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ON THE PENALTY APPROXIMATION OF QUADRATIC PROGRAMMING PROBLEM

ZDENĚK DOSTÁL

An upper bound for the difference of the exact solution of the problem of minimization of quadratic functional on a subspace and its penalty approximation has been given. The paper is supplied with a numerical example.

1. INTRODUCTION

It is an easy consequence of the general theory of the penalty method [1] that the solution \bar{x} of the problem

$$\min \{f(x): x \in \mathcal{V}\}, \quad (1)$$

where $f(x) = \frac{1}{2}x^T Ax - b^T x$ is a convex quadratic functional and \mathcal{V} is a subspace of the real Euclidean space \mathbb{R}^n , may be approximated by the solution \bar{x}_ε of the problem

$$\min \{f(x) + \frac{1}{2}\varepsilon^{-1}x^T Px: x \in \mathbb{R}^n\} \quad (2)$$

where $\varepsilon > 0$ and P is any $n \times n$ matrix with the range \mathcal{V} .

Even though there is a number of exact methods for transformation of the problem (1) to the unconstrained one [3], hardly any of them is as simple and cheap from the point of view of numerical realisation as the penalty approximation (2). In a special case when P is diagonal its application consists simply in overwriting corresponding diagonal entries of A by a large number and is widely used [2]. In a more general case, when values of constrained variables are not fully determined by constraints, the application is complicated by the fact that adding a too large matrix to A may destroy the corresponding part of A . This has motivated us to examine in detail the difference $\bar{x} - \bar{x}_\varepsilon$ and to carry out some numerical experiments in order to assess the applicability of the penalty approach.

2. AN ESTIMATE

Let $f(x) = \frac{1}{2}x^T Ax - b^T x$ where $A \in \mathbb{R}^{n \times n}$ is a real positive semidefinite matrix and $b \in \mathbb{R}^n$. Denote by \mathcal{V} and \mathcal{U} a subspace of \mathbb{R}^n and its orthogonal complement.

Let $V \in \mathbb{R}^{n \times m}$ and $U \in \mathbb{R}^{n \times (n-m)}$ be two matrices whose columns are formed by the orthonormal bases of \mathcal{V} and \mathcal{U} , respectively. Denote by P a conjugate projector on \mathcal{V} . Let us remind that a matrix P is a conjugate projector on a subspace \mathcal{V} iff $P^2 = P$ and $P^T A(I - P) = 0$. In this case $I - P$ is a conjugate projector on the null space of P . Notice that AP is symmetric and positive semidefinite as $P^T A = P^T AP$.

Theorem. Let f, U, V, \mathcal{V} be defined as above, let \bar{x} be the solution of the problem $\min f(x)$, and let \bar{x}_ε denote for each $\varepsilon > 0$ the solution of the problem $\min (f(x) + \frac{1}{2}\varepsilon^{-1}x^T U U^T x)$.

If $V^T A V$ is invertible, then

$$|\bar{x} - \bar{x}_\varepsilon| \leq \varepsilon(|P - V V^T| + 1)^2 |b|, \quad (3)$$

where $P = V(V^T A V)^{-1} V^T A$ is a conjugate projector on \mathcal{V} .

Proof. Under the assumption of the theorem, obviously $\bar{x} = V(V^T A V)^{-1} V^T b$ and $A + \varepsilon^{-1} U U^T$ is positive definite.

Now notice that $Q = (V, U)$ is an $n \times n$ orthogonal matrix, so that

$$(A + \varepsilon^{-1} U U^T)^{-1} = Q(Q^T(A + \varepsilon^{-1} U U^T)Q)^{-1} Q^T.$$

As

$$Q^T(A + \varepsilon^{-1} U U^T)Q = \begin{pmatrix} V^T A V & V^T A U \\ U^T A V & U^T A U + \varepsilon^{-1} I \end{pmatrix},$$

we can apply the formula for the inverse of 2×2 block matrix to get

$$\begin{aligned} \bar{x}_\varepsilon &= (V, U) \begin{pmatrix} (V^T A V)^{-1} + C Z C^T & -C Z \\ -Z C^T & Z \end{pmatrix} \begin{pmatrix} V^T \\ U^T \end{pmatrix} b = \\ &= \bar{x} + V C Z^T V^T b - V C Z U^T b - U Z C^T V^T b + U Z U^T b, \end{aligned} \quad (4)$$

where

$$C = (V^T A V)^{-1} V^T A U$$

and

$$Z = (U^T A U + \varepsilon^{-1} I - U^T A V (V^T A V)^{-1} V^T A U)^{-1}.$$

Let us now estimate the norm of Z . First notice that

$$Z = \varepsilon(I - \varepsilon U^T A (V (V^T A V)^{-1} V^T A - I) U)^{-1}. \quad (5)$$

Since $P = V(V^T A V)^{-1} V^T A$ is a conjugate projector on \mathcal{V} , we can rewrite Z into the form

$$Z = \varepsilon(I + \varepsilon U^T A (I - P) U)^{-1},$$

where the inverted matrix is obviously positive definite with all the eigenvalues greater or equal to 1. It follows that

$$|Z| \leq \varepsilon.$$

Further, notice that

$$\begin{aligned} |VC| &= |V(V^TAV)^{-1}V^TAU| = |PU| = |P(I - VV^T)| = \\ &= |P - PVV^T| = |P - VV^T|. \end{aligned}$$

Taking into account that both $|U| \leq 1$ and $|V| \leq 1$, we get

$$\begin{aligned} |x - x_\varepsilon| &= |VCZC^TV^Tb - VCZU^Tb - UZC^TV^Tb + UZU^Tb| \leq \\ &\leq (|VC|^2 + 2|VC| + 1)|Z||b| \leq \varepsilon(|VC| + 1)^2|b| = \\ &= \varepsilon(|P - VV^T| + 1)^2|b|. \end{aligned}$$

Consider the following trivial example: $A = I$, $b = (1, 0 \dots)^T$, \mathcal{V} is the set of all vectors with the first coordinate equal to 0. Then obviously $|b| = 1$,

$$|\bar{x} - \bar{x}_\varepsilon| = \varepsilon(1 + \varepsilon)^{-1},$$

while (3) yields

$$|\bar{x} - \bar{x}_\varepsilon| \leq \varepsilon.$$

This shows that the constant $(|P - VV^T| + 1)^2$ can not be improved without additional assumptions. \square

The following corollary gives us an idea about the constant in (3) in terms of the spectrum of A .

Corollary. Let f , U , V , A , \bar{x} and \bar{x}_ε be those of the theorem, and let A be invertible. Then

$$|\bar{x} - \bar{x}_\varepsilon| \leq \varepsilon(\kappa(A) + 1)^2|b|, \quad (4)$$

where $\kappa(A)$ is the spectral condition number of A .

Proof. Notice that

$$|P - VV^T| = |(P - VV^T)(UU^T + VV^T)| = |PUU^T| \leq |P|$$

and

$$|P| = |V(V^TAV)^{-1}V^TA| \leq |(V^TAV)^{-1}||A| \leq |A^{-1}||A|.$$

Thus $|P - VV^T| \leq \kappa(A)$, which substituted into (3) yields (4). \square

3. NUMERICAL EXAMPLE

As an illustration, consider the problem

$$\min_{x \in \mathcal{V}} \frac{1}{2}x^T Ax - b^T x,$$

where

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix},$$

$$\mathcal{V} = \{x: x_1 + x_3 = 0\}.$$

Direct computations show that $\kappa(A) = (1 + \sqrt{2})/(1 - \sqrt{2})$, so that we get the estimate

$$|x - x_\varepsilon| \leq 80.77\varepsilon.$$

The values of both sides are tabulated in Table 1. All the penalty computations were carried out in double precision and no serious effect of computer arithmetics has

Table 1.

ε	10^{-1}	10^{-4}	10^{-7}
$ x - x_\varepsilon $	1.716E-1	1.837E-4	1.837E-7
$\varepsilon(1 + \kappa(A))^2 b $	8.077E0	8.077E-3	8.077E-6

been observed with $\varepsilon^{-1} \leq 10^{10}$. We conclude that the estimate (4) may give us some idea about the error of penalty computations, though in many cases it will be too pessimistic.

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