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AN APPROXIMATION OF THE PRESSURE FOR THE TWO-DIMENSIONAL ISING MODEL

MARTIN JANŽURA

A sequence of pressure functions corresponding to some one-dimensional models is used to approximate the pressure function of the two-dimensional Ising model. The rate of convergence is derived and the method is demonstrated with a numerical study.

1. INTRODUCTION

The two-dimensional Ising model is the simplest non-trivial Gibbs random field. Namely, a probability measure μ on the space $\{0, 1\}^{\mathbb{Z}^2}$ is called to agree with the Ising model if its one-dimensional conditional distributions satisfy the "nearest-neighbor" property and can be expressed in the following way

$$\mu(x_t | x_{Z^2 \setminus \{t\}}) = \mu(x_t | x_{\partial t}) = \Pi_t(x_t | x_{Z^2 \setminus \{t\}})$$

for every $t \in Z^d$ and a. e. $x \in \{0, 1\}^{\mathbb{Z}^d}$, where

$$\Pi_t(x_t | x_{Z^2 \setminus \{t\}}) = \frac{\exp\{-x_t(h + J_1(x_{t+u} + x_{t-u}) + J_2(x_{t+v} + x_{t-v}))\}}{1 + \exp\{-h - J_1(x_{t+u} + x_{t-u}) - J_2(x_{t+v} + x_{t-v})\}}$$

are called the local characteristics,

$$\partial t = \{s \in Z^2; \|t - s\| = 1\} = \{u, -u, v, -v\}, \quad u = (1, 0), \quad v = (0, 1),$$

and h, J_1, J_2 are arbitrary constants.

In general, the system $\{\Pi_t(\cdot | \cdot)\}_{t \in Z^2}$ depending on the triplet (h, J_1, J_2) does not determine the probability measure μ uniquely. The existence, uniqueness, and other properties of the Ising model are closely related to the function called the pressure and defined by the limit

$$\begin{aligned} & \lim_{V \nearrow Z^2} |V|^{-1} \log \sum_{x_V \in \{0,1\}^V} \exp \left\{ -h \sum_{t \in V} x_t - J_1 \sum_{t \in V \cap (V-u)} x_t x_{t+u} - J_2 \sum_{t \in V \cap (V-v)} x_t x_{t+v} \right\} = \\ & = p(h, J_1, J_2) \end{aligned}$$

where $V \nearrow Z^2$ means the expansion ensuring $|V|^{-1} |V \cap (V - t)| \rightarrow 1$ for every $t \in Z^2$. (By $|V|$ we denote the cardinality.)

But, with the exception of the famous Onsager's result (cf. [3]), concerning a special case of the problem, no direct way of calculating the pressure p is known. Therefore various approximative methods, using mostly some kind of expansion, are applied. Here, we propose a new approximative method based on an approximation of the pressure of the two-dimensional model by the pressure of some properly chosen one-dimensional models, for which the transfer matrix method is available (cf. [2]).

As will be seen later, the method works quite well in the "high temperature" area (i. e. for "small" parameters h, J_1, J_2) and even in the neighborhood of the critical point it seems to give satisfactory results.

2. BASIC LEMMA

For a fixed positive integer R and a real γ let us consider the two-dimensional model with the state space $\bar{X} = \{0, 1\}^R$ and the "nearest-neighbor" local characteristics given by

$$\bar{\Pi}_t^{\gamma}(\bar{x}_t | \bar{x}_{Z^2 \setminus \{t\}}) = \frac{\exp \left\{ -U_{\gamma}^0(\bar{x}_t) - \sum_{s \in \partial t} U_{\gamma}^s(\bar{x}_t, \bar{x}_{t+s}) \right\}}{\sum_{\bar{y}_t \in \bar{X}} \exp \left\{ -U_{\gamma}^0(\bar{y}_t) - \sum_{s \in \partial t} U_{\gamma}^s(\bar{y}_t, \bar{x}_{t+s}) \right\}}$$

for every $\bar{x}_t \in \bar{X}, \bar{x}_{\partial t} \in \bar{X}^{\partial t}$, where

$$\begin{aligned} U_{\gamma}^0(\bar{x}) &= h \cdot \sum_{i=1}^R \bar{x}^i + J_1 \sum_{i=1}^{R-1} \bar{x}^i \bar{x}^{i+1}, \\ U_{\gamma}^u(\bar{x}, \bar{y}) &= \gamma \cdot J_1 \cdot \bar{x}^R \bar{y}^1, \quad U^{-u}(\bar{x}, \bar{y}) = U^u(\bar{y}, \bar{x}), \\ U_{\gamma}^v(\bar{x}, \bar{z}) &= J_2 \cdot \sum_{i=1}^R \bar{x}^i \bar{z}^i + (1 - \gamma) J_1 \bar{x}^R \bar{z}^1, \quad U^{-v}(\bar{x}, \bar{z}) = U^v(\bar{z}, \bar{x}), \end{aligned}$$

for every $\bar{x}, \bar{y}, \bar{z} \in \bar{X}$.

Let us denote by $G_t(\gamma)$ the set of translation invariant probability distributions on \bar{X}^{Z^2} with the one-dimensional conditional distributions equal a. s. to the local characteristics $\{\bar{\Pi}_t^{\gamma}\}_{t \in Z^2}$.

Finally, we denote by

$$\begin{aligned} \bar{p}_{\gamma}(h, J_1, J_2) &= \\ &= \lim_{V \nearrow Z^2} |V|^{-1} \log \sum_{\bar{x}_V \in \bar{X}^V} \exp \left\{ - \sum_{t \in V} U_{\gamma}^0(\bar{x}_t) - \sum_{t \in V \cap (V-u)} U_{\gamma}^u(\bar{x}_t, \bar{x}_{t+u}) - \sum_{t \in V \cap (V-v)} U_{\gamma}^v(\bar{x}_t, \bar{x}_{t+v}) \right\} \end{aligned}$$

the pressure corresponding to above defined model.

Lemma. Let $\gamma^* \in [0, 1]$ be the point at which the function

$$F(\gamma) = \gamma \bar{p}_1(h, J_1, J_2) + (1 - \gamma) \bar{p}_0(h, J_1, J_2) - \bar{p}_\gamma(h, J_1, J_2)$$

assumes its maximum. Then there exists

$$\mu^* \in G_I(\gamma^*)$$

such that

$$\bar{p}_1(h, J_1, J_2) - \bar{p}_0(h, J_1, J_2) = J_1 [\mu^*(\bar{x}_0^R \cdot \bar{x}_u^1 = 1) - \mu^*(\bar{x}_0^R \cdot \bar{x}_v^1 = 1)]$$

holds.

Proof. The statement follows immediately from the equivalence between translation invariant Gibbs states and tangent functionals to the convex functional p (cf. [4], Thm. 8.3) and the general subdifferential calculus (cf. e. g. [5], Sec. 5).

3. MAIN RESULT

Now, let us make clear what was the aim of introducing the models with the “aggregated” state space \bar{X} in the preceding section.

Directly from the definitions it is easy to see that

$$\bar{p}_1(h, J_1, J_2) = R \cdot p(h, J_1, J_2)$$

holds for every triplet (h, J_1, J_2) .

Since for $\gamma = 0$ there is no horizontal interaction, i. e. the model consists of mutually independent columns, we may view the model as a one-dimensional one. And, considering all \bar{x}_t , $t \in Z$ as the corresponding segments of a sequence $x_Z = \{x_s\}_{s \in Z} \in \{0, 1\}^Z$ (we put $x_s = \bar{x}_t^1$ for $s = t \cdot R + i$), we conclude that

$$\bar{p}_0(h, J_1, J_2) = R \cdot p^R(h, J_1, J_2),$$

where

$$p^R(h, J_1, J_2) = \lim_{n \rightarrow \infty} [2n + 1]^{-1} \log \sum_{x_{[-n, n]} \in \{0, 1\}^{[-n, n]}} \exp \left\{ -h \sum_{j=-n}^n x_j - J_1 \sum_{i=-n}^{n-1} x_i x_{i+1} - J_2 \sum_{i=-n}^{n-R} x_i x_{i+R} \right\}$$

is the pressure of the one-dimensional model with the state space $\{0, 1\}$ and the local characteristics

$$\Pi_t^0(x_t | x_{Z \setminus \{t\}}) = \frac{\exp \{-h x_t - J_1 x_t (x_{t+1} + x_{t-1}) - J_2 (x_{t-R} + x_{t+R})\}}{1 + \exp \{-h - J_1 (x_{t+1} + x_{t-1}) - J_2 (x_{t-R} + x_{t+R})\}},$$

for every $t \in Z$, $x_t \in \{0, 1\}$, $x_{Z \setminus \{t\}} \in \{0, 1\}^{Z \setminus \{t\}}$.

Now, we may formulate the main result on the approximation.

Theorem. For every triplet (h, J_1, J_2) it holds

$$|p(h, J_1, J_2) - p^R(h, J_1, J_2)| \leq (2R)^{-1} |J_1|,$$

and therefore

$$p(h, J_1, J_2) = \lim_{R \rightarrow \infty} p^R(h, J_1, J_2).$$

Proof. The statement follows from Lemma and the considerations above if we realize that the probability measures

$$\nu_u(x, y) = \mu^*(\bar{x}_0^R = x, \bar{x}_u^1 = y), \quad x, y \in \{0, 1\},$$

and

$$\nu_v(x, y) = \mu^*(\bar{x}_0^R = x, \bar{x}_v^1 = y), \quad x, y \in \{0, 1\}$$

have the same marginals, and therefore

$$|\nu_u(1, 1) - \nu_v(1, 1)| \leq \frac{1}{2}.$$

□

Remark. The values of p^R may be calculated with the aid of the transfer matrix (for details see e.g. [2], Section I.2.1). Of course, actually we are able to calculate p^R for rather small R only. But the convergence is, in fact, quite fast, and even $R = 6$ or $R = 7$, especially in high temperature area (i.e. for rather small interactions), give nice results.

4. NUMERICAL STUDY

Now, we try to demonstrate the method with a particular case which has been chosen in order to make possible a comparison of the obtained results with the rigorous Onsager's one.

Therefore, let $J_1 = J_2 = J \geq 0$ and $h = -2J$.

For $R = 4, 5, 6, 7$ and some $J \in [0, 2]$ the values of $p^R(-2J, J, J)$ obtained by the transfer matrix method (cf. [2], Section I.2.1) are given in the table.

	$J = 0$	$J = 0.5$	$J = 1.0$	$J = 1.5$	$J = 2 \cdot \log(1 + \sqrt{2})$	$J = 2$
$R = 4$	0.6931	0.9589	1.2579	1.5916	1.7800	1.9568
$R = 5$	0.6931	0.9590	1.2595	1.6085	1.8213	2.0320
$R = 6$	0.6931	0.9590	1.2590	1.5999	1.7968	1.9841
$R = 7$	0.6931	0.9590	1.2591	1.6051	1.8158	2.0297

Here, for the critical point $J_c = 2 \log(1 + \sqrt{2})$ the exact Onsager's solution gives

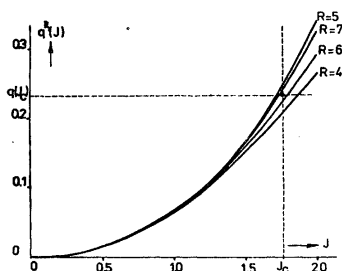
$$p(-2J_c, J_c, J_c) = \log(1 + \sqrt{2}) + \log 2/2 + 2 \cdot G/\pi \doteq 1.8110692$$

($G = 0.915965594$ is the Catalan's constant).

Trying to make differences between the functions p^R for various R 's more evident, we deal with their deviations

$$q^R(J) = p^R(-2J, J, J) - \log 2 - J/2$$

from the line $\log 2 + J/2$ (i. e. their common tangent in $J = 0$) in the following figure.



Similarly, we denote $q(J_c) = p(-2J_c, J_c, J_c) - \log 2 - J_c/2 \doteq 0.2365$.

5. CONCLUDING REMARK

Approximation of the described type was at first derived in [1] for purpose of application in mathematical statistics. But here a completely different proof is used, which yields a stronger result and deeper insight into the problem.

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