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Ewaryst Rafajłowicz
Optimization of measurements for state estimation in parabolic distributed systems

OPTIMIZATION OF MEASUREMENTS FOR STATE ESTIMATION IN PARABOLIC DISTRIBUTED SYSTEMS

EWARYST RAFAJŁOWICZ

The aim of this paper is to consider the optimal, from the viewpoint of a state estimation ac-
curacy, measurement location problem in linear distributed-parameter systems with white Gaus-
sian noises. In contrary to earlier works the proposed approach consists of two steps. Namely,
to find closed form solution for the optimal measurement weighting function (MWF), assuming
measurements continuous in space, and then to approximate this solution using point sensors.
At the first step it was also shown that the Green function of the system can be used as a subopti-
mal MWF. State estimation in one-dimensional heat transfer problem illustrates the results.

1. INTRODUCTION

Theory of optimal state estimation for linear distributed parameter systems (DPS)
has reached a certain degree of completeness (see [4] and the bibliography therein).
As it is pointed out by several authors (see e.g. [4], [8] and confirmed by simulation
studies [8], [1]), applications of this theory are strongly dependent on measurements
type and locations. A problem of sensors locations, from the viewpoint of observ-
ability and state estimation accuracy, has been considered in a number of papers
(see e.g. [1], [2], [3], [7], [8]). A common feature of methods applied in these
papers is their computationally-oriented character, i.e. they present different numeri-
cal algorithms for an estimation accuracy maximization with respect to spatial
location of sensors. An approach of this kind, although valuable from a practical
point of view, can give only limited insight into a nature of the problem.

The aim of this paper is to propose an alternative approach, which consists of
two steps. Namely, to find explicit (or almost explicit) formulas for an optimal weight-
ing functions of measurement devices acting continuously in space and then to ap-
proximate them basing on point observations. Realisation of the first step is possible
if we assume that both measurement and equation errors are the Gaussian white
noises and that the system is described by the linear parabolic partial differential
equation (PDE) with point spectrum and complete set of eigenfunctions. These

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assumptions are precisely described in Section 2, where the problem of choice of measurements weighting functions (MWF) so as to minimize spatially averaged asymptotic variance of a state estimator is also stated. In Section 3 the optimal solution is expressed in the form of an infinite series and uncoupled algebraic equations for its coefficients are derived. Approximate solutions of these equations lead to the suboptimal MWF, which occur to be directly proportional to the system Green's function. This important relationship between the suboptimal MWF and properties of the system itself is a consequence of the fact that the noises, being white, do not corrupt internal characteristics of the system. The results of Section 3 are used in Section 4 in order to find positions of point sensors for which basic properties of the optimal (or suboptimal) MWF are approximately retained. An illustrative example is also given in this section.

2. ACCURACY OF STATE ESTIMATION AND PROBLEM STATEMENT

System description. Consider a linear stochastic DPS described by

$$(2.1) \qquad \frac{\partial q(x,t)}{\partial t} = A_x q(x,t) + w(x,t) : t > 0, x \in \Omega$$

defined on the open spatial domain $\Omega \subset \mathbb{R}^k$. Above $q(x,t)$ is one dimensional state of the system, A_x is a linear spatial differential operator whose parameters may depend on x but not on t (further restrictions on A_x will be imposed later). $w(x,t)$ is the Gaussian stochastic process with zero mean and the covariance function

$$(2.2) \qquad E[w(x,t) \cdot w(y,\tau)] = \delta(t-\tau) \cdot \delta(x-y)$$

where $\delta(t)$ and $\delta(x)$ are the Dirac delta functions in time and space. The initial and boundary conditions for (2.1) are of the form:

$$(2.3) \qquad q(x,0) = q_0(x) \quad \text{for} \quad x \in \Omega$$

$$(2.4) \qquad B_x q(x,t) = 0 \quad \text{for} \quad t > 0, x \in \Gamma$$

where Γ is the boundary of Ω, B_x is a linear time-invariant spatial differential operator while $q_0(x)$ is the Gaussian function with the mean $\tilde{q}_0(x)$ and the covariance function:

$$(2.5) \qquad E[(q_0(x) - \tilde{q}_0(x))(q_0(y) - \tilde{q}_0(y))] = P_0(x,y).$$

It is assumed that the solution of (2.1), (2.3), (2.4), understood in the weak sense [4], exists and is unique.

Measurements. The state $q(x,t)$ is estimated from measurements:

$$(2.6) \qquad s(x,t) = \int_\Omega c(x,\varkappa) q(\varkappa,t) \, d\varkappa + v(x,t) : t \in (0,T_0), x \in \Omega$$

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where $c: \Omega \times \Omega \to R$ is a measurement operator kernel (MOK) to be described later. $v(x, t)$ is the Gaussian white noise with zero mean and

(2.7) $$\mathsf{E}[v(x, t) \cdot v(y, \tau)] = \delta(t - \tau)\,\delta(x - y).$$

It is also assumed that the stochastic processes w, v and q_0 are mutually independent.

Estimation accuracy. Let $\hat{q}(x, t)$ denote the optimal (in the mean square sense) estimate of $q(x, t)$ based on measurements $s(x, \tau)$, $x \in \Omega\ \tau \in (0, t)$. As is known (see [4], [8]) $\hat{q}(x, t)$ can be obtained by using equations of the Kalman-Bucy filter and the estimate covariance function, defined as $\bar{P}(x, y, t) \triangleq \mathsf{E}[(q(x, t) - \hat{q}(x, t)) \cdot (q(y, t) - \hat{q}(y, t))]$, fulfills the following equation:

(2.8) $$\frac{\partial \bar{P}(x, y, t)}{\partial t} = A_x \bar{P}(x, y, t) + A_y \bar{P}(x, y, t) + \delta(x - y) -$$

$$- \int_\Omega \int_\Omega \bar{P}(x, \varkappa, t)\, e(\varkappa, \eta)\, \bar{P}(y, \eta, t)\, \mathrm{d}\varkappa\, \mathrm{d}\eta : t > 0,\ x, y \in \Omega$$

with the initial and boundary conditions:

(2.9) $$\bar{P}(x, y, 0) = P_0(x, y);\ x, y \in \Omega$$

(2.10) $$B_x \bar{P}(x, y, t) = 0,\ x \in \Gamma,\ y \in \Omega \cup \Gamma,\ t > 0$$

where

(2.11) $$e(\varkappa, \eta) \triangleq \int_\Omega c(\varkappa, v) \cdot c(v \cdot \eta)\, \mathrm{d}v.$$

We assume that the steady state covariance function, defined as $P(x, y) \triangleq$ $\triangleq \lim\limits_{t \to \infty} \bar{P}(x, y, t)$ exists, is unique and fulfills the equation:

(2.12) $$0 = A_x P(x, y) + A_y P(x, y) + \delta(x - y) - \int_\Omega \int_\Omega P(x, \varkappa)\, e(\varkappa, \eta)\, P(\eta, y)\, \mathrm{d}\varkappa\, \mathrm{d}\eta$$

for $x, y \in \Omega$ with the boundary condition: $B_x P(x, y) = 0$, $x \in \Gamma$, $y \in \Omega \cup \Gamma$. Sufficient conditions for existence of $P(x, y)$ can be found in [4].

The following restrictions are imposed on the operator A_x:

A$_1$) The operator A_x is defined on the Hilbert space V, which is dense subset of $L^2(\Omega)$ with elements fulfilling the boundary condition $B_x v = 0$, $v \in V$.

A$_2$) Derivatives in A_x are understood in the generalized sense and $A_x v \in L^2(\Omega)$ for $v \in V$.

A$_3$) A_x is symmetric and eigenvalues $\lambda_1, \lambda_2, \ldots$ of the equation:

(2.13) $$A_x \varphi = -\lambda \cdot \varphi, \quad \varphi \in V$$

are real and positive, while associated eigenfunctions φ_1, φ_2, ... form complete and orthonormal sequence in $L^2(\Omega)$.

Remark 1. A typical example of A_x and V, for which A$_1$), A$_2$), A$_3$) hold is the

Laplace operator with homogenous boundary condition of the first kind if V is the Sobolev space $H_0^1(\Omega)$ (see [5], where sufficient conditions for operators A_x to possess complete set of eigenfunctions can also be found).

Problem formulation. Our aim is to minimize the averaged steady-state estimation error defined as

$$(2.14) \qquad \int_\Omega P(x, x)\, dx$$

by suitable choice of MOK $c(x, y)$. It is assumed that the choice of c is restricted to a class of functions of the form:

$$(2.15) \qquad c(x, y) = \sum_{i=1}^{\infty} c_i \cdot \varphi_i(x) \cdot \varphi_i(y)$$

and such that.

$$(2.16) \qquad \sum_{i=1}^{\infty} c_i^2 \leqq N_e$$

where $c_i \neq 0$, $i = 1, 2, \ldots$ are real numbers to be chosen, while $0 < N_e < \infty$ is a given number.

Remark 2. If the signal $g \in L^2(\Omega)$ is measured without a noise by using MOK of the form (2.15) then the resulting signal w is given by

$$(2.17) \qquad w(x) = \int_\Omega c(x, y)\, g(y)\, dy = \sum_{i=1}^{\infty} c_i \cdot g_i \cdot \varphi_i(x)$$

where

$$g_i = \int_\Omega g(x) \cdot \varphi_i(x)\, dx, \quad i = 1, 2, \ldots.$$

As one can see from (2.17) kernels of the form (2.15) do not lead to aliasing between different modes in w, i.e. each mode of g has its own amplification factor c_i. Furthermore, reconstruction of g from given w is a simple task since

$$g_i = \int_\Omega w(x)\, \varphi_i(x)\, dx \,/\, c_i, \quad i = 1, 2, \ldots.$$

Note that (2.16) and the Schwarz inequality implies that $w \in L^2(\Omega)$.

3. PROBLEM SOLUTION

From (2.12) it can be seen that $P(x, y)$ depends on $c(x, y)$ only through $e(x, y)$ defined by (2.11). Hence it is convenient to minimize (2.14) with respect to $e(x, y)$ and then to find the corresponding $c(x, y)$. To this end, we note that (2.15), (2.16) and (2.11) yield

$$(3.1) \qquad e(x, y) = \sum_{i=1}^{\infty} e_i \cdot \varphi_i(x) \cdot \varphi_i(y)$$

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$$(3.2) \qquad\qquad \sum_{i=1}^{\infty} e_i \leqq N_e$$

$$(3.3) \qquad\qquad e_i = c_i^2 > 0 , \quad i = 1, 2, \dots .$$

Now, the problem can be reformulated as follows: Find $e^* \in E$, for which the steady-state covariance function exists and

$$(3.4) \qquad\qquad \min_{e \in E} \int_{\Omega} P(x, x) \, \mathrm{d}x$$

is attained, where P depends on $e \in E$ through the constraint (2.12). In the above E denotes the class of functions, for which (3.1), (3.2), (3.3) hold.

Our approach to the optimization problem (3.4) is based on eigenfunction expansion of the function $P(x, y)$.

Let $e \in E$ be arbitrary but fixed. Taking (3.1) into account, we look for the solution of (2.12) in the form:

$$(3.5) \qquad\qquad P(x, y) = \sum_{k=1}^{\infty} p_k \cdot \varphi_k(x) \cdot \varphi_k(y)$$

where p_k, $k = 1, 2, \dots$ are chosen in such a way that (2.12) holds. Substitution of (3.5) into (2.12), multiplication of both sides by $\varphi_j(x) \cdot \varphi_j(y)$ and integration twice over Ω yield

$$(3.6) \qquad\qquad -2 \cdot p_j \cdot \lambda_j + 1 - p_j^2 \cdot e_j = 0 , \quad j = 1, 2, \dots .$$

We have to find nonnegative solutions of (3.6) since $P(x, y)$ is nonnegative definite. These are of the form:

$$(3.7) \qquad\qquad p_j = \left[-\lambda_j + \sqrt{(\lambda_j^2 + e_j)} \right]/e_j , \quad j = 1, 2, \dots .$$

Thus, for every $e \in E$, for which (3.3) holds, the unique solution of (2.12) is of the form (3.5) with p_j given by (3.7). We impose (3.3) since if $c_i = 0$ then the information concerning ith mode is not gained.

The above considerations and substitution of (3.5) into (2.14) imply that our problem can be equivalently reformulated as follows: Find $e_j^*, j = 1, 2, \dots$, which are positive and such that

$$(3.8) \qquad\qquad \min_{\{e_j > 0\}} \left\{ \sum_{j=1}^{\infty} \left[-\lambda_j + \sqrt{(\lambda_j^2 + e_j)} \right]/e_j \right\}$$

is attained under the constraint (3.2).

It is easy to verify that each member of the sum (3.8) is decreasing function of e_j for $e_j > 0$, $j = 1, 2, \dots$. Thus it is sufficient to consider the optimization problem under the constraint

$$(3.9) \qquad\qquad \sum_{j=1}^{\infty} e_j = N_e .$$

Using the Lagrange multiplier method we can decompose (3.8) as follows

(3.10)
$$\min_{e_j > 0} \left[L_j(e_j) \right], \quad j = 1, 2 \dots$$

where

(3.11)
$$L_j(e_j) = \left[-\lambda_j + \sqrt{(\lambda_j^2 + e_j)} \right]/e_j + \gamma e_j$$

while γ is the Lagrange multiplier to be determined from (3.9).

Now, we can look for the solutions of the optimization problems (3.10), (3.11), (3.9) using different approaches. We discuss only two of them.

Optimal solution. Analysis of the function (3.11) shows that the value $e_k^*(\gamma)$ which minimizes (3.11) for a given γ is a root of the equation:

(3.12)
$$\frac{dL_k(e_k)}{de_k} = \frac{\lambda_k \sqrt{(\lambda_k^2 + e_k)} - \lambda_k^2 - 0 \cdot 5 e_k}{e_k^2 \sqrt{(\lambda_k^2 + e_k)}} + \gamma = 0$$

and γ is determined from the condition (3.9), i.e.

(3.13)
$$\sum_{k=1}^{\infty} e_k^*(\gamma) = N_e.$$

One can notice that by substitution $z_k = \sqrt{(\lambda_k^2 + e_k)}$ the equation (3.12) can be converted into a third order algebraic equation. Closed form formulas for roots of such equations, although available, are too complicated for analysis and this approach is recommended when high precision of numerical results is necessary.

Suboptimal solution. The fact that in typical cases $\lim_{k \to \infty} \lambda_k = \infty$ together with (3.9) implies that $\lim_{k \to \infty} e_k = 0$. Thus, the term $\sqrt{(\lambda_k^2 + e_k)}$ in (3.12) can be approximated by λ_k, $k = 1, 2 \dots$. This yields

(3.14)
$$\hat{e}_k = 1/(2\hat{\gamma}\lambda_k), \quad k = 1, 2, \dots$$

further called the suboptimal solution, where $\hat{\gamma}$ is determined by

(3.15)
$$\hat{\gamma} = (2N_e)^{-1} \cdot \sum_{j=1}^{\infty} 1/\lambda_j.$$

In order to evaluate $|e_k^* - \hat{e}_k|$ one can use two terms of Taylor's expansion of the function $dL_k(e_k)/de_k$ at the point e_k^*. This and bounds, upper for $dL_k(e_k)/de_k$ and lower for the second derivative, lead to the inequality $|e_k^* - \hat{e}_k| \leq C/\lambda_k$, $k = 1, 2 \dots$, $C > 0$. Summarizing, the approximate solution of the problem (2.14) is

(3.16)
$$\hat{e}(x, y) = \sum_{k=1}^{\infty} \hat{e}_k \varphi_k(x) \, \varphi_k(y)$$

where $\hat{e}_k$ is given by (3.14). On the other hand, the Green function $G(x, y)$ of the

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operator A_x, which is the solution of $A_x G(x, y) = \delta(x - y)$, is of the form:

$$(3.17) \qquad G(x, y) = \sum_{k=1}^{\infty} \varphi_k(x)\, \varphi_k(y)/\lambda_k .$$

Comparing (3.16), (3.14) and (3.17) we have:

Collorary. The suboptimal solution $\hat{e}(x, y)$ is directly proportional to the Green function of the system elliptic operator A_x.

Using (2.11) and (3.16) it is easy to verify that the suboptimal MOK $\hat{c}(x, y)$, corresponding to $\hat{e}(x, y)$, is given by

$$(3.18) \qquad \hat{c}(x, y) = \sum_{k=1}^{\infty} \sqrt{(\hat{e}_k)}\, \varphi_k(x) \cdot \varphi_k(y).$$

It is clear that the optimal solution of (2.14) is also of the form (3.16) with $\hat{e}_k$ replaced by e_k^*, $k = 1, 2 \ldots$, while for the corresponding optimal MOK we have

$$(3.19) \qquad c^*(x, y) = \sum_{k=1}^{\infty} \sqrt{(e_k^*)}\, \varphi_k(x) \cdot \varphi_k(y).$$

4. POINT MEASUREMENTS AND ILLUSTRATIVE EXAMPLE

The above results form a basis for finding sensors positions and a method of discrete in space measurements processing, which ensure the estimation accuracy close to the optimal one i.e. attainable for continuous in space measurements. The idea of proposed discretization is based on the fact that the sequence e_k^*, $k = 1, 2, \ldots$ approaches rapidly to zero. Hence, nearly the same estimation accuracy can be expected if the positions $\xi_i \in \Omega$, $i = 1, 2, \ldots$, of point sensors and the symmetric and non-negative definite matrix $D = [d_{im}]_{i,m=1,2 \ldots M}$ are chosen in such a way that for the function:

$$(4.1) \qquad e_d(x, y) = \sum_{i,m=1}^{M} d_{im} \delta(x - \xi_i)\, \delta(y - \xi_m)$$

we have

$$(4.2) \qquad \int_{\Omega}\int_{\Omega} e_d(x, y)\, \varphi_k(x) \cdot \varphi_l(y)\, dx\, dy = \begin{cases} 0 & \text{for } k \neq l \\ e_k^* & \text{for } k = l \end{cases}$$

where $k, l = 1, 2, \ldots M$. To this end it is sufficient to choose ξ_i for which

$$(4.3) \qquad \sum_{i=1}^{M} \varphi_k(\xi_i)\, \varphi_l(\xi_i) = \begin{cases} 0 & \text{for } k \neq l \, ; \\ \beta_k & \text{for } k = l \end{cases} \quad k, l = 1, 2 \ldots M$$

and the matrix D of the form:

$$(4.4) \qquad D = \sum_{k=1}^{M} \frac{e_k^*}{\beta_k}\, \bar{\varphi}_i \bar{\varphi}_i^{\mathsf{T}}$$

where $\bar{\varphi}_i^T = \left[\varphi_i(\xi_1), \varphi_i(\xi_2), \cdots, \varphi_i(\xi_M)\right]$. It is worthwhile to notice that for typical eigenfunctions (e.g. Jacobi polynomials and sinusoidal functions) the points ξ_i, $i = 1, 2, \ldots, M$, for which (4.3) holds are known. It is clear that if e_k^*, $k = 1, 2, \ldots, M$, in (4.2), (4.4) are replaced by $\hat{e}_k$ then the above considerations are valid for the sub-optimal $\hat{e}(x, y)$. It should be noticed that for the function (4.1) the condition (2.16) does not hold and in practical applications the Dirac delta function must be replaced by a square integrable approximation.

Illustrative example. Let us consider the system described by (2.1) with $A_x = \mathrm{d}^2/\mathrm{d}x^2$, $\Omega = (0, \pi)$ and the boundary conditions: $q(0, t) = q(\pi, t) = 0$ for $t > 0$. Eigenfunctions and eigenvalues of this operator are of the form: $(2/\pi)^{1/2} \sin kx$ and k^2, $k = 1, 2, \ldots$. Substitution of $\lambda_k = k^2$ into (3.12) and (3.14) allows to calculate the Fourier coefficients e_k^* and $\hat{e}_k$ of the optimal and suboptimal solution, respectively. These values, obtained for $N_e = 65$ with $\gamma^* = 0.001$ and $\hat{\gamma} = 0.01$, are summarized in Table 1, together with the corresponding values p_k^* and $\hat{p}_k$ defined by (3.7).

Table 1.

k	1	2	3	4	5	6	7	8	9	10
e_k^*	50·5	14·5	$2 \cdot 10^{-5}$	0+	0+	0+	0+	0+	0+	0+
p_k^*	·1223	·1050	·0550	·0312	·0200	·0130	·0102	·0078	·0055	·0050
$\hat{e}_k$	41·38	10·12	4·63	2·61	1·66	1·15	0·85	0·65	0·51	0·42
$\hat{p}_k$	·1333	·1094	·0547	·0300	·0199	·0138	·0102	·0077	·0060	·0049

The symbol 0+ in Table 1 means that the corresponding number is not greater than 10^{-6}. Fig. 1 shows the optimal and suboptimal solution of the problem (2.14) at the point $x = \pi/2$ i.e. $e^*(\pi/2, y)$ and $\hat{e}(\pi/2, y)$. As one can see the values e_k^* and $\hat{e}_k$, $k = 1, 2, \ldots$, as well as the functions $e^*(\pi/2, y)$ and $\hat{e}(\pi/2, y)$ are not very close to each other. Nevertherless, the corresponding values of the optimality criterion, which are equal to $Q^* = 0.3764$ and $\hat{Q} = 0.3787$, are almost the same.

In order to approximate $e^*(x, y)$ by point measurement we use (4.1), (4.2), (4.3), (4.4) for $M = 2$, $\varphi_1(x) = \sin x$, $\varphi_2(x) = \sin 2x$ and $\xi_1 = \pi/3$, $\xi_2 = 2\pi/3$ what yields

$$(4.5) \qquad D = \begin{bmatrix} 41·5 & 15·59 \\ 15·59 & 23·5 \end{bmatrix}$$

We remark that e_d defined by (4.1) with D given by (4.5) provides only a basis for finding a realizable method of estimation from point measurements. A necessity of dealing with Dirac delta functions implies that such a method can lead to MOK $c_d(x, y)$, for which the equality

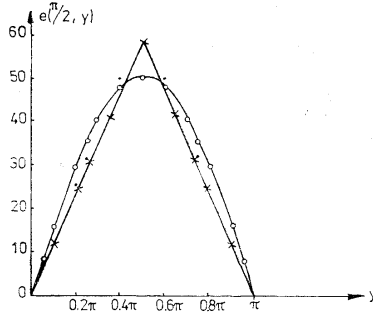$$(4.6) \qquad e_d(x, y) = \int_\Omega c_d(x, v) \cdot c_d(v, y)\, \mathrm{d}v$$

Fig. 1. Optimal ∘ and suboptimal × measurement operator kernel in Example 4.

holds only approximately. In order to illustrate a way of finding such approximation let us look for c_d of the following form:

$$(4.7) \qquad c_d(x, v) = \sum_{i,j=1}^{M} r_{ij} \cdot h_i(x) \cdot \delta(v - \xi_j)$$

where r_{ij} and $h_i(x)$; $i, j = 1, 2, \ldots, M$, are to be chosen. Hence

$$\int_\Omega c_d(x, v)\, c_d(v, y)\, dv = \sum_{i,j,k,m=1}^{M} r_{ij} r_{km} h_i(x)\, h_k(\xi_j) \cdot \delta(y - \xi_m).$$

If we choose $h_i(x) = \delta_\varepsilon(x - \xi_i)$, $i = 1, 2, \ldots, M$, where

$$\delta_\varepsilon(x) = \begin{cases} 1/\varepsilon & \text{for} \quad |x| \leq \varepsilon/2 \\ 0 & \text{for} \quad |x| > \varepsilon/2 \end{cases}$$

then for sufficiently small $\varepsilon > 0$, such that $|\xi_i - \xi_k| > \varepsilon$ for $i \neq k$; $i, k = 1, 2, \ldots, M$, we obtain

$$(4.8) \qquad \int_\Omega c_d(x, v)\, c_d(v, y)\, dv = \frac{1}{\varepsilon} \sum_{i,m=1}^{M} \delta_\varepsilon(x - \xi_i) \cdot \delta(y - \xi_m) \sum_{j=1}^{M} r_{ij} r_{jm}.$$

One can notice that (4.8) approximates $e_d(x, y)$ if we take r_{ij}; $i, j = 1, 2, \ldots, M$, such that

$$(4.9) \qquad d_{im} = \frac{1}{\varepsilon} \sum_{j=1}^{M} r_{ij} r_{jm}, \quad i, m = 1, 2, \ldots, M,$$

It is to be noticed that symmetry of D implies that factorization (4.9) can always be done. In particular the matrix $R = [r_{ij}]$ can be chosen lower triangular. For D given by (4.5) equalities (4.9) hold for

$$(4.10) \qquad r_{11} = 6\cdot44 \sqrt{\varepsilon}\,, \quad r_{12} = 0\,, \quad r_{21} = 2\cdot42 \sqrt{\varepsilon}\,, \quad r_{22} = 4\cdot2 \sqrt{\varepsilon}$$

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and the desired MOK is given by

$$(4.11) \qquad c_d(x, v) = \sum_{i,j=1}^{2} r_{ij} \delta_\varepsilon (x - \xi_i) \, \delta(v - \xi_j)$$

We remark that exactly the same way can be used for approximation of $\hat{e}(x, y)$ by point measurements. In this case however M has to be greater in order to retain comparable accuracy.

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Dr. Ewaryst Rafajlowicz, Institute of Engineering Cybernetics, Technical University of Wrocław, Wybrzeże St. Wyspiańskiego 27, 50—370 Wrocław, Poland.