulation with the set $H(s)$ is, hence, limited to

(a) generating $H_0(s) \subset H(s)$,

(b) assuring that any formula occurs in $H(s)$ at most once (this condition represents a certain weakening of the local principle and will be discuted below).

Polynomial principle: The number of operations and decisions, necessary to make the final decision concerning the $H(s)$ and its possible modification (or concerning the joining of a new formula with $H(s)$) does not exceed a polynomial function of the length of the entering data no matter how the operations or decision units are counted. This principle excludes, e.g., any decision procedure requesting, even in the potential form, to search all the subsets of $H_0(s)$ (such a procedure is of exponential complexity in general). In recent mathematical papers some arguments can be found in favour of the hypothesis identifying the "reasonable" or "applicable" algorithms with those of at most polynomial complexity. No precise justification has been given yet (and is it possible to give one?), however, we shall take this idea into consideration here (some discussion can be found in [1]).

Clearly, the polynomial and local principles are not the only demands the testing procedure is to satisfy. On the other hand the test must allow to generate a formal representation which is powerful enough to be used for the intended goals, i.e. to plan a goal-tending activity of the automaton in the environment. Influenced by the results from [3] or [4] we might be tempted to consider the consistency as the only criterion to which $H(s)$ is subjected. To exaggerate the priority of this demand too much would lead, however, to some undesirable consequence, as the following example shows:

Consider three formal representations, every of them containing 100 formulas. First of them consists only of propositional tautologies, the second contains 99 valid and valuable pieces of information on the environment the 100-th formula being the negation of one of the 99 former formulas. Finally, in the third case $H(s)$ contains fifty formulas and their negations. Now, if the consistency is our only critetion, the first case will be preferred, the other two will be refused and the difference between them disappears. From the intuitive point of view, probably, the first and the third sets will be refused and practically identified as saying nothing about the world and the second representation will be preferred as the relatively best, even if not the ideal one.

Let us introduce, now, several points of view from which the quality of a formal representation can be judged together with some explicit criteria. Some notations and notions seem to be inevitable.

For any set A denote by $c(A)$ its cardinal. If A is a finite set of formulas denote by Cons (A) the largest (with respect to the number of formulas) subset of A, which is consistent, by Ax (A) the smallest set of formulas from A from which all the formulas from A can be deduced (again with respect to cardinality). Clearly, Cons (A) and Ax (A) are not defined uniquely by these conditions, but their cardinals are and only these numbers will be used below, so a choosing rule defining the two sets precisely is of no importance for us.

Definition 1. Let S be the set of situations, let ϱ be a binary function defined on $S \times \times S$ as follows:

$$\varrho(s, s') = k + l,$$

where $s = \beta_1\beta_2 \ldots \beta_k\alpha_1\alpha_2 \ldots \alpha_n s_0$, $s' = \gamma_1\gamma_2 \ldots \gamma_l\alpha_1\alpha_2 \ldots \alpha_n s_0$, $\beta_k \neq \gamma_l$. The value $\varrho(s, s')$ can be called *degree* or *measure of freshness* between s and s'.

Theorem 1. The function ϱ is a metric in S.

Proof. If $s = s'$, then $k = l = 0$ and $\varrho(s, s') = 0$, symmetry follows from the symmetry of the addition. In order to prove the triangle inequality consider three situations:

$$s = \beta_1 \ldots \beta_k\alpha_1 \ldots \alpha_n s_0 \in S,$$

$$s' = \gamma_1 \ldots \gamma_m\alpha_1 \ldots \alpha_n s_0 \in S,$$

$$s'' = \delta_1 \ldots \delta_l\alpha_1 \ldots \alpha_n s_0 \in S.$$

Suppose that $\beta_k = \gamma_m = \delta_l$ does not hold, without any loss of generality we may also suppose that $\beta_k \neq \delta_l$ (renaming the situations if necessary). Hence $\varrho(s, s'') = k + l$ and the two possibilities occur:

1) $\gamma_m = \beta_k, \gamma_{m-1} = \beta_{k-1}, \ldots, \gamma_{m-i} = \beta_{k-i}, \gamma_{m-i-1} \neq \beta_{k-i-1}$,

   $i \leqq m, k$. Then $\gamma_m \neq \delta_l$, hence, $\varrho(s, s') = (m - i) + (k - i)$, $\varrho(s', s'') = m + l$, which gives

$$\varrho(s', s'') + \varrho(s, s') = (m - i) + (k - i) + m + l = k + l + 2(m - i) \geqq k + l = $$
$$= \varrho(s, s'').$$

2) $\gamma_m \neq \beta_k, \gamma_m = \delta_l, \gamma_{m-1} = \delta_{l-1}, \ldots, \gamma_{m-p} = \delta_{l-p}, \gamma_{m-p-1} \neq \delta_{l-p-1}$,

   this case is analogous to the first one, now $\varrho(s', s'') = l + m$, $\varrho(s, s') = (k - p) + (l - p)$ and the inequality holds again. Q.E.D.

We are in a position, now, to discuss the criteria of quality for a formal representation.

1) *Consistency*

The desirability of consistency is clear, the danger of exaggerating this demand has been discussed. Hence, we propose the following degree of consistency:

$$\frac{c(H(s))}{c(\text{Cons }(H(s)))}.$$

This value equals to 1 if and only if $H(s)$ is consistent, our aim is to minimize this value if possible. An intuitive interpretation of this criterion offers the following theorem.

Theorem 2. Let $(\Omega, \mathscr{S}, P)$ be a probability space, let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be mutually independent and equally distributed random variables defined on $(\Omega, \mathscr{S}, P)$, taking

their values in $H(s)$ and such that

$$P(\{\omega : \omega \in \Omega, \ \alpha_1(\omega) = p\}) = (c(H(s)))^{-1}$$

for any $p \in H(s)$. Then for any n holds: if

$$c(H(s)) \cdot (c(\mathrm{Cons}(H(s))))^{-1} \to 1 \ ,$$

then

$$P(\{\omega : \omega \in \Omega, \ \{\alpha_1(\omega), \ldots, \alpha_n(\omega)\} \ is \ consistent\}) \to 1 \ .$$

Proof. Clearly,

$$P(\{\omega : \omega \in \Omega, \ \alpha_i(\omega) \in \mathrm{Cons} \ (H(s))\}) = c(\mathrm{Cons} \ (H(s))) \cdot (c(H(s)))^{-1} \ .$$

If $\alpha_i(\omega) \in \mathrm{Cons} \ (H(s))$ for all $n \geq i$, then the sampled formulas form a consistent set. This fact and the statistical independence of the random variables gives

$$P(\{\omega : \omega \in \Omega, \ \{\alpha_1(\omega), \ldots, \alpha_n(\omega)\} \ is \ consistent\}) \geq c(\mathrm{Cons}(H(s))) \cdot (c(H(s)))^{-1}$$

from which the assertion immediately follows. Q.E.D.

2) Logical independence

This criterion expresses our wish to minimize the extent of $H(s)$ under the condition that the information on the environment contained, at least implicitly, in $H(s)$ will not be diminished. From this point of view, if $H(s) = \mathrm{Ax} \ (H(s))$, the ideal state is achieved (clearly, formulas from $H(s)$ are logically independent). So we propose to measure the degree of logical independence using the value

$$\frac{c(H(s))}{c(\mathrm{Ax}(H(s)))} \ .$$

Minimization of this value is desirable and its minimal and optimal value is one. As far as the logical independence of formulas in a random sample is considered an analogous theorem to Theorem 2 could be formed and proved, giving an intuitive justification of the proposed criterion.

3) Semantic completeness

The representation $H(s)$ is requested to describe the environment as completely as possible in the sense that any interesting and valid formula describing the environment should be derivable from $H(s)$. If $H(s)$ is consistent and complete, it describes adequately the environment in the sense that just the valid formulas (and no else) are derivable. If the set $H(s)$ contained a formula A not valid in the environment, then $H(s)$ must contain non A as well, as non A is valid, so $H(s)$ would be inconsistent. This relation among completeness, consistency and adequacy is the reason for our not having introduced a special adequacy criterion for $H(s)$.

A numerical criterion of semantic completeness can be derived as follows: Consider a probability distribution on the set of all valid formulas in our language. If $H(s)$ is not semantically complete, then there is a positive probability of sampling a valid but from $H(s)$ not deducible formula. On the other hand, to enlarge the set $H(s)$ involves a possibility to add some new and logically independent formulas, which implies a possibility to deduce from $H(s)$ some not deduced yet formulas. Keeping in mind the strong simplifying connected with the following assumption we can say: Let us assume that there is a real, $0 < K < 1$, such that

$$P(\{\omega : \omega \in \Omega, \ \alpha(\omega) \ \textit{is not deducible from} \ H(s) \cup \{x\}\}) =$$
$$= K \cdot P(\{\omega : \omega \in \Omega, \ \alpha(\omega) \ \textit{is not deducible from} \ H(s)\}) ,$$

where α is the random variable sampling formulas, K depends on α and on the considered theory. So we shall take the value

$$K^{c(H(s))}$$

as the numerical degree of completeness with the aim to minimalize this value. Probably the value $K^{c(Ax(H(s)))}$ would express better our idea, however, using the criterion would bring us to a contradiction with the local and polynomial principles as will be shown later.

4) Freshness

It can be easily seen that, in general, $H(s)$ contains formulas expressing properties of various situations, not only that one in which the automaton is just situated. However, we admit the possibility of random influences which are only from time to time enregistered and stored in $H(s)$. This gives: the "older" a formula is, the lesser is its reliability, in another words, the more fresh the formulas from $H(s)$ are, the better. Denoting by $\varrho(x, s)$ the value $\varrho(s', s)$, where x is $V(a, i, s')$, s is the actual situation and ϱ is a metric on S we have the value

$$(c(H(s)))^{-1} \sum_{x \in H(s)} (\varrho(x, s) + c_1) , \quad c_1 \geqq 0 ,$$

as a possible quantitative degree of freshness; the aim is the minimization of this value. Here c_1 expresses a degree of "getting old" the formulas in one step (realize that formulas from $H(s)$ are used in the situation φs resulting from s by an action φ).

5) Importance

There may be many reasons even outside the process of formal representation and the automaton itself for which some formulas can be considered as more important than others (e.g. the formers speak about more important objects or properties).

In our paper we shall limit ourselves to the assumption that there is a function g_0, $g_0 : L \to \langle 0, \infty)$, where L is the alphabet of our language, i.e. if c_i is a constant

of the language then $g_0(c_i)$ can be understood as the importance of the object denoted by c_1. If $x = x_1 x_2 \ldots x_n$ is a formula, $x_i \in L$, then g_0 can be extended by the relations

$$g(x) = \sum_{i=1}^{n} g_0(x_i)$$

or

$$g(x) = n^{-1} \sum_{i=1}^{n} g_0(x_i) \,.$$

Using the universal form $V(a, i, s)$ for the formulas we may define g_0 on the set $I^2 \cup N$ and then write

$$g(V(a, i, s)) = g_0(a) \cdot g_0(i)$$

or

$$g(V(a, i, s)) = g_0(a) + g_0(i)$$

or in another way. We shall always suppose that the importance of a formula does not depend on its situation term, i.e.

$$g(V(a, i, s)) = g(V(a, i, s')) \,, \quad a \in I^2 \,, \quad i \in N \,, \quad s, s' \in S \,.$$

In other words, importance of a situation is expressed only by the mean of the value $\varrho(s, s')$.

A quantitative measure of the importance of a formal representation $H(s)$ can be, hence, expressed by the value

$$\left(c(H(s)) \right)^{-1} \sum_{x \in H(s)} g(x) \,,$$

with the aim to maximize this value if possible.

We are in a position, now, to suggest and study a criterion, joining and unifying in a compromise way the five criteria mentioned above.

Definition 2. For any formal representation the real $F(H(s))$, defined by the relation

$$F(H(s)) = \left[\frac{c(H(s))}{c(\mathrm{Cons}\,(H(s)))} \right] \cdot \left[\frac{c(H(s))}{c(\mathrm{Ax}\,(H(s)))} \right] \cdot \left[\frac{\sum\limits_{x \in H(s)} (\varrho(x, s) + c_1)}{\sum\limits_{x \in H(s)} g(x)} \right] \cdot K^{c(H(s))}$$

will be called the F-quality of $H(s)$. A formal representation $H_1(s')$ is called F-better than a formal representation $H_2(s)$, if $F(H_1(s')) \langle F(H_2(s))$.

The particular criteria, as well as the criterion F, can be subjected to some comments the most important among which may be:

1) a non-negative weight $w(x)$ can be ascribed to any formula x and instead of $c(H(s))$, $\varrho(x, s)$ and $g(x)$ the values $\sum\limits_{x \in H(s)} w(x)$, $\varrho(x, s) \cdot w(x)$, $g(x) \cdot w(x)$ can be considered, analogously for $c(\mathrm{Ax}\,(H(s)))$ and $c(\mathrm{Cons}\,(H(s)))$.

2) instead of defined criteria any increasing functions of them can be considered, also when defining F instead of product any other increasing function of the five particular criteria may be used.

3) instead of using the notion of deducibility in the definition of $\mathrm{Ax}\,(H(s))$ we can introduce, somehow, the notion of immediate deducibility defining $\mathrm{Ax}\,(H(s))$ as the smallest set of formulas from which all the formulas from $H(s)$ are immediately deducible.

On the other hand, there are also some reasons favorizing our criterion F. Defining it on the level general enough to take into consideration the previous remarks we would not be able to obtain effective decision rules applicable when $H(s)$ formed, what is our aim in this paper. On the other hand, our concrete choose of such and such weights or such and such increasing functions can be justified only by a concrete problem, not at a general level. The same is the problem how to define the "immediate deducibility". However, to limit ourselves to a concrete case would request, first, to explain why this case is worth enough to be studied and, moreover, to find the weight function and other functions corresponding to this case. Not wanting to deal with such problems we limit ourselves to the particular criteria and F-criterion as defined above.

3. GLOBAL DECISION RULES

The papers [3] and [4] show that under certain conditions the automaton activity can be modeled in the form of a random walk the next action of the automaton being a random event not depending on the previous actions (if certain simplification is admitted). Using this way of reasoning also here we may start with this model:

Let $\alpha_1, \alpha_2, \ldots$ be random variables, mutually independent and equally distributed, defined on a probability space $(\Omega, \mathscr{S}, P)$ and taking its values in the space of formulas of the form $V(a, i, s)$. These random variables represent a flow of formulas following from the action performed by the automaton. In the model investigated in [3] and [4] these formulas entered immediately the set $H(s)$. At the same time, the formulas in question are to represent, in a sense, the environment (considered as a four-dimensional space the space of situations being the fourth dimension). Hence, our task is as follows: How to choose, from the flow $\alpha_1(\omega), \alpha_2(\omega), \ldots$ of entering formulas a subsequence $\alpha_{i_1}(\omega), \alpha_{i_2}(\omega), \ldots, \alpha_{i_n}(\omega), \ldots$ in such a way that the value

$$F(H(s_n)) = F(H(s_n, \omega)) = F\left(\bigcup_{i_j < n} \{\alpha_{i_j}(\omega)\}\right)$$

were small as possible; s_n denotes the situation when $\alpha_n(\omega)$ enters the automaton in order to be or not be joined with $H(s_n)$.

In order not to violate our polynomial principle we shall limit ourselves to a confrontation of $\alpha_n(\omega)$ with the set $H(s_n)$. However, for some reasons which will become clear later we do not take into consideration, in this chapter, the local principle

and we shall confront $\alpha_n(\omega)$ with all the formulas from $H(s_n)$. So we have to find a decision function $\mathscr{D}$ with two values 0 (not to admit) and 1 (admit) satisfying:

$$\mathscr{D}(H(s_n), \alpha_n(\omega)) = 1, \quad \text{if} \quad F(H(s_n) \cup \{\alpha_n(\omega)\}) < F(H(s_n)),$$

$$\mathscr{D}(H(s_n), \alpha_n(\omega)) = 0, \quad \text{if} \quad F(H(s_n) \cup \{\alpha_n(\omega)\}) \geqq F(H(s_n)).$$

The following theorems give a possibility how to obtain such a decision function.

Theorem 3. Let $x = V(a, i, s)$ be a formula, let the formal representation $H(s)$ not imply x neither *non* x. If

$$g(x) \geqq \frac{\sum\limits_{x' \in H(s)} g(x')}{\sum\limits_{x' \in H(s)} (\varrho(x', s) + c_1)} \cdot (\varrho(s, s'') + c_1),$$

then

$$F(H(s) \cup \{x\}) < F(H(s)).$$

If $s = s''$ and $c_1 = 0$, then the latter inequality is always valid.

Proof. The relation

$$g(x) \geqq \frac{\sum\limits_{x' \in H(s)} g(x')}{\sum\limits_{x' \in H(s)} (\varrho(x', s) + c_1)} \cdot (\varrho(s, s'') + c_1)$$

gives, after an easy calculation,

$$\frac{\left(\sum\limits_{x' \in H(s)} (\varrho(x', s) + c_1)\right) + \varrho(s, s'') + c_1}{\left(\sum\limits_{x' \in H(s)} g(x')\right) + g(x)} \leq \frac{\sum\limits_{x' \in H(s)} (\varrho(x', s) + c_1)}{\sum\limits_{x' \in H(s)} g(x')}.$$

$H(s)$ does not imply x neither *non* x, i.e. $H(s)$ as well as $H(s) \cup \{x\}$ are consistent. $\mathrm{Ax}\,(H(s) \cup \{x\}) = \mathrm{Ax}\,(H(s)) \cup \{x\}$, which gives

$$\frac{c(H(s) \cup \{x\})}{c(\mathrm{Ax}\,(H(s) \cup \{x\}))} = \frac{c(H(s)) + 1}{c(\mathrm{Ax}\,(H(s))) + 1} \leqq \frac{c(H(s))}{c(\mathrm{Ax}\,(H(s)))},$$

hence,

$$\left(\frac{c(H(s) \cup \{x\})}{c(\mathrm{Cons}\,(H(s) \cup \{x\}))}\right) \cdot \left(\frac{c(H(s) \cup \{x\})}{c(\mathrm{Ax}\,(H(s) \cup \{x\}))}\right) \cdot$$

$$\cdot \frac{\left(\sum\limits_{x' \in H(s)} (\varrho(x', s) + c_1)\right) + \varrho(s, s'') + c_1}{\left(\sum\limits_{x' \in H(s)} g(x')\right) + g(x)} \cdot K^{c(H(s) \cup \{x\})} <$$

$$< \left(\frac{c(H(s))}{c(\mathrm{Cons}(H(s)))}\right) \left(\frac{c(H(s))}{c(\mathrm{Ax}\,(H(s)))}\right) \frac{\sum\limits_{x' \in H(s)} (\varrho(x', s) + c_1)}{\sum\limits_{x' \in H(s)} g(x')} \cdot K^{c(H(s))}.$$

The left side equals to $F(H(s) \cup \{x\})$, the right one is just $F(H(s))$, so the first assertion is proved, the other follows immediately when setting $s = s''$, i.e. $\varrho(s, s'') = 0$ and $c_1 = 0$. Q.E.D.

Theorem 4. Let x be a formula deducible from $H(s)$ but not belonging to $H(s)$, $x = V(a, i, s'')$, suppose that $c(\text{Ax}\,(H(s) \cup \{x\})) = c(\text{Ax}\,(H(s)))$. Let

$$g(x) \geqq \Big(\sum_{x' \in H(s)} g(x') \Big) \cdot \left[\Big(1 + \frac{\varrho(s, s'') + c_1}{\sum\limits_{x' \in H(s)} (\varrho(x', s) + c_1)} \Big) \cdot \big(1 + (c(H(s)))^{-1} \big)^2 - 1 \right],$$

then $F(H(s) \cup \{x\}) < F(H(s))$. If $s = s''$, $c_1 = 0$, then the condition

$$g(x) \geqq 3 \, (c(H(s)))^{-1} \sum_{x' \in H(s)} g(x')$$

assures the inequality $F(H(s) \cup \{x\}) < F(H(s))$.

Proof. The relation

$$g(x) \geqq \Big(\sum_{x' \in H(s)} g(x') \Big) \left[\Big(1 + \frac{\varrho(s, s'') + c_1}{\sum\limits_{x' \in H(s)} (\varrho(x', s) + c_1)} \Big) \Big(1 + (c(H(s)))^{-1} \Big)^2 - 1 \right]$$

gives

$$(c(H(s)) + 1)^2 \frac{\Big(\sum\limits_{x' \in H(s)} (\varrho(x', s) + c_1) \Big) + \varrho(s, s'') + c_1}{\Big(\sum\limits_{x' \in H(s)} g(x') \Big) + g(x)} \leqq$$

$$\leqq (c(H(s)))^2 \frac{\sum\limits_{x' \in H(s)} (\varrho(x', s) + c_1)}{\sum\limits_{x' \in H(s)} g(x')}$$

and

$$K^{c(H(s))+1} \Big(\frac{c(H(s)) + 1}{c(\text{Cons}(H(s)))} \Big) \Big(\frac{c(H(s)) + 1}{c(\text{Ax}\,(H(s)))} \Big) \frac{\Big(\sum\limits_{x' \in H(s)} (\varrho(x', s) + c_1) \Big) + \varrho(s, s'') + c_1}{\Big(\sum\limits_{x' \in H(s)} g(x') \Big) + g(x)} \leqq$$

$$\leqq K^{c(H(s))} \Big(\frac{c(H(s))}{c(\text{Cons}(H(s)))} \Big) \Big(\frac{c(H(s))}{c(\text{Ax}(H(s)))} \Big) \frac{\sum\limits_{x' \in H(s)} (\varrho(x', s) + c_1)}{\sum\limits_{x' \in H(s)} g(x')} \; .$$

According to our assumption $c(\text{Ax}(H(s) \cup \{x\})) = c(\text{Ax}\,(H(s)))$. We do not know, whether $\text{Cons}\,(H(s) \cup \{x\}) = \text{Cons}\,(H(s))$ or $\text{Cons}\,(H(s) \cup \{x\}) = \text{Cons}\,(H(s)) \cup \{x\}$, nevertheless, the following is valid

$$K^{c(H(s) \cup \{x\})} \Big(\frac{c(H(s) \cup \{x\})}{c(\text{Cons}\,(H(s) \cup \{x\}))} \Big) \Big(\frac{c(H(s) \cup \{x\})}{c(\text{Ax}\,(H(s) \cup \{x\}))} \Big) \; .$$

$$\cdot \frac{\left(\sum_{x' \in H(s)} \left(\varrho(x', s) + c_1\right)\right) + \varrho(s, s'') + c_1}{\left(\sum_{x' \in H(s)} g(x')\right) + g(x)} \leqq$$

$$\leqq K^{c(H(s))} \left(\frac{c(H(s))}{c(\mathrm{Cons}\,(H(s)))}\right) \left(\frac{c(H(s))}{c(\mathrm{Ax}\,(H(s)))}\right) \frac{\sum_{x' \in H(s)} \left(\varrho(x', s) + c_1\right)}{\sum_{x' \in H(s)} g(x')},$$

but this is nothing else than the inequality $F(H(s) \cup \{x\}) \leqq F(H(s))$. Setting $\varrho(s, s'') = 0$, $c_1 = 0$, we obtain

$$g(x) \geqq \left(\sum_{x' \in H(s)} g(x')\right) \left[\left(1 + \left(c(H(s))\right)^{-1}\right)^2 - 1\right] = \sum_{x' \in H(s)} g(x') \left[\frac{2}{c(H(s))} + \frac{1}{\left(c(H(s))\right)^2}\right]$$

and this inequality holds, clearly, if

$$g(x) \geqq 3 \left(\sum_{x' \in H(s)} g(x')\right) \cdot \left(c(H(s))\right)^{-1}$$

and the proof is finished. Q.E.D.

Theorem 5. Let the conditions of Theorem 4 hold, let

$$g(x) \quad \geqq 7 \left(c(H(s))\right)^{-1} \sum_{x' \in H(s)} g(x'),$$

$$\varrho(s, s'') \leqq \left(c(H(s))\right)^{-1} \sum_{x' \in H(s)} \varrho(x', s).$$

Then $F(H(s) \cup \{x\}) \leqq F(H(s))$.

Proof. The supposed inequality for $\varrho(s, s'')$ assures that

$$1 + \frac{\varrho(s, s'') + c_1}{\sum_{x' \in H(s)} \left(\varrho(x', s) + c_1\right)} \leqq 1 + \left(c(H(s))\right)^{-1}$$

and this gives that the right side of the inequality for $g(x)$ in Theorem 4 does not exceed

$$\left(\sum_{x' \in H(s)} g(x')\right) \cdot \left[\left(1 + \left(c(H(s))\right)^{-1}\right)^3 - 1\right] \leqq 7 \left(c(H(s))\right)^{-1} \left(\sum_{x' \in H(s)} g(x')\right).$$

From this inequality and from the assertion of Theorem 4 the result follows. Q.E.D.

Theorem 6. Let $x = V(a, i, s)$ be a formula deducible from $H(s)$ but not belonging to $H(s)$, suppose that $c(\mathrm{Ax}\,(H(s) \cup \{x\})) = c(\mathrm{Ax}\,(H(s)))$ and that $H(s)$ is consistent. Let at least one of the two following pairs of conditions hold:

(I) $\quad \varrho(s, s'') \leqq \left(c(H(s))\right)^{-1} \sum_{x' \in H(s)} \varrho(x', s), \quad g(x) \geqq 3 \left(c(H(s))\right)^{-1} \sum_{x' \in H(s)} g(x'),$

or

(II) $\qquad s = s''$, $\quad c_1 = 0$, $\quad g(x) \geqq (c(H(s)))^{-1} \sum\limits_{x' \in H(s)} g(x')$.

Then $F(H(s) \cup \{x\}) \leqq F(H(s))$.

Proof. The conditions assure that $\text{Cons}(H(s)) = H(s)$, $\text{Cons}(H(s) \cup \{x\}) = H(s) \cup \{x\}$, $c(\text{Ax}(H(s) \cup \{x\})) = c(\text{Ax}(H(s)))$. Hence,

$$F(H(s) \cup \{x\}) = K^{c(H(s) \cup \{x\})} \left(\frac{c(H(s) \cup \{x\})}{c(\text{Cons}(H(s) \cup \{x\}))} \right) \left(\frac{c(H(s) \cup \{x\})}{c(\text{Ax}(H(s) \cup \{x\}))} \right) \cdot$$

$$\cdot \frac{\left(\sum\limits_{x' \in H(s)} (\varrho(x', s) + c_1) \right) + \varrho(s, s'') + c_1}{\left(\sum\limits_{x' \in H(s)} g(x') \right) + g(x)},$$

$$F(H(s)) = K^{c(H(s))} \left(\frac{c(H(s))}{c(\text{Cons}(H(s)))} \right) \left(\frac{c(H(s))}{c(\text{Ax}(H(s)))} \right) \frac{\sum\limits_{x' \in H(s)} (\varrho(x', s) + c_1)}{\sum\limits_{x' \in H(s)} g(x')}$$

and the desired inequality between F-qualities is assured if

$$g(x) \geqq \left(\sum\limits_{x' \in H(s)} g(x') \right) \left(\frac{c(H(s)) + 1}{c(H(s))} \right) \frac{\left(\sum\limits_{x' \in H(s)} (\varrho(x', s) + c_1) \right) + \varrho(s, s'') + c_1}{\sum\limits_{x' \in H(s)} (\varrho(x', s) + c_1)} -$$

$$- \sum\limits_{x' \in H(s)} g(x').$$

If

$$\varrho(s, s'') \leqq (c(H(s)))^{-1} \sum\limits_{x' \in H(s)} \varrho(x', s),$$

then

$$\frac{\left(\sum\limits_{x' \in H(s)} (\varrho(x', s) + c_1) \right) + \varrho(s, s'') + c_1}{\sum\limits_{x' \in H(s)} (\varrho(x', s) + c_1)} \leqq 1 + (c(H(s)))^{-1},$$

hence, the validity of the inequality

$$g(x) \geqq 3(c(H(s)))^{-1} \cdot \left(\sum\limits_{x' \in H(s)} g(x') \right) \geqq (c(H(s)))^{-1} \cdot \left(\sum\limits_{x' \in H(s)} g(x') \right) \cdot (2 + (c(H(s)))^{-1})$$

assures that $F(H(s) \cup \{x\}) \leqq F(H(s))$. If $c_1 = 0$, $s = s''$, then the inequality for $g(x)$ takes the form

$$g(x) \geqq (c(H(s)))^{-1} \sum\limits_{x' \in H(s)} g(x')$$

and the proof is finished. Q.E.D.

Let us join two remarks. In some of the theorems we have just proved we used the condition that x follows from $H(s)$ but $c(\text{Ax}(H(s) \cup \{x\})) = c(\text{Ax}(H(s)))$. This means, the assertions of the theorems in question do not hold, if x depends on $H(s)$ and enables to reduce the number of axioms. Clearly, in this case joining x with $H(s)$ the ratio $c(H(s))/c(\text{Ax}(H(s)))$ is enlarged and the quality of $H(s)$ is worse now (the number of "useless" formulas increased). On the other hand we feel that joining such x to $H(s)$ is of certain use, as it gives the possibility to eliminate some formulas from $H(s)$ in future as they will follow from x. Trying to propose a criterion taking into consideration this aspect we shall arrive at the conclusion that it is not possible without neglecting the polynomial principle. It is why we shall follow the compromise way of decision: we shall use the criterion following from the theorems in question even in case the condition $c(\text{Ax}(H(s))) = c(\text{Ax}(H(s) \cup \{x\}))$ is not valid.

The foregoing theorems give some sufficient conditions for the joining of x to $H(s)$ to improve the F-quality or at least not to make it worse. As can be easily seen the conditions are of a common form:

$$g(x) \geqq f(x, c(H(s)), \bar{g}_{H(s)}, \bar{\varrho}_{H(s)}, d_1, d_2)$$

with $\bar{g}_{H(s)} = (c(H(s)))^{-1} \sum_{x' \in H(s)} g(x')$, $\bar{\varrho}_{H(s)} = (c(H(s)))^{-1} \sum_{x' \in H(s)} \varrho(x', s)$, d_1 and d_2 are two two-valued parameters first of them expressing the fact whether x is implied by $H(s)$ or not and the other expressing the fact whether $H(s)$ is consistent or not. The other parameters c_1 and K are not important now. The conditions are sufficient but not necessary, they do not satisfy, hence, the demands imposed to the decision function $\mathscr{D}$.

However, the situation is far from being simple and the criteria given above have some justification and are, in a sense, optimal. They are optimal among all the criteria based on the particular five criteria and satisfying the polynomial principle. Clearly, there is a real $h(x)$ for any formula x such that $g(x) \geqq h(x)$ if and only if $F(H(s) \cup \{x\}) \leqq F(H(s))$, however, in such a case $h(x)$ depends not only on the arguments given in the description of the function f but also on arguments $\text{Cons}(H(s))$ and $\text{Ax}(H(s))$. As there does not exist, in general, a polynomial procedure enabling to find these subsets of $H(s)$ the criterion based on the value $h(x)$ does not satisfy the polynomial principle.

This way of reasoning makes clear, why we use the value $K^{c(H(s))}$ and not $K^{c(\text{Ax}(H(s)))}$ as the degree of semantic completeness. Using the last, more intuitive one, we should come either to the criteria we have obtained or to criteria not satisfying the polynomial principle.

The obtained criteria have another advantage, namely from the point of view of local principle. In the following chapter we shall solve the problem how to use our criteria not having at our disposal all the set $H(s)$ but just a random sample $H_0(s, \omega) \subset H(s)$. Using the averages $\bar{g}_{H_0(s,\omega)}$ and $\bar{\varrho}_{H_0(s,\omega)}$ instead of $\bar{g}_{H(s)}$ and $\bar{\varrho}_{H(s)}$ we can apply well-known results of probability theory and investigate the probability

with which the decision taken with respect to a random sample $H_0(s, \omega)$ improves the F-quality of the formal representation $H(s)$. This is connected with another advantages of our criteria; when computing the treshold value we do not need to compute the value of the F-criterion neither for $H(s) \cup \{x\}$ nor for $H(s)$. Such a computation would be of exponential complexity, in general, and as such is not allowed to be a part of our decision procedure.

4. A STATISTICAL APPROXIMATION OF THE DECISION RULES

Let us return to the local principle neglected in the foregoing chapter and try to modify the obtained criteria to be consistent with this principle. Let $\beta_1, \beta_2, \ldots$ be random variables defined on the probability space $(\Omega, \mathscr{S}, P)$, taking their values in the set $H(s)$ of formulas, mutually independent, equally distributed and such that

$$P(\{\omega : \beta_1(\omega) = x\}) = (c(H(s)))^{-1}$$

for any $x \in H(s)$. More precisely, we should write $\beta_{s,i}$ instead of β_i. Define random variables

$$g_i(\omega) \quad = g(\beta_i(\omega)),$$
$$\varrho_i(s, \omega) = \varrho(s, \beta_i(\omega)),$$
$$r_i(s, \omega) = \varrho_i(s, \omega) + c_1.$$

Using the usual notations for expected values and dispersions we have:

$$Eg_i = \sum_{x \in H(s)} g(x) . P(\{\omega : \beta_i(\omega) = x\}) = (c(H(s)))^{-1} \sum_{x \in H(s)} g(x),$$
$$E\varrho_i = (c(H(s)))^{-1} \sum_{x \in H(s)} \varrho(x, s),$$
$$Er_i = (c(H(s)))^{-1} \sum_{x \in H(s)} (\varrho(x, s) + c_1).$$

These values do not depend on i, so we can write Eg, Er and $E\varrho$. The criteria obtained in the foregoing chapter may be written in the form

$$g(x) \geq f(Eg, E\varrho, \ldots).$$

The local principle does not allow to use the values Eg, Er, $E\varrho$, however, the immediate idea is to approximate these values by random means, i.e. by their average values computed from random samples. In general, the values of the random variables ϱ_i and g_i can be correlated, so we set, for a natural n:

$$\bar{g}_n = n^{-1} \sum_{i=1}^{n} g(\beta_i(\omega)),$$
$$\bar{r}_n = n^{-1} \sum_{i=n+1}^{2n} r(\beta_i(\omega)).$$

Our aim, in this chapter, will be to find for any criterion derived above, a constant $K \geqq 1$ or an $\varepsilon \geqq 0$ such that either

$$P(\{\omega : K\, f(\bar{g}_n(\omega), \bar{r}_n(\omega), \ldots) \geqq f(Eg, Er, \ldots)\}) \geqq 1 - \delta$$

or

$$P(\{\omega : f(\bar{g}_n(\omega), \bar{r}_n(\omega), \ldots) + \varepsilon \geqq f(Eg, E_r, \ldots)\}) \geqq 1 - \delta$$

with δ given a priori.

Theorem 7.

$$P\left(\left\{\omega : \frac{1 + \sqrt{[(D^2g/(Eg)^2 + D^2r/(Er)^2)\,(2\delta n)^{-1}]}}{1 - \sqrt{[(D^2g/(Eg)^2 + D^2r/(Er)^2)\,(2\delta n)^{-1}]}}\,\left(\varrho(s, s'') + c_1\right)\frac{\bar{g}_n(\omega)}{\bar{r}_n(\omega)} \geqq\right.\right.$$
$$\left.\left. \geqq \left(\varrho(s, s'') + c_1\right)\frac{Eg}{Er}\right\}\right) \geqq 1 - \delta\,.$$

Proof. If X is a random variable with a finite expected value EX and a finite dispersion D^2X, if $\varepsilon > 0$, then the well-known Tchebyshev inequality sounds:

$$P(\{\omega : |\overline{X} - EX| < \varepsilon\}) \geqq 1 - (D^2X)/n\varepsilon^2\,,$$

where $\overline{X}(\omega) = n^{-1}\sum_{i=1}^{n} X_i(\omega)$ is the sample average value obtained on the basis of n mutually independent realizations of the random variable X. In a proof of this inequality, see, e.g. [2], we can find the two following inequalities which are also valid:

$$P(\{\omega : \overline{X}(\omega) - EX < \varepsilon\}) \geqq 1 - (2^{-1}D^2X)/n\varepsilon^2\,,$$
$$P(\{\omega : EX - \overline{X}(\omega) < \varepsilon\}) \geqq 1 - (2^{-1}D^2X)/n\varepsilon^2\,.$$

Hence,

$$P(\{\omega : \bar{g}_n(\omega) > (1 - \varepsilon)\,Eg\}) = P(\{\omega : \bar{g}_n(\omega) > Eg - \varepsilon Eg\}) \geqq 1 - \frac{1}{2}\left(\frac{D^2g}{n(\varepsilon Eg)^2}\right)\,.$$

and, in an analogous way, for the random variable r. The statistical independence of g_n and r_n gives

$$P\left(\left\{\omega : \frac{1 + \varepsilon}{1 - \varepsilon}\left(\varrho(s, s'') + c_1\right)\frac{\bar{g}_n}{\bar{r}_n} > \left(\varrho(s, s'') + c_1\right)\frac{Eg}{Er}\right\}\right) =$$

$$= P\left(\left\{\omega : \frac{1 + \varepsilon}{1 - \varepsilon}\frac{\bar{g}_n}{\bar{r}_n} > \frac{Eg}{Er}\right\}\right) \geqq P(\{\omega : \bar{g}_n > (1 - \varepsilon)\,Eg\})\, P(\{\omega : \bar{r}_n < (1 + \varepsilon)\,Er\}) \geqq$$

$$\geqq \left[1 - \frac{1}{2}\left(\frac{D^2g}{n(\varepsilon Eg)^2}\right)\right]\left[1 - \frac{1}{2}\left(\frac{D^2r}{n(\varepsilon Er)^2}\right)\right] \geqq 1 - \frac{1}{2n\varepsilon^2}\left(\frac{D^2g}{(Eg)^2} + \frac{D^2r}{(Er)^2}\right)\,.$$

The last expression is at least $1 - \delta$ if and only if

$$\varepsilon \geqq \sqrt{[(2\delta n)^{-1} (D^2 g \mid (Eg)^2 + D^2 r \mid (Er)^2)]}$$

and, substituting this value into the inequality

$$P\left(\left\{\omega : \frac{1 + \varepsilon}{1 - \varepsilon} \left(\varrho(s, s'') + c_1\right) \frac{\bar{g}_n}{\bar{r}_n} > \left(\varrho(s, s'') + c_1\right) \frac{Eg}{Er}\right\}\right) \geqq 1 - \delta$$

we obtain the desired result. Q.E.D.

For the sake of a simplification let us denote

$$K(r, g, \delta, n) = \frac{1 + \sqrt{[(D^2 g \mid (Eg)^2 + D^2 r \mid (Er)^2)(2\delta n)^{-1}]}}{1 - \sqrt{[(D^2 g \mid (Eg)^2 + D^2 r \mid (Er)^2)(2\delta n)^{-1}]}}.$$

Theorem 8. Let $\beta_1, \beta_2, \ldots$ be the same random variables as above, let $n_1 = n_1(\omega)$ denote the number of different formulas among $\beta_1(\omega), \beta_2(\omega), \ldots, \beta_{2n}(\omega)$. Then

$$P\left(\left\{\omega : K(r, g, \delta/2, n) \left(1 + n_1^{-1}\right)^2 \left(\varrho(s, s'') + c_1\right) \frac{\bar{g}_n}{\bar{r}_n} + \left(2 + 2n_1^{-1}\right) \bar{g}_n + \right.\right.$$

$$+ \left(2 + 2n_1^{-1}\right) \sqrt{[D^2 g (\delta n)^{-1}]} \geqq$$

$$\geqq \left(\sum_{x' \in H(s)} g(x')\right) \left[\left(1 + \frac{\varrho(s, s'') + c_1}{\sum_{x' \in H(s)} (\varrho(x', s) + c_1)}\right) \left(1 + (c(H(s)))^{-1}\right)^2 - 1\right]\right\}\right) \geqq 1 - \delta.$$

Proof. A simple calculation gives

$$\left(\sum_{x' \in H(s)} g(x')\right) \left[\left(1 + \frac{\varrho(s, s'') + c_1}{\sum_{x' \in H(s)} (\varrho(x', s) + c_1)}\right) \left(1 + (c(H(s)))^{-1}\right)^2 - 1\right] =$$

$$= \left(\frac{Eg}{Er}\right) \left[\frac{2c_2}{c(H(s))} + c_2 + \frac{c_2}{(c(H(s)))^2}\right] + Eg \left(2 + \frac{2}{c(H(s))}\right),$$

where $c_2 = \varrho(s, s'') + c_1$. As $n_1 \leqq c(H(s))$, this expression can be dominated by

$$\left(\frac{Eg}{Er}\right) c_2 \left(1 + n_1^{-1}\right) + Eg(2 + 2n_1^{-1}).$$

Now, if we choose K_0 and K_2 such that, simultaneously,

$$P\left(\left\{\omega : K_0 \left(1 + n_1^{-1}\right)^2 c_2 \frac{\bar{g}_n}{\bar{r}_n} \geqq \left(1 + n_1^{-1}\right)^2 c_2 \frac{Eg}{Er}\right\}\right) \geqq 1 - (\delta/2),$$

$$P\left(\left\{\omega : \left(2 + 2n_1^{-1}\right) \bar{g}_n + K_2 \geqq \left(2 + 2n_1^{-1}\right) Eg\right\}\right) \geqq 1 - (\delta/2),$$

then, with a probability greater than or equal to $1 - \delta$ the inequalities hold simultaneously, which assures the validity of the theorem. The only thing which rests is to find K_0 and K_2. Clearly $K_0 = K(r, g, \delta/2, n)$ copes with the demands. Tchebyshev inequality gives

$$P(\{\omega : (2 + 2n_1^{-1})\, \bar{g}_n + K_2 \geqq (2 + 2n_1^{-1})\, Eg\}) \geqq 1 - \tfrac{1}{2} D^2 g\big(n K_2^2 (2 + 2n_1^{-1})^2\big)^{-1}$$

and our demands are satisfied, if

$$K_2 \geqq (2 + 2n_1^{-1}) \sqrt{[D^2 g(\delta n)^{-1}]}\,.$$

Setting these values into the derived inequality we obtain the assertion. Q.E.D.

Theorem 9.
$$P(\{\omega : 3\bar{g}n + 3(\sqrt{2})^{-1} \sqrt{[D^2 g(\delta n)^{-1}]} > 3Eg\}) \geqq 1 - \delta\,,$$
$$P(\{\omega : \bar{g}n + (\sqrt{2})^{-1} \sqrt{[D^2 g(\delta n)^{-1}]} > Eg\}) \geqq 1 - \delta\,.$$

Proof. Analogous with that of the second part of the foregoing theorem, just δ instead of $\delta/2$ is used. Q.E.D.

The three following assertions are immediate consequences of the foregoing theorems and those of the last chapter.

Theorem 10. Let the conditions of Theorem 3 hold, let $\delta < 0$ be given. If
$$g(x) \geqq K(r, g, \delta, n)\, (\varrho(s, s'') + c_1)\, \bar{g}_n (\bar{r}_n)^{-1}\,,$$
then with a probability at least $1 - \delta$ $F(H(s) \cup \{x\}) \leqq F(H(s))$.

Proof. An immediate consequence of Theorems 3 and 7. Q.E.D.

Theorem 11. Let the conditions of Theorems 4 and 8 hold, let $\delta > 0$ be given. If
$$g(x) \geqq K(r, g, \delta/2, n)\, (1 + n_1^{-1})^2\, (\varrho(s, s'') + c_1) \frac{\bar{g}_n}{\bar{r}_n} + $$
$$+ (2 + 2n_1^{-1})\, (\bar{g}_n + \sqrt{[D^2 g(\delta n)^{-1}]}\,,$$
then with a probability at least $1 - \delta$ $F(H(s) \cup \{x\}) \leqq F(H(s))$.

Proof. An immediate consequence of Theorems 4 and 8. Q.E.D.

Theorem 12. Let the conditions of Theorem 6 hold, let as least one of the two following pairs of conditions hold:

(I) $$\varrho(s, s'') \leqq (c(H(s)))^{-1} \sum_{x' \in H(s)} \varrho(x', s)\,,$$

$$g(x) \geqq 3\bar{g}_n + 3(\sqrt{2})^{-1} \sqrt{[D^2 g(\delta n)^{-1}]}\,,$$

(II) $$s = s'', g(x) \geqq \bar{g}_n + (\sqrt{2})^{-1} \sqrt{[D^2 g(\delta n)^{-1}]}\,.$$

Let $\delta > 0$ be given. Then with a probability at least $1 - \delta$ $F(H(s) \cup \{x\}) \leqq F(H(s))$.

146   Proof. An immediate consequence of Theorems 6 and 9. Q.E.D.

The treshold values of the criteria derived above depend on Eg, Er, D^2g and D^2r the a priori knowledge of which cannot be assumed. This fact is a weak point of the criteria in question and we propose two possibilities how to overcome it, at least partially.

(I) Let the random variables g and r be positive and bounded, i.e.

$$0 < \min\{g(x) : x \in H(s)\} \leq \max\{g(x) : x \in H(s)\} < \infty,$$

and the same for r. The value D^2g would be maximal, if

$$P(\{\omega : g_1(\omega) = \min\{g(x)\}\}) = P(\{\omega : g_1(\omega) = \max\{g(x)\}\}) = 1/2,$$

so

$$D^2g \leq (1/4)(\max\{g(x)\} - \min\{g(x)\})^2.$$

Setting this upper bound instead of D^2g and $\min\{g(x)\}$ instead of Eg into the treshold values in Theorems $10-12$ (and analogously for D^2r and Er) we obtain that the assertion $F(H(s) \cup \{x\}) \leq F(H(s))$ holds again with a probability at least $1 - \delta$.

(II) The other possibility consists in a former estimation of D^2g, Eg, D^2r and Er on the basis of random samples and average values originating from them. Practically, such an approach is justifiable if the extent of the random samples is large enough to give a certain assurance for the validity of such conclusions. From theoretic point of view, however, the probability with which the F-quality of $H(s)$ is improved or not is itself a random variable some estimations of which request, again, the knowledge of D^2g, Eg, D^2r and Er. The algorithm proposed below is constructed in such a way that the various possibilities of approximating the values D^2g, Eg, D^2r and Er can be choosen by an appropriate choice of a parameter.

Our criteria, derived and investigated above, have a common feature which can be called stability principle. New formulas are joined with $H(s)$ only in case we are sure that $F(H(s))$ will not become worse (global criteria) or at least the probability of this event is great enough. In other cases we systematically prefer not to change the formal representation.

In Chapter 2 we mentioned the possibility to transform, after a number of steps, all the formal representation $H(s)$ in order to improve its F-quality. We may transform this case to that of sequential testing one formula after another in such a way that the random variables α_1, α_2, ... are allowed to take their values also in the set $H(s)$. If $\alpha_n(\omega) \in H(s) - H_n(s, \omega)$, take $H'(s) = H(s) - \{\alpha_n(\omega)\}$ and apply the criteria to $H'(s)$ and $\alpha_n(\omega)$. Now, the decision "do not join $\alpha_n(\omega)$ with $H'(s)$" is nothing else than "erase $\alpha_n(\omega)$ from $H(s)$". Such a way of reasoning enables, e.g. to erase from $H(s)$ some formulas which were, in the past, joined with $H(s)$ but now there are superfluous being consequences of a more general formula joined with $H(s)$ later. On the other hand even some "individual" and easily derivable formulas can be let in $H(s)$ supposing they deserve it because of their extremal freshness or importance.

Before giving the decision schema two remarks seem to be appropriate. In the foregoing explanation we used the fact that the values Eg and Er are statistically estimable by their sample means. At the same time the changes of the formal representation $H(s)$ are always of local character — one formula is joined or deleted if there is any change at all. Immediately the idea arises to use the sample values $\bar{g}_n$ and $\bar{r}_n$ several times supposing the cardinality of $H(s)$ is aleready "large enough".

Let s, s', $s < s'$ be two situations; $\beta_1(s')$, $\beta_2(s')$, ..., $\beta_n(s')$ are random variables generating the random sample $H_0(s')$, $\beta_1(s)$, $\beta_2(s)$, ..., $\beta_n(s)$ are random variables generating the random sample $H_0(s)$. Then

$$\bar{g}_n(s) = n^{-1} \sum_{i=1}^{n} g(\beta_i(s, \omega)),$$

$$\bar{g}_n(s') = n^{-1} \sum_{i=1}^{n} g(\beta_i(s', \omega)),$$

similarly for $\bar{r}_n(s)$ and $\bar{r}_n(s')$.

Let us limit ourselves to the case when g and r are limited, i.e.

$$0 \leqq \min \{g(x)\} \leqq g(x) \leqq \max \{g(x)\} < \infty,$$

$$0 \leqq \min \{r(x)\} \leqq r(x) \leqq \max \{r(x)\} < \infty, \quad x \in H(s) \cup H(s').$$

Then $|\bar{g}_n(s) - \bar{g}_n(s')|$ is maximal just if the number of steps leading from s' to s i.e. $l(s) - l(s') = m$, is even, in $m/2$ steps a formula with the maximal value of g is deleted from $H(s)$ and in $m/2$ steps a formula with the minimal value of g is joined with $H(s)$ (of course, also the opposite case is possible). In such a case

$$\left|E_{H(s)}g - E_{H(s')}g\right| \leqq m\big(c(H(s')) + m\big)^{-1} (\max \{g\} - \min \{g\}),$$

which gives

$$\bar{g}_n(s') - \frac{(m - [m/2]) (\max \{g\} - \min \{g\})}{c(H(s')) + m - [m/2]} \leqq \bar{g}_n(s) \leqq \bar{g}_n(s') -$$

$$- \frac{(m - [m/2]) (\max \{g\} - \min \{g\})}{c(H(s')) + m - [m/2]}.$$

Analogous assertions can be found also for r and ϱ. From this and from Theorems 10−12 immediately follows

Theorem 13. Let us set

$$\bar{g}_n(s') + \frac{(m - [m/2]) (\max \{g\} - \min \{g\})}{c(H(s')) + m - [m/2]}$$

instead of $\bar{g}_n = \bar{g}_n(s)$ and

$$\bar{r}_n(s') + \frac{(m - [m/2]) (\max \{r\} - \min \{r\})}{c(H(s')) + m - [m/2]}$$

instead of $\bar{r}_n = \bar{r}_n(s)$ into inequalities derived in Theorems $10-12$. Then, again, the probability of not making the F-quality worse will be at least $1 - \delta$.

On the basis of these considerations we introduce into our procedure two parameters, K_0 and K_1, and we shall compute the sample means $\bar{g}_n(s)$ and $\bar{r}_n(s)$ only if $l(s) \leq K_1$ or if $l(s)$ is divisible by K_0; in other cases the former values $\bar{g}_n(s')$ and $\bar{r}_n(s')$ will be used.

Another problem is how to test the inconsistency and logical dependence in a formal representation $H(s)$. Let m_1, n be integers, consider random variables β_1, β_2, ..., $\beta_{m_1 n}$ and denote by B_i the formula

$$\beta_{(i-1)n+1}(\omega) \wedge \beta_{(i-1)n+2}(\omega) \wedge \ldots \wedge \beta_{in}(\omega).$$

Test, for any $i \leq m_1$, whether $B_i(\omega) \to F$, i.e. non B_i, is provable. If it is the case, $H(s)$ is inconsistent and any formula, including the tested formula x, is derivable from $H(s)$. If we are not able to prove $B_i(\omega) \to F$ no matter which $i \leq m_1$ is taken,
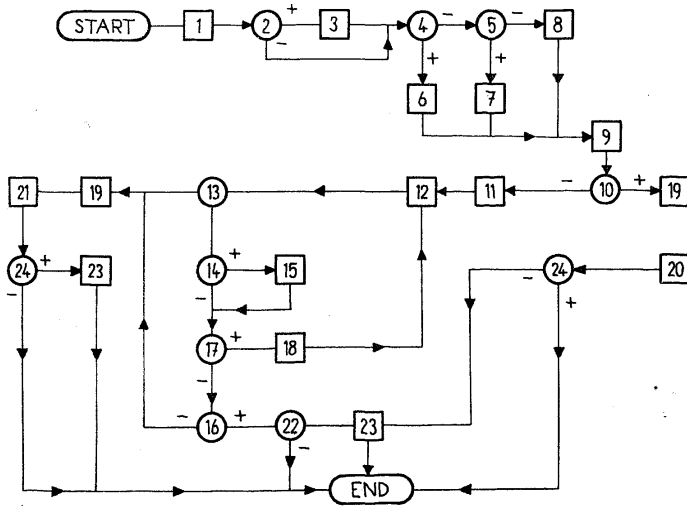

Fig. 1.

we try to test $B_i \to x$; if this answer is positive, we know that x follows from $H(s)$.
If we are not able to prove $B_i \to x$ no matter which $i \leqq m_1$ is taken, we proclaim $H(s)$ to be consistent and we proclaim, as well, that x does not follow from $H(s)$.

If $H(s)$ is proclaimed to be inconsistent or if x is proclaimed to follow from $H(s)$ both the decisions are, clearly, correct. In the opposite case the decisions are not always correct and there is a probability, in general positive, of error. However, this probability depends on $\mathrm{Ax}\left(H(s)\right)$ and $\mathrm{Cons}\left(H(s)\right)$. As these sets cannot be found, in general, using procedures of at most polynomial complexity, the computing of the probability or probabilities of error are not allowed to form a part of our decision procedure, hence, the decision procedure must not depend on these probabilities. It is why no expressions or estimation of these probabilities are studied in this paper.

The decision procedure is given in a form of a flowchart. Instructions are in squares, decisions in circles $(+:$ the condition is satisfied, $-:$ the condition is not satisfied). The procedure starts with entering of a new formula $\alpha(s, \omega)$ and ends with its (possible) joining with $H(s)$.

$\boxed{1}$ $x \leftarrow \alpha(s, \omega)$,
     $j \leftarrow l(s)$,
     $H(s) \leftarrow H(s) - \{x\}$.

2   Is $j \leqq K_1$ or j divisible by K_0?

$\boxed{3}$ $y_i \leftarrow \beta_i(\omega)$, $i = 1, 2, \ldots, n$,
     $z_i \leftarrow \beta_{i+n}(\omega)$, $i = 1, 2, \ldots, n$,

$$\bar{g}_n \leftarrow n^{-1} \sum_{i=1}^{n} g(y_i),$$

$$\bar{r}_n \leftarrow n^{-1} \sum_{i=1}^{n} \left(\varrho(z_i, s) + c_1\right).$$

4   $I = 1$?

     Comment: I is a formal parameter the possible value of which are 1, 2 or 3. This value must be given a priori and shows by which way the necessary estimations of Eg, Er, D^2g and D^2r are obtained.

5   $I = 2$?

$\boxed{6}$ $Eg \leftarrow (Eg)_0$, $Er \leftarrow (Er)_0$, $D^2g \leftarrow (D^2g)_0$, $D^2r \leftarrow (D^2r)_0$.

$\boxed{7}$ $Eg \leftarrow \min\{g\}$, $D^2g \leftarrow (1/4)\left(\max\{g\} - \min\{g\}\right)^2$,
     $Er \leftarrow \min\{r\}$, $D^2r \leftarrow (1/4)\left(\max\{r\} - \min\{r\}\right)^2$.

$\boxed{8}$ $Eg \leftarrow \bar{g}_n$, $D^2g \leftarrow n^{-1} \sum_{i=1}^{n} \left(g(y_i) - \bar{g}_n\right)^2$,

     $Er \leftarrow \bar{r}_n$, $D^2r \leftarrow n^{-1} \sum_{i=1}^{n} \left(r(z_i) - \bar{r}_n\right)^2$.

$\boxed{9}$ $n_1 \leftarrow \mathrm{card}\left(\bigcup_{i=1}^{n} \{z_i, y_i\}\right)$.

Comment: n_1 is the number of different formulas in the random sample $\{y_1, y_2, \ldots, y_n, z_1, z_2, \ldots, z_n\}$.

10 $\vdash \left(\bigwedge\limits_{i=1}^{n} y_i\right) \to F?$

11 $k \leftarrow 1, L \leftarrow 0.$

12 $z_{kj} \leftarrow \beta_{nk+j}(\omega), \quad j = 1, 2, \ldots, n.$

13 $\vdash \left(\bigwedge\limits_{j=1}^{n} z_{kj}\right) \to F?$

14 $\vdash \left(\bigwedge\limits_{j=1}^{n} z_{kj}\right) \to x?$

15 $L \leftarrow 1$

Comment: L is a formal parameter without any interpretation.

16 $L = 1?$

17 $k < m?$

18 $k \leftarrow k + 1$

19 $K \leftarrow \dfrac{1 + \sqrt{\left[(D^2 g \mid (Eg)^2 + D^2 r \mid (Er)^2\right)(2\delta n)^{-1}\right]}}{1 - \sqrt{\left[(D^2 g \mid (Eg)^2 + D^2 r \mid (Eg)^2\right)(2\delta n)^{-1}\right]}}$

20 $M \leftarrow K(1 + n_1^{-1})^2 \left(\varrho(s, s'') + c_1\right) \bar{g}_n(\bar{r}_n)^{-1} + 2(1 + n_1^{-1}) \left(\bar{g}_n + \sqrt{\left[D^2 g (\delta n)^{-1}\right]}\right)$

Comment: s'' in 20 – 22 denotes the situation term in the formula x.

21 $M \leftarrow K(\varrho(s, s'') + c_1) \bar{g}_n(\bar{r}_n)^{-1}.$

22 $\{(\varrho(s, s'') \leqq Er - c_1) \wedge (g(x) \geqq 3(\bar{g}_n + (\sqrt{2})^{-1} \sqrt{[D^2 g (\delta n)^{-1}]}))\} \vee \{(s = s'') \wedge$
     $\wedge (g(x) \geqq (\sqrt{2})^{-1} \sqrt{[D^2 g (\delta n)^{-1}]})\}?$

23 $H(s) \leftarrow H(s) \cup \{x\}.$

24 $g(x) \geqq M?$

(Received February 3, 1976.)

REFERENCES

[1] M. O. Rabin: Theoretical Impediments to Artificial Intelligence. In: Information Processing 1974. North Holland Publ. Comp., Amsterdam, 1974, 615—619.

[2] A. Rényi: Teorie pravděpodobností. Academia, Praha 1972.

[3] I. Kramosil: A Probabilistic Approach to Automaton-Environment Systems. Kybernetika *11* (1975), 3, 173—206.

[4] I. Kramosil: Pravděpodobnostní model vztahu mezi automatem a okolím. In: Aplikovaná robotika. České Budějovice, 1975, vol. II, XI—XII.

Dr. Ivan Kramosil, CSc., Ústav teorie informace a automatizace ČSAV (Institute of Information Theory and Automation — Czechoslovak Academy of Sciences), Pod vodárenskou věží 4, 180 76 Praha 8. Czechoslovakia