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AN APPROACH FOR POLE ASSIGNMENT BY GAIN OUTPUT FEEDBACK IN SQUARE SINGULAR SYSTEMS

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This study deals with the problem of (finite) pole assignment in a linear, time-invariant, multivariable, singular system $E\dot{x} = Ax + Bu$, with output $y = Cx$, via a gain output feedback of the form $u = Ky + r$ that preserves the uniqueness property. It is shown that the problem of pole assignment in singular and regular systems are closely related from both analysis and synthesis points of view. The use of an appropriate transformation group enables one to apply the following approach: first to design a gain output feedback in a regular (rather than in a singular) system, and then to incorporate the output feedback into the original singular system while preserving the spectra obtained in the regular system. The present approach bridges the gap between the relevant theory in regular and singular systems, and simplifies the mechanism for the evaluation of a suitable gain output feedback in a given singular system.

1. INTRODUCTION

The basic theory of singular systems is by now fairly well established. Various concepts such as controllability, stabilization, observability and duality, and pole assignment by state feedback, have been extended from regular system theory to singular systems.

However, the problem of (finite) pole assignment by gain output feedback in singular systems, which is of great importance, has not received much attention. Since the complete state observation is not available in most practical situations, it is essential to find the conditions under which the system is pole-assignable with incomplete state measurements. To some extent, the use of an observer can provide an approach for assigning the closed-loop poles. In that regard [1] constructs an observer, which can measure the states of a strongly observable singular system in an asymptotic sense. In [2] a compensator design procedure is presented for strongly controllable and strongly observable singular systems, which eliminates impulsive modes and assigns the closed-loop poles to the specified points on the complex plane. Results concerning the problem of eigenstructure assignment by output feedback in singular systems have been considered in [3]. However, a general solution of the problem of pole assignment by gain output feedback only which ensures the uniqueness property, i.e., closed-loop regularity, is yet unknown.

The purpose of this study is to derive an approach for solving the problem of pole assignment by gain output feedback. An equivalent relation between the problem of pole assignment by gain output feedback in singular and regular systems has been established. Practically, this relation implies that the design of an admissible output gain feedback in a given singular system that satisfies a closed-loop regularity can be achieved through a design of output feedback in an appropriate regular system. (The use of regular system model for solving the problem of pole assignment by gain state feedback in singular system, has been presented by Ailon [4].) The significance of this approach is that well established results (Davison [5], Kimura [6], Davison and Wang [7]) and algorithms from the regular systems theory become useful tools for pole assignment by gain output feedback in singular systems.

Consider a linear singular system

$$E\dot{x} = Ax + Bu \tag{1.1a}$$

$$y = Cx \tag{1.1b}$$

where $x \in \mathbb{R}^n$; $u \in \mathbb{R}^m$; $y \in R^p$; $E, A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$; $C \in \mathbb{R}^{p \times n}$. We assume that the pair $\{E, A\}$ is solvable, i.e.,

$$\det(\lambda E - A) \neq 0\,, \quad \text{for almost all } \lambda\,. \tag{1.2}$$

Definition 1.1. A gain output feedback

$$u = Ky + r\,, \tag{1.3}$$

is admissible if and only if the pair $\{E, A + BKC\}$ is solvable.

The following definitions are taken from Yip and Sincovec [8]:

Definition 1.2. The system (1.1) is completely controllable (C-controllable) if one can reach any state from any initial state.

Definition 1.3. The system (1.1) is controllable within the set of reachable states (R-controllable) if one can reach any state in the set of reachable states from any admissible initial state.

Definition 1.4. We say that the system (1.1) is observable if and only if, for $t \geqq 0$, $x(t)$ can be computed from $E, A, B, C, y(t^\wedge)$ and $u(t^\wedge)$ for any $t^\wedge \in [0, b]$, $b > 0$.

Theorem 1.1. [8] a) The system (1.1) is R-controllable if and only if the augmented matrix $[\lambda E - AB]$ is of full rank. b) The system (1.1) is C-controllable if and only if the augmented matrices $[\lambda E - A\ B]$ and $[E\ B]$ are of full rank. c) The system (1.1) is observable if and only if the augmented matrix $[(\lambda E - A)^{\mathrm{T}}\ C^{\mathrm{T}}]$ is of full rank.

The following well known theorem from complex-function theory, is useful in this study:

Rouche's theorem. If $f(\lambda)$ and $g(\lambda)$ are analytic inside and on a simple closed curve Γ and if $|g(\lambda)| < |f(\lambda)|$ on Γ, then $f(\lambda) + g(\lambda)$ and $f(\lambda)$ have the same number of zeroes inside Γ.

2. THE MAIN RESULTS

Let $P_f, P_r \in \mathbb{R}^{n \times n}$ be regular matrices, and define

$$P_f E P_r \dot{z} = P_f A P_r z + P_f B u ;$$
$$y = C P_r z .$$

Since $\det(\lambda_i P_f E P_r - (P_f A P_r + P_f BKCP_r)) = 0 \Leftrightarrow \det(\lambda_i E - (A + BKC)) = 0$, we may assume without loss of generality, that the matrices E, A, B, and C in (1.1) either have the following form

$$\{E, A, B, C\} = \left\{ \begin{bmatrix} I_{n-k} & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}, \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, [C_1, C_2] \right\} \tag{2.1}$$

where $n - k = \operatorname{rank} E$, $I_{n-k} \in \mathbb{R}^{(n-k)\times(n-k)}$ is the identity matrix, $A_1 \in \mathbb{R}^{(n-k)\times(n-k)}$, $A_4 \in \mathbb{R}^{k \times k}$, $B_2 \in \mathbb{R}^{k \times m}$, and $C_2 \in \mathbb{R}^{p \times k}$, or they can be brought by allowed transformation into this form.

The rest of this section will be divided into two parts. In the first part (*case a*), the main results will be established for the case $\operatorname{rank}[E\ B] = \operatorname{rank}[E^T\ C^T] = n$. An extension of the results will then be presented in the second part (*case b*), where it will be assumed that (either) $\operatorname{rank}[E\ B] < n$, and (or) $\operatorname{rank}[E^T\ C^T] < n$.

Case a: $\operatorname{rank}[E\ B] = \operatorname{rank}[E^T\ C^T] = n$

We shal show first that in the present case, there is no loss of generality by assuming that A in (2.1) is given by

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & -I_k \end{bmatrix}. \tag{2.2}$$

Since

$$\operatorname{rank}[E\ B] = \operatorname{rank}[E^T\ C^T] = n \tag{2.3}$$

one clearly has (observe (2.1))

$$\operatorname{rank} B_2 = \operatorname{rank} C_2 = k . \tag{2.4}$$

Hence there exist nonsingular matrices $L \in \mathbb{R}^{m \times m}$, $R \in \mathbb{R}^{p \times p}$ such that

$$B_2 L = [0\ B_{2L}] ; \quad RC_2 = \begin{bmatrix} 0 \\ C_{2R} \end{bmatrix} \tag{2.5}$$

where $B_{2L} \in \mathbb{R}^{k \times k}$, $C_{2R} \in \mathbb{R}^{k \times k}$ are nonsingular, and we may define

$$K_4^* := B_{2L}^{-1}(-I_k - A_4)\, C_{2R}^{-1} ; \quad K_4^* \in \mathbb{R}^{k \times k} , \tag{2.6a}$$

which implies that

$$-I_k - A_4 = B_{2L} K_4^* C_{2R} . \tag{2.6b}$$

Next, we define K by

$$K := L\begin{bmatrix} 0 & 0 \\ 0 & K_4^* \end{bmatrix} R \tag{2.7a}$$

where R an L satisfy (2.5). Clearly (2.5) and (2.7a) imply that

$$B_2 L L^{-1} K R^{-1} R C_2 = [0 \;\; B_{2L}] \begin{bmatrix} 0 & 0 \\ 0 & K_4^* \end{bmatrix} \begin{bmatrix} 0 \\ C_{2R} \end{bmatrix}. \tag{2.7b}$$

From (2.5), (2.6), and (2.7) we have

$$B_2 K C_2 = B_{2L} K_4^* C_{2R} = -I_k - A_4 \quad \text{or} \quad B_2 K C_2 + A_4 = -I_k . \tag{2.8}$$

Recalling (2.1) and using (2.7)–(2.8) one gets

$$A + BKC = \begin{bmatrix} A_1 + B_1 K C_1 & A_2 + B_1 K C_2 \\ A_3 + B_2 K C_1 & A_4 + B_2 K C_2 \end{bmatrix} := \begin{bmatrix} A_1^{\#} & A_2^{\#} \\ A_3^{\#} & -I_k \end{bmatrix},$$

and it is clear that there exist nonsingular matrices P_f^* and P_r^* such that

$$A^* := P_f^*(A + BKC)\, P_r^* = \begin{bmatrix} A_1^* & 0 \\ 0 & -I_k \end{bmatrix}, \quad P_f^* E P_r^* = E, \tag{2.9}$$

where E is given in (2.1). Since the pair of maps $\{E, A^*\}$ is solvable, (2.2) indeed represents the general case for the problem under consideration, as claimed.

Assume rank $(E) = n - k < 0$. We define a regular system

$$\dot{\xi} = A_1 \xi + B_1 u \,; \quad \xi \in R^{n-k}, \quad u \in \mathbb{R}^m \tag{2.10a}$$

$$\eta = C_1 \,; \qquad \eta \in R^p, \tag{2.10b}$$

(A_1, B_1, C_1 are given by (2.1)–(2.2)), with output feedback

$$u = \Phi \eta + r \,; \quad \Phi \in \mathbb{R}^{m \times p}, \tag{2.11}$$

which plays an essential role in this study.

Definitions 2.1. The sets of zeros (listed according to multiplicity) of $\Pi_\Phi(\lambda, \Phi) := := \det(\lambda I_{n-k} - (A_1 + B_1 \Phi C_1))$ and $\Pi_k(\lambda, K) := \det(\lambda E - (A + BKC))$ are denoted respectively by $\Omega_\Phi = \{\lambda_1^*, \lambda_2^*, \ldots, \lambda_{n-k}^*\}$ and $\Omega_k = \{\lambda_1, \lambda_2, \ldots, \lambda_s\}$, $s \leqq n - k$.

Theorem 2.1. Consider the system (1.1) and assume that (2.3) holds, and $n - k > 0$. Suppose without loss generality that A is given by (2.2). Then the following results hold.

(i) For every given $\Phi \in \mathbb{R}^{m \times p}$ in (2.11) we have:

(i. 1) If $\det(I_k + B_2 \Phi C_2) \neq 0$, (2.12a)

there exists an admissible $K \in \mathbb{R}^{m \times p}$ such that $\Omega_k = \Omega_\Phi$, where $\Omega_{(\cdot)}$ is the set of zeroes of $\Pi_{(\cdot)}(\lambda, (\cdot))$ in Definition 2.1;

(i.2) if $\det(I_k + B_2 \Phi C_2) = 0$, (2.12b)

then, for every $\delta > 0$ there exists an admissible matrix $K(\delta)$ such that $\Omega_{k(\delta)} = = \{\lambda_1, \lambda_2, \ldots, \lambda_{n-k}\}$, with $|\lambda_i - \lambda_i^*| < \delta$, where $\lambda_i^* \in \Omega_\Phi$ for $i = 1, 2, \ldots, n - k$, (counting multiplicity).

(ii) For every given admissible $K \in \mathbb{R}^{m \times p}$ in (1.1), the following holds:

(ii.1) If $\deg[\lambda E - (A + BKC)] = n - k$, then there exists Φ such that $\Omega_\Phi = \Omega_k$;

(ii.2) If $0 < \deg[\lambda E - (A + BKC)] = r < n - k$, then, for every $\delta > 0$ there exists $\Phi(\delta)$ such that $\Omega_{\Phi(\delta)} = \{\lambda_1^*, \ldots, \lambda_r^*, \ldots, \lambda_{n-k}^*\}$ with $|\lambda_i^* - \lambda_i| < \delta$, $i = 1, 2, \ldots, r$.

Proof. Part (i).

(i.1) Consider the regular system (2.10) and let Φ be a given feedback matrix in (2.11) for which (2.12a) holds. Using (2.1) and (2.2), one gets

$$A + BKC = \begin{bmatrix} A_1 + B_1KC_1 & B_1KC_2 \\ B_2KC_1 & -I_k + B_2KC_2 \end{bmatrix} \tag{2.13}$$

Consider the matrix $\lambda E - (A + BKC)$. By observing (2.13) one concludes that to prove (i.1), it is sufficient to establish that there exists $K \in \mathbb{R}^{m \times p}$ and a nonsingular constant matrix $N \in \mathbb{R}^{n \times n}$ that fulfill the following two equations

$$N \begin{bmatrix} \lambda I_{n-k} - (A_1 + B_1KC_1) & -B_1KC_2 \\ -B_2KC_1 & I_k - B_2KC_2 \end{bmatrix} =$$

$$= \begin{bmatrix} \lambda I_{n-k} - (A_1 + B_1\Phi C_1) & 0 \\ -B_2KC_1 & I_k - B_2KC_2 \end{bmatrix}, \tag{2.14a}$$

$$\det(I_k - B_2KC_2) \neq 0 . \tag{2.14b}$$

The following form is suggected for $N \in \mathbb{R}^{n \times n}$:

$$N = \begin{bmatrix} I_{n-k} & N_2 \\ 0 & I_k \end{bmatrix}; \quad N_2 \in \mathbb{R}^{(n-k) \times k}, \tag{2.15}$$

where N_2 is yet unknown. Using (2.15), N and K solve (2.14a) if the following equation is satisfied

$$N \begin{bmatrix} -B_1KC_1 & -B_1KC_2 \\ -B_2KC_1 & I_k - B_2KC_2 \end{bmatrix} = \begin{bmatrix} -B_1\Phi C_1 & 0 \\ -B_2KC & I_k - B_2KC_2 \end{bmatrix}, \tag{2.16}$$

or equivalently, the following two equations hold

$$-B_1KC_1 - N_2B_2KC_1 = -B_1\Phi C_1 \tag{2.17a}$$

$$-B_1KC_2 + N_2(I_k - B_2KC_2) = 0 \tag{2.17b}$$

It will be shown in the sequel that if N_2 is given by

$$N_2 = B_1\Phi C_2 , \tag{2.18}$$

then, there exists a matrix K that fulfills (2.16)–(2.17), (provided (2.12a) holds). Substituting (2.18) for N_2 in (2.17a) and (2.17b) respectively, one gets

$$-B_1KC_1 - B_1\Phi C_2B_2KC_1 = -B_1\Phi C_1 \tag{2.19a}$$

$$-B_1KC_2 - B_1\Phi C_2B_2KC_2 = -B_1\Phi C_2 . \tag{2.19b}$$

Obviously, if K satisfies

$$-B_1K - B_1\Phi C_2B_2K = -B_1\Phi , \tag{2.20}$$

it satisfies both (2.19a) and(2.19b). Further, if K satisfies

$$-K - \Phi C_2 B_2 K = -\Phi \tag{2.21}$$

then, K satisfies (2.20) as well. Hence, to establish part (i.1) it is sufficient to show that there exists K that solves (2.21) and satisfies (2.14b).

Define

$$L^{-1}K := K^* = \begin{bmatrix} K_1^* \\ K_2^* \end{bmatrix}, \quad L^{-1}\Phi := \Phi^* = \begin{bmatrix} \Phi_1^* \\ \Phi_2^* \end{bmatrix}, \tag{2.22}$$

where L is given by the first equation in (2.5); $[\cdot]_1^* \in \mathbb{R}^{(m-k)\times p}$ and $[\cdot]_2^* \in \mathbb{R}^{k\times p}$. Premultiplying (2.21) by L^{-1} and using (2.22) and (2.5) we have

$$\begin{bmatrix} K_1^* \\ K_2^* \end{bmatrix} + \begin{bmatrix} \Phi_1^* \\ \Phi_2^* \end{bmatrix} \quad C_2[0 \; B_{2L}] \begin{bmatrix} K_1^* \\ K_2^* \end{bmatrix} = \begin{bmatrix} \Phi_1^* \\ \Phi_2^* \end{bmatrix} \tag{2.23}$$

or

$$K_1^* + \Phi_1^* C_2 B_{2L} K_2^* = \Phi_1^* \tag{2.24a}$$

$$K_2^* + \Phi_2^* C_2 B_{2L} K_2^* = \Phi_2^* . \tag{2.24b}$$

From (2.5) and (2.22)

$$B_2 \Phi = B_{2L} \Phi_2^* . \tag{2.25}$$

Hence, using (2.25), (2.24b) becomes

$$(I_k + B_{2L}^{-1} B_2 \Phi C_2 B_{2L}) K_2^* = \Phi_2^* ,$$

which is equivalent to

$$B_{2L}^{-1}(I_k + B_2 \Phi C_2) B_{2L} K_2^* = \Phi_2^* ,$$

or to

$$(I_k + B_2 \Phi C_2) B_{2L} K_2^* = B_{2L} \Phi_2^* . \tag{2.26}$$

Using (2.12a), (2.26) yields

$$B_{2L} K_2^* = (I_k + B_2 \Phi C_2)^{-1} B_{2L} \Phi_2^* ,$$

and following (2.25)

$$K_2^* = B_{2L}^{-1}(I_k + B_2 \Phi C_2)^{-1} B_2 \Phi . \tag{2.27a}$$

Substituting (2.27a) for K_2^* in (2.24a) one obtains

$$K_1^* = \Phi_1^*(I_k - C_2(I_k + B_2 \Phi C_2)^{-1} B_2 \Phi) . \tag{2.27b}$$

Hence (2.27) determines K^* and K solves (2.21), where

$$K = L \begin{bmatrix} K_1^* \\ K_2^* \end{bmatrix} . \tag{2.28}$$

To complete the proof of this part, it remains to show that K in (2.28) satisfies (2.14b). Post- and pre-multiplications of (2.21) by B_2 and C_2 respectively imply that

$$B_2 K C_2 + B_2 \Phi C_2 B_2 K C_2 = B_2 \Phi C_2 . \tag{2.29}$$

Using (2.12a), we have from (2.29)

$$B_2KC_2 = (I_k + B_2\Phi C_2)^{-1} B_2\Phi C_2 . \tag{2.30}$$

Substracting both sides of (2.30) from I_k, one gets

$$I_k - B_2KC_2 = I_k - (I_k + B_2\Phi C_2)^{-1} B_2\Phi C_2 ,$$

which, after a pre-multiplication by the matrix $(I_k + B_2\Phi C_2)$, becomes

$$(B_2\Phi C_2 + I_k)(I_k - B_2KC_2) = (B_2\Phi C_2 + I_k) - B_2\Phi C_2 = I_k .$$

The last result obviously asserts that $(I_k - B_2KC_2)$ is nonsingular, and statement (i.1) follows.

(i.2) Assume that Φ in (2.11) satisfies (2.12b) and define

$$\Phi_{\bullet} = \Phi + \varepsilon\Phi , \tag{2.31}$$

and the following two polynomials

$$\Pi_{\Phi}(\lambda) := \det(\lambda I_{n-k} - (A_1 + B_1\Phi C_1)) \tag{2.32a}$$

$$\Pi_{\bullet}(\lambda) := \det(\lambda I_{n-k} - (A_1 + B_1\Phi_{\bullet} C_1)) . \tag{2.32b}$$

Assume that $\Pi_{\Phi}(\lambda)$ has q distinct eigenvalues, i.e.,

$$\Pi_{\Phi}(\lambda) = (\lambda - \lambda^*_{i(1)})^{m(1)} (\lambda - \lambda^*_{i(2)})^{m(2)} \dots (\lambda - \lambda^*_{i(q)})^{m(q)} , \tag{2.33}$$

where $m(1) + m(2) + \dots + m(q) = n - k$. From (2.31) $B_1\Phi_{\bullet}C_1 = B_1\Phi C_1 + \varepsilon B_1\Phi C_1$, and hence (2.32) implies

$$\Pi_{\bullet}(\lambda) = \Pi_{\Phi}(\lambda) + \varepsilon \Pi_1(\lambda) + \varepsilon^2 \Pi_2(\lambda) + \dots + \varepsilon^{n-k} \Pi_{n-k}(\lambda) , \tag{2.34}$$

where $\Pi_{\Phi}(\lambda)$ is given by (2.32a) and $\Pi_i(\lambda)$, $i = 1, 2, \dots, n - k$ are polynomials in λ.

Since Φ satisfies (2.12b), there exists a sufficiently small $\varepsilon^* > 0$ such that

$$\det(I_k + B_2(\Phi + \varepsilon\Phi) C_2) \neq 0 , \quad \forall \varepsilon \in (0, \varepsilon^*] \tag{2.35}$$

Now, let Γ_i be a circle defined by $|\lambda_i - \lambda^*_i| = \delta$, where $\lambda^*_i \in \Omega_{\Phi}$ and

$$0 < \delta < \min_i \{\min_j \{|\lambda^*_i - \lambda^*_j| : \lambda^*_i \neq \lambda^*_j \in \Omega_{\Phi}\}\} . \tag{2.36}$$

(Thus, $\delta > 0$ is selected sufficiently small such that $\lambda^*_j \neq \lambda^*_i$ implies $\lambda^*_j \notin V_i$, where V_i is a delta-neighborhood of λ^*_i.) Let $\varepsilon_{\bullet} \in (0, \varepsilon^*]$ be chosen such that

$$|\Pi_{\Phi}(\lambda)| < |\varepsilon_{\bullet} \Pi_1(\lambda) + \varepsilon_{\bullet}^2 \Pi_2(\lambda) + \dots + \varepsilon_{\bullet}^{n-k} \Pi_{n-k}(\lambda)| ; \quad \forall \lambda \text{ on } \Gamma_i, \ \forall \Gamma_i . \tag{2.37}$$

With this selection of $\varepsilon_{\bullet}$, the functions $g(\lambda) := \varepsilon_{\bullet} \Pi_1(\lambda) + \varepsilon_{\bullet}^2 \Pi_2(\lambda) + \dots + \varepsilon_{\bullet}^{n-k} \Pi_{n-k}(\lambda)$ and $f(\lambda) := \Pi_{\Phi}(\lambda)$, satisfy the condition in Rouche's theorem with respect to all of the contours Γ_i, i.e., $|f(\lambda)| < |g(\lambda)|$ at each point λ on Γ_i, for every Γ_i. Therefore $\Pi_{\Phi}(\lambda) = f(\lambda)$ and $\Pi_{\bullet}(\lambda) = f(\lambda) + g(\lambda)$ have the same number of zeroes inside Γ_i, and this is true for every Γ_i. In addition the feedback matrix $\Phi(\delta) = \Phi + \varepsilon_{\bullet}\Phi$ satisfies (2.12a). By recalling statement (i.1) in the teorem, there exists $K(\delta)$ such that $\Omega_{k(\delta)} = \Omega_{\Phi(\delta)}$. Since this result holds for any sufficiently small $\delta > 0$, (i.2) follows and this completes the proof of (i.2), and hence of part (i).

Part (ii). (ii.1) Let K be a matrix such that $\deg[\lambda E - (A + BKC)] = n - k$. This implies that $\mathrm{rank}(I_k - B_2KC_2) = k$ in (2.14a), i.e., $I_k - B_2KC_2$ is nonsingular. We present

$$N_2 = B_1KC_2(I_k - B_2KC_2)^{-1} . \tag{2.38}$$

For this definition of N_2 (2.17b) is satisfied. It is also observed that any solution Φ of the equation

$$B_1K + N_2B_2K = B_1\Phi , \tag{2.39}$$

satisfies (2.17a). Substituting (2.38) for N_2 (2.39) yields

$$B_1K + B_1KC_2(I_k - B_2KC_2)^{-1} B_2K = B_1\Phi . \tag{2.40}$$

Both sides of the last equation belong to the column space of B_1, and thus (2.40) can be solved for Φ. By definition (note that N_2 satisfies (2.17b) and Φ satisfies (2.39) and hence (2.17a)), (2.14a) holds, and the statement of (ii.1) in the theorem follows.

(ii. 2) Let K be a matrix for which

$$0 < \deg[\lambda E - (A + BKC)] = r < n - k , \tag{2.41}$$

where E, A, B and C are given by (2.1) and (2.2). Define

$$K_{\bullet} = K + \varepsilon K . \tag{2.42}$$

From (2.1) and (2.2)

$$\lambda E - (A + BK_{\bullet}C) = \begin{bmatrix} \lambda I_{n-k} - (A_1 + B_1K_{\bullet}C_1) & - B_1K_{\bullet}C_2 \\ - B_2K_{\bullet}C_1 & I_k - B_2K_{\bullet}C_2 \end{bmatrix} \tag{2.43}$$

The rest of the proof is mainly relies on the approach presented in the proof of part (i.2) (see equations (2.31)–(2.37)). First it is observed that for a sufficiently small $\varepsilon^* > 0$

$$\mathrm{rank}(I_k - B_2(K + \varepsilon K) C_2) = k ; \quad \forall\varepsilon \in (0, \varepsilon^*] , \tag{2.44a}$$

and from (2.43)

$$\deg[\lambda E - (A + B(K + \varepsilon K) C)] = n - k ; \quad \forall\varepsilon \in (0, \varepsilon^*] . \tag{2.44b}$$

Next, consider the following two polynomials:

$$\Pi_k(\lambda) := \det(\lambda E - (A + BKC)) \tag{2.45a}$$

$$\Pi_{\bullet}(\lambda) := \det(\lambda E - (A + BK_{\bullet}C)) . \tag{2.45b}$$

By observing (2.42) through (2.45), one obtains

$$\Pi_{\bullet}(\lambda) = \Pi_k^{\bullet}(\lambda) + \varepsilon\, \Pi_1^{\bullet}(\lambda) + \dots + \varepsilon^k\, \Pi_k(\lambda) \tag{2.46}$$

where $\Pi_i(\lambda)$ are polynomials in λ. Assume that $\Pi_k(\lambda)$ has p distinct eigenvalues, i.e.,

$$\Pi_k(\lambda) = (\lambda - \lambda_{i(1)})^{m(1)} (\lambda - \lambda_{i(2)})^{m(2)} \dots (\lambda - \lambda_{i(p)})^{m(p)} , \tag{2.47}$$

where $\lambda_i \in \Omega_k$, $p \leqq r$, and $m(1) + m(2) + \dots + m(p) = r < n - k$.

We select a sufficiently small $\delta > 0$ such that if V_i is delta-neighborhood of $\lambda_i \in \Omega_k$, then $\lambda_i \neq \lambda_j$ implies $\lambda_j \notin V_i$. Finally, a sufficiently small $\varepsilon_{\bullet} \in (0, \varepsilon^*]$ is selected such that $g(\lambda) := \varepsilon_{\bullet} \Pi_1(\lambda) + \varepsilon_{\bullet}^2 \Pi_2(\lambda) + \ldots + \varepsilon_{\bullet}^k \Pi(\lambda)$ and $f(\lambda) := \Pi_k(\lambda)$ satisfy the condition of Rouche's theorem with respect to every contour Γ_i of V_i, i.e., $|f(\lambda)| < |g(\lambda)|$ at each point λ on any selected Γ_i. Therefore $\Pi_k(\lambda) = f(\lambda)$ and $\Pi_{\bullet}(\lambda) = f(\lambda) + g(\lambda)$ have the same number of zeroes inside every selected Γ_i. In addition $K(\delta) = K + \varepsilon_{\bullet} K$ satisfies (2.44b), and hence, from the proof of statement (ii.1) above, there exists $\Phi(\delta)$ such that $\Omega_{\phi(\delta)} = \Omega_{k(\delta)}$. Recalling that m(i) is the multiplicity of $\lambda_i \in \Omega_k$, one concludes that m(i) roots of $\Pi_{\phi(\delta)}(\lambda)$ are inside Γ_i and with this result part (ii) has been verified, and the theorem follows. □

Comment 2.1. So far we did not consider the case $n - k = 0$ (i.e., $E = 0$). However, this is a trivial case since then, $\Omega_k = \emptyset$ (the empty set) for any admissible K. To be consistent with previous notations we say that in this particular case the regular system (2.10) vanishes.

Case b: rank $[E\ B] < n$ and/or rank $[E^T\ C^T] < n$

Theorem 2.2. Assume rank $[E\ B] < n$ and/or rank $[E^T\ C^T] < n$ in (1.1). Suppose that there exists at least one K for which $\Omega_k \neq \emptyset$, where Ω_k is the set of roots of $\det(\lambda E - (A + BKC))$. Then, there exists a reduced order system:

$$E^{\bullet}\dot{x}^{\bullet} = A^{\bullet}x^{\bullet} + B^{\bullet}u\ ; \quad x^{\bullet} \in \mathbb{R}^q\,, \quad u \in \mathbb{R}^m \tag{2.48a}$$

$$y^{\bullet} = C^{\bullet}x^{\bullet}\ ; \quad y^{\bullet} \in \mathbb{R}^p \tag{2.48b}$$

(which is obtained from (1.1) by realizing a finite sequence of matrix operations), with the following properties:

(i) $0 <$ ran $[E^{\bullet}\ B^{\bullet}] =$ rank $[E^{\bullet T}\ C^{\bullet T}] = q < n$, (i.e., the matrices $[[\cdot][\cdot]]$ are of full rank);

(ii) for any K, $\det(\lambda E - (A + BKC)) = \eta \det(\lambda E^{\bullet} - (A^{\bullet} + B^{\bullet}KC^{\bullet})$, $\eta \neq 0$;

(iii) for any admissible K, $C(\lambda E - (A + BKC))^{-1} B = C^{\bullet}(\lambda E^{\bullet} - (A^{\bullet} + B^{\bullet}KC^{\bullet}))^{-1} B^{\bullet}$.

Proof. The proof will be by construction.

Step 1. If rank $[E\ B] = n$, then, from the conditions of the theorem rank $[E^T\ C^T] < n$. Without loss of generality we may assume that rank $[E\ B] = n - r(1) > 0$. (Note that $E \neq 0$, since otherwise $\Omega_k = \emptyset$ for any admissible K.) Let $G_1 \in R^{r(1) \times n}$ be a maximal rank matrix subject to

$$G_1[E\ B] = 0\,. \tag{2.49}$$

Using (2.49), one can define a matrix $G^1 \in \mathbb{R}^{(n-r(1)) \times n}$ such that

$$L_1 := \begin{bmatrix} G^1 \\ G_1 \end{bmatrix} \in \mathbb{R}^{n \times n}$$

is nonsingular, and using (2.49)

$$L_1[E, B, A] = \begin{bmatrix} E^1 & B^1 & G^1 A \\ 0, & 0, & G_1 A \end{bmatrix}. \tag{2.50}$$

Since the pair $\{E, A\}$ is solvable, G_1A in (2.50) is of full rank. Hence there exists a nonsingular matrix R_1 such that

$$G_1AR_1 = [0\ A_{14}] \tag{2.51}$$

where $A_{14} \in \mathbb{R}^{r(1)\times r(1)}$ is nonsingular. Using (2.50)–(2.51):

$$\{L_1[ER_1, B, AR_1], CR_1\} = \left\{\begin{bmatrix} E_{11} & E_{12} & B_{11} & A_{11} & A_{12} \\ 0 & 0, & 0, & 0 & A_{14}\end{bmatrix}, [C_{11}\ C_{12}]\right\} \tag{2.52}$$

where (see (2.50)) $[E_{11}\ E_{12}] := E^1R_1$; $E_{11} \in \mathbb{R}^{(n-r(1))\times(n-r(1))}$; $B_{11} := B^1$; $[C_{11}\ C_{12}] := CR_1$; $C_{11} \in \mathbb{R}^{p\times(n-r(1))}$, and it is observed that $\operatorname{rank}(B) = \operatorname{rank}(B_{11})$.

From (2.52) we have

$$\det(\lambda(E - A) = \eta_1 \det(\lambda E_{11} - A_{11}) \tag{2.53}$$

where $\eta_1 = \det(-A_{14})\det(L_1^{-1})\det(R_1^{-1}) \neq 0$. Hence, solvability of the pair $\{E, A\}$, inplies solvability of $\{E_{11}, A_{11}\}$. In addition, for any given K we have using (2.52)

$$\begin{aligned} &\det(\lambda E - (A + BKC)) = \\ &= \det(L_1^{-1})\det(R_1^{-1})\begin{bmatrix} \lambda E_{11} - (A_{11} + B_{11}KC_{11}) & \lambda E_{12} - (A_{12} + B_{11}KC_{12}) \\ 0 & -A_{14}\end{bmatrix} \\ &= \eta_1 \det(\lambda E_{11} - (A_{11} + B_{11}KC_{11}))\,, \end{aligned} \tag{2.54}$$

and for any admissible K

$$\begin{aligned} &C(\lambda E - (A + BKC))^{-1} B = \\ &[C_{11}C_{12}]\, R_1^{-1}R_1 \begin{bmatrix} [*] & [*](\lambda E_{12} - (A_{12} + B_{11}KC_{12})(A_{14})^{-1} \\ 0 & (-A_{14})^{-1}\end{bmatrix} L_1L_1^{-1}\begin{bmatrix} B_{11} \\ 0\end{bmatrix} \\ &= C_{11}(\lambda E_{11} + B_{11}KC_{11})^{-1} B_{11}\,, \end{aligned} \tag{2.55}$$

where $[*] = (\lambda E_{11} - (A_{11} + B_{11}KC_{11}))^{-1}$. To this end we define the following solvable (see (2.54)) system:

$$E_{11}\dot{x}_1 = A_{11}x_1 + B_{11}u\,, \quad x_1 \in \mathbb{R}^{n-r(1)} \tag{2.56a}$$

$$y_1 = C_{11}x_1\,, \qquad y_1 \in \mathbb{R}^p \tag{2.56b}$$

Now, if the set of maps $\{E_{11}, A_{11}, B_{11}, C_{11}\}$ satisfies condition (i), we define $E^{\cdot} := E_{11}$; $A^{\cdot} := A_{11}$; $B^{\cdot} := B_{11}$, $C^{\cdot} := C_{11}$, $q := n - r(1)$, and since (see (2·53)–(2.55)), (2.56), satisfies conditions (ii) and (iii) the process is ended. If $\{E_{11}, A_{11}, B_{11}, C_{11}\}$ does not satisfy (i) we turn to the next step.

Step 2. Without loss of generality we may assume that $\operatorname{rank}[E_{11}\ B_{11}] = n - - r(1) - r(2)$, $r(2) > 0$. Let $G_2 \in \mathbb{R}^{r(2)\times(n-r(1))}$ be a maximal rank matrix subject to $G_2[E_{11}\ B_{11}] = 0$, and define a matrix $G^2 \in \mathbb{R}^{(n-r(1)-r(2))\times(n-r(1))}$ such that

$$L_2 := \begin{bmatrix} G^2 \\ G_2 \end{bmatrix} \in \mathbb{R}^{(n-r(1))\times(n-r(1))} \tag{2.57}$$

is nonsingular. Then, (compare with (2.50))

$$L_2[E_{11}, B_{11}, A_{11}] = \begin{bmatrix} E^2 & B^2 & G^2A_{11} \\ 0, & 0, & G_2A_{11} \end{bmatrix}. \tag{2.58}$$

Solvability of the system (2.56) implies that G_2A_{11} in (2.58) is of full rank. Therefore there exists a nonsingular matrix R_2 such that $G_2A_{11}R_2 = [0\ A_{24}]$, where $A_{24} \in \in \mathbb{R}^{r(2)\times r(2)}$ is nonsingular. Hence,

$$\{L_2[E_{11}R_2, B_{11}, A_{11}B_2], C_{11}R_2\} = \left\{\begin{bmatrix} E_{21} & E_{22} & B_{21} & A_{21} & A_{22} \\ 0 & 0, & 0, & 0 & A_{24} \end{bmatrix}, [C_{21}\ C_{22}]\right\} \tag{2.59}$$

where (see (2.58)) $[E_{21}\ E_{22}] = E^1R_2$, $E_{21} \in \mathbb{R}^{(n-r(1)-r(2))\times(n-r(1)-r(2))}$, $[C_{21}\ C_{22}] = = C_{11}R_2$, $C_{21} \in \mathbb{R}^{p\times(n-r(1)-r(2))}$, and $\operatorname{rank}(B_{21}) = \operatorname{rank}(B_{11}) = \operatorname{rank}(B)$. Using (2.59) we have $\det(\lambda E_{11} - A_{11}) = \eta_2^* \det(\lambda E_{21} - A_{21})$, and for any admissible K

$$C_{11}(\lambda E_{11} - (A_{11} + B_{11}KC_{11}))^{-1} B_{11} =$$
$$= C_{21}(\lambda E_{21} - (A_{21} + B_{21}KC_{21}))^{-1} B_{21}.$$

Therefore one obtains, using previous arguments, the following results (see (2.53) through (2.55)):

$$\det(\lambda E - A) = \eta_1 \det(\lambda E_{11} - A_{11}) = \eta_2 \det(\eta E_{21} - A_{21}), \quad \eta_2 \neq 0, \tag{2.60}$$

$$\det(\lambda E - (A + BKC)) = \eta_1 \operatorname{drt}(\lambda E_{11} - (A_{11} + B_{11}KC_{11})) =$$
$$= \eta_2 \det(\lambda E_{21} - (A_{21} + B_{21}KC_{21})), \tag{2.61}$$

$$C(\lambda E - (A + BKC))^{-1} B = C_{11}(\lambda E_{11} + B_{11}KC_{11}))^{-1} B_{11} =$$
$$= C_{21}(\lambda E_{21} - (A_{21} + B_{21}KC_{21}))^{-1} B_{21}. \tag{2.62}$$

We define now the following singular system

$$E_{21}\dot{x}_2 = A_{21}x_2 + B_{21}u\,; \quad x_2 \in \mathbb{R}^{n-r(1)-r(2)}, \quad u \in \mathbb{R}^m, \tag{2.63a}$$

$$y_2 = C_{21}x_2\,; \quad y_2 \in \mathbb{R}^p. \tag{2.63b}$$

which, following (2.60), is solvable with $\operatorname{rank}(B) = \operatorname{rank}(B_{11}) = \operatorname{rank}(B_{21})$. Again, if the maps $\{E_{21}, A_{21}, B_{21}, C_{21}\}$ satisfies condition (i), the system (2.48) is associated with (2.63), one has $q := n - r(1) - r(2)$, and the process is completed. Otherwise we continue to step 3.

Now, this phase of the procedure proceeds through to steps 3, 4, ... But since $\operatorname{rank}(B) = \operatorname{tank}(B_{11}) = \operatorname{rank}(B_{21}) = \ldots$, one must have after, say, s steps, $0 \leqq \leqq s \leqq n - \operatorname{rank}(B)$:

$$\operatorname{rank}[E_{s1}\ B_{s2}] = n - r(1) - r(2) - \ldots - r(s), \tag{2.64}$$

i.e., $[E_{s1}\ B_{s1}]$ is of full rank, and one obtains after s steps

$$E_{s1}\dot{x}_s = A_{s1}x_s + B_{s1}u\,; \quad x_s \in R^{n-r(1)-r(2)-\ldots-r(s)}\,; \quad u \in R^m \tag{2.65a}$$

$$y_s = C_{s1}x_s\,; \quad y_s \in \mathbb{R}^p. \tag{2.65b}$$

Using previous arguments the pair of maps $\{E_{s1}, A_{s1}\}$ is solvable, $\text{rank}(B) = \text{rank}(B_{11}) = \text{rank}(B_{21}) = \ldots = \text{rank}(B_{s1})$, and according to the previous steps (see (2.60)–(2.62))

$$\det(\lambda E - (A + BKC)) = \ldots = \eta_s \det(\lambda E_{s1} - (A_{s1} + B_{s1}KC_{s1})) \quad (2.66)$$

$$C(\lambda E - (A + BKC))^{-1} B = \ldots = C_{s1}(\lambda E_{s1} - (A_{s1} + B_{s1}KC_{s1}))^{-1} B_{s1}, \quad (2.67)$$

for any admissible K. Since $[E_{s1}\ B_{s1}]$ is of full rank, it remains to check whether or not the matrix $[E_{s1}^{\mathrm{T}}\ C_{s1}^{\mathrm{T}}]$ satisfies condition (i). If this condition is satisfied the process is ended with the definitions $E^{\cdot} := E_{s1}$; $A^{\cdot} := A_{s1}$; $B^{\cdot} := B_{s1}$; $C^{\cdot} := C_{s1}$; $q := n - r(1) - r(2) - \ldots - r(s)$, and using (2.66) and (2.67) the statement of the theorem follows.

Assume condition (i) is not fulfilled yet, i.e., $0 < \text{rank}[E_{s1}^{\mathrm{T}}\ C_{s1}^{\mathrm{T}}] < n - r(1) - r(2) - \ldots - r(s)$. (Note that since there exists at least one K for which $\Omega_k \neq \emptyset$, $\text{rank}\, E_{s1} > 0$.) This leads us to the second phase of the proof which is 'dual' to the first one. In fact we start by determining nonsingular matrices $R_1^{\#}$ and $L_1^{\#}$ that satisfy the following (compare with (2.50)–(2.52)):

$$\{(L_1^{\#})^{\mathrm{T}} E_{s1}^{\mathrm{T}}(R_1^{\#})^{\mathrm{T}}, C_{s1}^{\mathrm{T}}, A_{s1}^{\mathrm{T}}(R_1^{\#})^{\mathrm{T}}], B_{s1}^{\mathrm{T}}(R_1^{\#})^{\mathrm{T}}\} =$$
$$= \left\{\begin{bmatrix} E_{(s+1)1}^{\mathrm{T}} & E_{(s+1)2}^{\mathrm{T}} & C_{(s+1)1}^{\mathrm{T}} & A_{(s+1)1}^{\mathrm{T}} & A_{(s+1)2}^{\mathrm{T}} \\ 0 & 0, & 0, & 0 & A_{(s+1)4}^{\mathrm{T}} \end{bmatrix}, [B_{(s+1)1} B_{(s+1)2}]\right\} \quad (2.68a)$$

or, equivalently

$$\{R_1^{\#}[E_{s1}L_1^{\#}, B_{s1}, A_{s1}L_1^{\#}], C_{s1}L_1^{\#}\} =$$
$$= \left\{\begin{bmatrix} E_{(s+1)1} & 0 & B_{(s+1)1} & A_{(s+1)1} & 0 \\ E_{(s+1)2} & 0, & B_{(s+1)2} & A_{(s+1)2} & A_{(s+1)4} \end{bmatrix}, [C_{(s+1)1}\ 0]\right\}. \quad (2.68b)$$

Again, by applying the previous approach and noting that (2.65) is solvable, the following system which is derived from (2.68b), is solvable:

$$E_{(s+1)1}\dot{x}_{(s+1)} = A_{(s+1)1}x_{(s+1)} + B_{(s+1)1}\, u, \quad (2.69a)$$

$$y_{(s+1)} = C_{(s+1)1}x_{(s+1)}. \quad (2.69b)$$

Using (2.66)–(2.68), (2.69) gains the following properties. For every K, (for reference consider the 'dual' forms of (2.54)–(2.55)),

$$\det(\lambda E - (A + BKC)) = \ldots = \eta_s \det(\lambda E_{s1} - (A_{s1} + B_{s1}KC_{s1})) =$$
$$= \eta_{s+1} \det(\lambda E_{(s+1)1} - (A_{(s+1)1} + B_{(s+1)1}KC_{(s+1)1})), \quad (2.70)$$

and (provided K is admissible),

$$C(\lambda E - (A + BKC))^{-1} B = \ldots = C_{s1}(\lambda E_{s1} - (A_{s1} + B_{s1}KC_{s1}))^{-1} B_{s1} =$$
$$= C_{(s+1)1}(\lambda E_{(s+1)1} - (A_{(s+1)1} + B_{(s+1)1}KC_{(s+1)1})^{-1} B_{(s+1)1}. \quad (2.71)$$

In addition, one concluded by observing (2.68) that since $[E_{s1}\ B_{s1}]$ if of full rank $[E_{(s+1)1}\ B_{(s+1)1}]$ is of full rank as well.

Using these results, the continuation of the proof is straightforward and one finally obtain the following system

$$E_{(s+v)1}\dot{x}_{s+v} = A_{(s+v)1}x_{s+v} + B_{(s+1)}u \tag{2.72a}$$

$$y_{s+v} = C_{(s+v)1}x_{s+v}, \tag{2.72b}$$

where $[E_{(s+v)1}B_{(s+v)1}]$ and $[E^{\mathrm{T}}_{(s+v)1}\ C^{\mathrm{T}}_{(s+v)1}]$ satisfy condition (i), and

$$\det(sE - A) = \dots = \eta_s \det(\lambda E_{s1} - A_{s1}) = \dots$$
$$\dots = \eta_{(s+v)} \det[\lambda E_{(s+v)1} - A_{(s+v)1}], \tag{2.73}$$

i.e., the pair $\{E_{(s+v)1}, A_{(s+v)1}\}$ is solvable. Also ,

$$\det(\lambda E - (A + BKC)) = \dots = \eta_s \det(\lambda E_{s1} + B_{s1}KC_{s1})) = \dots$$
$$\dots = \eta_{(s+v)} \det(\lambda E_{(s+v)1} - (A_{(s+v)} + B_{(s+v)1}KC_{(s+v)1})). \tag{2.74}$$

Finally, from (2.62), (2.67) and previous arguments we have

$$C(\lambda E - (A + BKC))^{-1} B = \dots = C_{s1}(\lambda E_{s1} - (A_{s1} + B_{s1}KC_{s1}))^{-1} B_{s1}$$
$$\dots = C_{(s+v)1}(\lambda E_{(s+v)1} - (A_{(s+v)1} + B_{(s+v)1}KC_{(s+v)1}))^{-1} B_{(s+v)1}. \tag{2.75}$$

By defining $E^{\bullet} := E_{(s+v)1}$, $A^{\bullet} := A_{(s+v)1}$, $B^{\bullet} := B_{(s+v)1}$, $C^{\bullet} := C_{(s+v)1}$, $x^{\bullet} := x_{s+v}$, $y^{\bullet} := {}_{s+v}$ and $q := n - r(1) - \dots - r(s) - \dots - r(s + v)$, we complete the proof.

The following theorem concludes the results of this section:

Theorem 2.3. Let the system (1.1) be denoted by $\{E, A, B, C\}$, and assume there is at least one K for which $\Omega_k \neq \emptyset$, where Ω_k is the set of roots of $\det(sE - (A + BKC))$. Then, there exist$_p$ a regular system $\{I_\sigma, \alpha, \beta, \gamma\}$ where $\sigma \leqq \operatorname{rank}(E)$, that satisfies the following. For any given feedback matrix Φ in the regular system, either there exists an admissible feedback matrix K such that $\Omega_k = \Omega_\Phi$ or (at least), for any $\delta > 0$ a matrix $K(\delta)$ can be selected for which every $\lambda_i \in \Omega_k$ is in delta-neighborhood of $\lambda_i^* \in \Omega_\Phi$. Conversely, for any admissible K either there exists Φ such that $\Omega_\Phi = \Omega_k$, or, for any $\delta > 0$ a feedback matrix $\Phi(\delta)$ can be selected for which every $\lambda_i \in \Omega_k$ is in delta-neighborhood of $\lambda_i^* \in \Omega'$ where $\Omega_{\Phi(\delta)} \supset \Omega'$.

Proof. The proof, as well as the procedure for deriving $\{I_\sigma, \alpha, \beta, \gamma\}$, the feedback matrices K (in the first part of the theorem), and Φ (in the second part), are directly obtained from Theorems 2.1, and 2.2. In particular, if $\operatorname{rank}[E\ B] = \operatorname{rank}[E^{\mathrm{T}}\ C^{\mathrm{T}}] = n$, one may introduce using (2.10)

$$\{I_\sigma, \alpha, \beta, \gamma\} = \{I_{n-k}, A_1, B_1, C_1\}, \tag{2.76a}$$

and for the case $\operatorname{rank}[E\ B] < n$ and/or $\operatorname{rank}[E^{\mathrm{T}}\ C^{\mathrm{T}}] < n$

$$\{I_\sigma, \alpha, \beta, \gamma\} = \{I_{q-k^{\bullet}}, A_1^{\bullet}, B_1^{\bullet}, C_1^{\bullet}\} \tag{2.76b}$$

where, (see (2.48)), $\operatorname{rank}(E^{\bullet}) = q - k^{\bullet}$, and $[\cdot]_1^{\bullet}$ are obtained from $A^{\bullet}$, $B^{\bullet}$, and $C^{\bullet}$ through equations (2.1)–(2.9) (while $[\cdot]^{\bullet}$ replaces $[\cdot]$ in these equations). (Note that it may happen that $C^{\bullet} = 0$ in (2.48). In this particular case, Theorem 2.2 simply implies that (if $\Omega_k \neq \emptyset$ for at least one K), $E^{\bullet}$ is nonsingular, Ω_k is invariant under output feedback, and for every selected admissible K the transfer-function matrix of (1.1) satisfies $T(\lambda, K) = 0$.)

3. CONCLUSIONS

Conclusion 3.1. Consider the system (1.1) and assume that $\operatorname{rank}[E\ B] = \operatorname{rank}[E^T\ C^T] = n$, and the triplets $\{E, A, B\}$ and $\{E, A, C\}$ are respectively controllable and observable. Then, the pairs $\{A_1, B_1\}$ and $\{A_1, C_1\}$ in (2.10) are respectively controllable and observable.

Proof. Since $[\lambda E - A\ B]$ is of full rank (see Theorem 1.1) for any finite λ, $[\lambda E - (A + BKC)\ B]$ is of full rank as well. Hence from (2.9) $[\lambda E - A^*\ P_f^* B]$ is of full rank, which implies (observe (2.1) and (2.9)) that $[\lambda I_{n-k} - A_1\ B_1]$ is of full rank, i.e., (2.10) is controllable. By duality one can prove using similar arguments that $[\lambda I_{n-k} - A_1^T\ B_1^T]$ is of full rank, and hence (2.10) is observable. □

Conclusion 3.2. Consider the system (1.1) and assume that $\operatorname{rank}[E\ B] < n$ and/or $\operatorname{rank}[E^T\ C^T] < n$, and the triplets $\{E, A, B\}$ and $\{E, A, C\}$ are respectively controllable and observable. Then, the triplets $\{E^{\bullet}, A^{\bullet}, B^{\bullet}\}$ and $\{E^{\bullet}, A^{\bullet}, C^{\bullet}\}$ in (2.48) are respectively controllable and observable.

Proof. Following the constructive proof of Theorem 2.2, one concludes that to establish the conclusion, it is sufficient to verify that the system (2.56) is both controllable and observable. Since $[\lambda E - A\ B]$ is of full rank, (2.52) implies that $[L_1(\lambda E - A) R_1\ L_1 B]$ and hence $[\lambda E_{11} - A_{11}\ B_{11}]$ are of full rank. By duality and using (2.52), we deduce that since $[\lambda E^T - A^T\ C^T$ is of full rank, $[\lambda R_1^T(E^T - A^T) L_1^T\ R_1^T C^T]$ and hence $[\lambda E_{11}^T - A_{11}^T\ C_{11}^T]$ are both of full rank. □

To obtain the next conclusions, we present the following notations and result from the study of Kimura [6]. Consider a regular system

$$\dot{\xi} = \alpha\xi + \beta v\ ; \quad \xi \in \mathbb{R}^{\sigma}, \quad v \in \mathbb{R}^m, \tag{3.1a}$$

$$\eta = \gamma\xi\ ; \qquad \eta \in \mathbb{R}^p, \tag{3.1b}$$

where α, β and γ, are matrices of appropriate dimensions. Let $\Omega = \{\lambda_1, \lambda_2, \ldots, \lambda_\sigma\}$ be a set of σ complex numbers. All numbers in Ω are (i) assumed to be distrinct and (ii) any complex λ_i is accompanied by its conjugate. The class of all sets satisfying these two conditions is denoted by Ξ_σ. If for any $\Omega \in \Xi_\sigma$ there exist λ_i^* in arbitrary neighborhood of $\lambda_i \in \Omega$ such that a matrix Φ exists which assigns $\Omega_\Phi = \{\lambda_1^*, \lambda_2^*, \ldots, \lambda_\sigma^*\} = \pi(\alpha + \beta\Phi\gamma)$, where $\pi(\cdot)$ is the set of eigenvalues of $(\cdot)$, the system (3.1) is called pole-assignable.

Assume

$$\sigma - m - p + 1 \leqq 0\,. \tag{3.2}$$

in (3.1). Then, the following holds.

Theorem 3.1 (cf. [6]). If the system (3.1) is both controllable and observable and satisfies the relation (3.2), then it is pole-assignable, i.e., for any $\Omega^* = \{\lambda_1^*, \lambda_2^*, \ldots \ldots, \lambda_\sigma^*\} \in \Xi_\sigma$ there exist numbers λ_i in arbitrary neighborhood of λ_i^* such that $\Omega = \{\lambda_1, \lambda_2, \ldots, \lambda_\sigma\} \in \Xi_\sigma$ and $\pi(\alpha + \beta\Phi\gamma) = \Omega$ for some real matrix Φ.

Back to singular systems, it is obvious that σ, the degree of the polynomial $\det(\lambda E - (A + BKC)$ satisfies $\sigma \leqq \operatorname{rank}(E)$. Therefore the value of σ is inherently part of the problem of pole-assignment in singular systems. We define an integer σ^* and a matrix K^* as follows:

$$\begin{aligned} &\deg[\det(\lambda E - (A + BK^*C))] = \sigma^* \geqq \\ &\geqq \deg[\det(\lambda E - A + BKC))] ; \quad \forall K \neq K^* . \end{aligned} \tag{3.3}$$

If $\sigma^* > 0$ then Ξ_{σ^*} is determined as above, and we say that (1.1) is pole-assignable if for any $\Omega \in \Xi_{\sigma^*}$ there exist λ_i^* in arbitrary neighborhood of $\lambda_i \in \Omega$ such that a matrix K exists which assigns $\Omega_k = \{\lambda_1^*, \lambda_2^*, \ldots, \lambda_\sigma^*\}$ where Ω_k is the set of roots of $\det(\lambda E - (A + BKC))$. (Note that if $E = 0$ (in case a), or $E^\bullet = 0$ (in case b), $\sigma^* = 0$ for any admissible K.)

Conclusion 3.3. Assume that $E \neq 0$ and $\operatorname{rank}[E\ B] = \operatorname{rank}[E^T\ C^T] = n$, and the triplets $\{E, A, B\}$ and $\{E, A, C\}$ are respectively controllable and observable. Further, suppose that $(n - k) - \operatorname{rank}(B_1) - \operatorname{rank}(C_1) + 1 \leqq 0$, where B_1 and C_1 are given in (2.10). Then the system (1.1) is pole-assignable.

Proof. By noting that ξ in (2.10) is $n - k (= \operatorname{rank}(E))$ vector, the conclusion follows from Theorems 2.1, 3.1 and Conclusion 3.1. □

Conclusion 3.4. Assume that $\operatorname{rank}[E\ B] < n$, and/or $\operatorname{rank}[E^T\ C^T] < n$, $E^\bullet \neq 0$ in (2.48), and triplets $\{E, A, B\}$ and $\{E, A, C\}$ are respectively controllable and observable. Further, suppose that $(q - k^\bullet) - \operatorname{rank}(B_1^\bullet) - \operatorname{rank}(C_1^\bullet) + 1 \leqq 0$, $(q - k^\bullet)$, $B_1^\bullet$, and $C_1^\bullet$ are given in (2.76b). Then, the system (1.1) is pole assignable.

Proof. The conclusion follows from Theorem 2.2, and conclusions 3.2 and 3.3. □

4. DISCUSSION

The equivalence relation, which has been established in this paper with respect to the problem of pole assignment in singular and regular systems is the main result of the paper. Using a group transformation, the present approach allows one to design a gain output feedback for a given singular system through an appropriate regular system. To illustrate the application of the results, assume rank $E = n - k$, and let the system satisfies the condition of *case a*. Now, suppose the system (1.1) is controllable and observable, and $m + p > n - k$. Then, according to Conclusion 3.3, every desired set Ω_k of $n - k$ distinct eigenvalues (with complex eigenvalues accompanied by its conjugate) can (almost) be achieved in the singular system (1.1) through an admissible gain output feedback of the form (1.3). If the system (1.1) does not satisfy the condition of *case a* then the derivation of K is obtained through the procedure presented for *case b*, and Conclusion 3.4.

We recall that this paper considers the problem of assigning the *finite* closed-loop poles of the singular system (1.1) while preserving the uniqueness property of the system. Assume $\operatorname{rank}[E\ B] = \operatorname{rank}[E^T\ C^T] = n$, (*case a*). Then, if the system satisfies the conditions in Conclusion 3.3, $(n - k)$ distinct closed-loop poles are assignable, and since $\operatorname{rank} E = n - k$, this is the maximal number of *finite* poles that can be assigned by gain output feedback. If *case b* holds, and the system satifies the conditions in Conclusion 3.4, $(q - k^*)$ distinct closed-loop poles are assignable, and since $\operatorname{rank} E^* = q - k^*$, one deduces from Theorem 2.2 that in this case $(q - k^*)$ is the maximal number of *finite* poles that can be assigned by gain output feedback. The combined problem of eliminating *impulsive modes* (the modes which are associated with natural system frequencies at infinity), and assigning *finite closed-loop poles* by output feedback, is a subject for further study.

Finally, this study presents an approach rather than developing a numerical algorithm for solving the problem under consideration. However, the proposed approach provides an appropriate framework for further research concerning the development of useful and numerically reliable algorithms, for various applications of gain output feedback in singular systems.

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