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Kybernetika, Vol. 31 (1995), No. 2, 207--211

Persistent URL: http://dml.cz/dmlcz/124420

KYBERNETIKA — VOLUME 31 (1995), NUMBER 2, PAGES 207-211

ON ONE NP-COMPLETE PROBLEM

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Let S be a finite set, and R be a set of three element subsets of S. An element r of R is interpreted as a production rule which enables to derive one of the elements of r from the others. A subset $X \subset S$ is conflicting if an element of S can be derived from X in two different ways. The problem of finding a largest non-conflicting subset is shown to be NP-complete.

Let S be a finite set; its elements will be called *constants*. Let R be a set of three element subsets of S. We interpret an element $r = \{a, b, c\} \in R$ as a *production rule*, which enables us to derive a value of any constant in r from the values of the remaining two constants.

Informally, we say that a subset of constants $X \subseteq S$ is conflicting if there is a constant which can be derived from X in two different ways. The problem treated here is to find, for a given set R of production rules, the largest non-conflicting set of constants. We show that this problem is NP-complete.

Let us point out that the problem is motivated by the study of models and useful constrains for qualitative physics. This is a new field of AI searching for an appropriate formalism supporting common sense reasoning, see [2] for a brief survey of this topic. The variables in the qualitative methodology are supposed to have only a fixed set of discrete values; mutual relations among variables are expressed by a limited set of dependencies (or constraints). The simplest constrains can be defined by the production rules mentioned above. The problem of existence of a non-conflicting set of a given size arises when trying to define a partially specified model for a given set of production rules, i.e. to find an evaluation of the set of variables corresponding to constraints given by production rules and the partial specification. The evaluation of a variable is called here a constant.

First, let us give some formal definitions. Let S be a non-empty finite set of constants, R be a set of production rules and X a non-empty subset of S. A *derivation* D *from* X is a finite sequence of ordered triples $\{(a_i, b_i, c_i)\}_{i=1}^{k}$ such that:

1. Members of each triple a_i, b_i, c_i form a production rule, i. e. $\{a_i, b_i, c_i\} \in R$. The third element, c_i, we consider to be derived from a_i, b_i.

2. Each of the first two members of any triple is either in X or has been derived earlier, i. e. $a_i, b_i \in X \cup \{c_j | 1 \leq j < i\}$.

The integer k is called the *length* of the derivation. An element $y \in S$ is *derived from* X by the derivation D if $y = c_i$ for some i. We say that all elements of X are derived from X by the empty derivation.

A *minimal derivation of an element* $y \in S$ *from* X is a derivation which derives y and it has no proper non-empty subderivation which derives y from X (i.e. we cannot omit any triples to get a smaller non-empty derivation of y from X). Every empty derivation is considered to be also a minimal one. Note that for every non-empty minimal derivation of y of the length k we have $y = c_k$, $k \leq |S|$ and $y \notin \{a_i, b_i, c_i | i < k\}$.

Two derivations are called *equivalent* if their sets of production rules are equal.

A set of constants X is called *conflicting* with respect to the set of rules R if there is an element of S which is derived by two non-equivalent minimal derivations from X.

Proposition 1. If there is an element $y \in X$ which is derived by a non-empty derivation from X then X is conflicting.

Proof. The proof is trivial; non-empty derivation of y contains a non-empty minimal one. The second minimal derivation is the empty one. □

Corollary 1. If there is a production rule $\{a, b, c\} \in R$ such that $\{a, b, c\} \subseteq X$ then X is conflicting with respect to R.

For a given set of constants $X \subset S$ and a set of production rules R the following simple polynomial algorithm decides whether X is conflicting with respect to R.

Algorithm 1.
{Input: sets S, R and X as described above.}
{Auxiliary variables:}
{Z is the set of constants that has been derived so far.}
{D is a derivation which derives all elements of Z.}
{*finished* is a boolean variable indicating end of computation.}
{*conflict* is a boolean variable indicating discovery of a conflict.}
begin
  $D := \emptyset$; $Z := X$;
  finished := false; *conflict* := false;
  while not *finished* do
    begin
      finished := true;
      for all $r \in R$ do
        if $|r \cap Z| = 3$ and r is not in D then *conflict* := true;
        else if $|r \cap Z| = 2$ then
          begin

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            denote a, b, c the elements of r such that {c} = r \ Z;
            append ordered triple (a, b, c) to D;
            Z := Z ∪ {c};
            finished := false;
          end;
    end;
    if conflict then write ("conflicting")
    else write ("non-conflicting");
end.
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Theorem 1. For a given set of constants $X \subseteq S$ and a set of production rules R the Algorithm 1 decides in polynomial time whether X is conflicting with respect to R.

Proof. The time bound follows from the fact that the while-loop is repeated at most $(|S \setminus X| + 1)$-times.

Let us prove the correctness.

a) Assume that the algorithm answered "conflicting". Let $r = \{a, b, c\}$ be the rule for which the variable *conflict* changed its value from false to true, i. e. $|r \cap Z| = 3$, so $r \subseteq Z$.

If $r \subseteq X$ then X is conflicting by Corollary 1.

Let $r \not\subseteq X$. Then, without loss of generality we can assume that $c \in Z \setminus X$ and each of a, b either belongs to X or was derived by D earlier than c. Denote $t = (p, q, c)$ the triple of D which derives c. Since $r \notin D$ it must hold $\{p, q\} \neq \{a, b\}$. Denote by D_1 the minimal non-empty derivation of c obtained from D by omitting some triples. Note that t is the last triple in D_1. The second minimal derivation D_2 of c we obtain from D by replacing t by (a, b, c) and then omitting unnecessary triples. Derivations D_1, D_2 are non-equivalent, hence X is conflicting.

b) Now, assume that the algorithm answered "non-conflicting". Then all constants which have a derivation from X are derived by D and all rules which can be used in any derivation from X are used in D. Let us prove that X is not conflicting in this case.

Assume for contrary that X is conflicting. Then there is a constant y with two non-equivalent minimal derivations D_1, D_2 from X. Without loss of generality we can assume that the sum of lengths of D_1, D_2 is minimal. Denote by B the set of all rules used in at least one of D_1, D_2.

Each constant which is contained in a rule from B is either in X or it is contained in at least two different rules of B. Indeed, for y it follows from the minimality of the sum of lengths: the last rules of D_1 and D_2 must be different. For other constants it follows from the minimality of derivations D_1, D_2: a constant $x \notin X$, $x \neq y$ is derived by a rule from B and (since $x \neq y$ and D_1, D_2 are minimal) is used by at least one other rule from B.

Let (a, b, c) be the last triple in D which is a use of a rule from B. Each of the constants a, b, c either is in X or it appeared in some earlier triple of D. So, the algorithm instead of appending (a, b, c) to D had to discover a conflict, a contradiction. □

Problem 1. Given a set of constants S, a set of rules R and an integer K. Decide whether there exists a non-conflicting set $X \subseteq S$ with respect to R with $|X| \geq K$.

Theorem 2. The Problem 1 is NP-complete.

Proof. First, the problem belongs to the class NP: One can non-deterministically guess a set X with at least K elements and use the above algorithm to verify (in a polynomial time) that X is non-conflicting with respect to R.

To prove that Problem 1 is NP-complete we show that the following well-known NP-complete problem [1] can be polynomially reduced to Problem 1.

> The Independent Set Problem: For a given undirected graph G and a given integer K, does there exist an independent set X of vertices with $|X| \geq K$. (A set of vertices is independent if it contains no two adjacent vertices.)

Let us have an undirected graph G and an integer K, we shall construct an instance of the Problem 1.

First, the Independent Set Problem can be easily reduced to a slightly restricted version in which the graph has no isolated vertices and $K \geq 3$. (Each isolated vertex can be replaced by a pair of adjacent vertices.)

Hence, let $G = (V, E)$, where V is the set of vertices, E is the set of undirected edges. Take three new elements $p, q, r \notin V$ and define a set of constants S and a set of production rules R as follows:

$$\begin{aligned} S &= V \cup \{p, q, r\} \\ R &= \{\{v, w, t\} | \{v, w\} \in E \text{ and } t \in \{p, q, r\}\} \\ &\quad \cup \{\{p, q, r\}\} \end{aligned}$$

To prove the Theorem it suffices to show that for every $X \subseteq S$ with at least three elements we have

(*) X is a non-conflicting set with respect to R if and only if $X \subseteq V$ and X is independent in G.

One implication is clear; any independent set $X \subseteq V$ in G with $|X| \geq 3$ is non-conflicting with respect to R since nothing can be derived from X.

Let us prove the other implication. Let $X \subseteq S$ be a non-conflicting set with respect to R and let $|X| \geq 3$.

a) First, we shall show that $X \subseteq V$. Since X is non-conflicting and $\{p, q, r\} \in R$, we get that $\{p, q, r\} \not\subseteq X$ (see Corollary 1). Since $|X| \geq 3$ we have that X contains at least one element v of V. Note that v is adjacent to at least one other vertex $w \in V$. Now, assume for contradiction, that $X \cap \{p, q, r\}$ is non-empty. Without loss of generality we assume that $p \in X$. If $w \in X$ then $\{v, w, p\} \subseteq X$, a contradiction (see Corollary 1). If $w \notin X$ consider the following two derivations from X:

(1) $(v, p, w), (v, w, q), (p, q, r)$

(2) $(v,p,w),(v,w,r)$

They are clearly minimal and non-equivalent. Thus X is conflicting, a contradiction. Therefore $X \cap \{p,q,r\}$ is empty and $X \subseteq V$.

b) It remains to prove that X is an independent set of vertices in G. Assume that there exist $v, w \in X$ with $\{v,w\} \in E$. Then the following two minimal derivations of r from X are non-equivalent:

(3) $(v,w,p),(v,w,q),(p,q,r)$

(4) (v,w,r)

Thus again X is conflicting, a contradiction.

Hence, we have proved $(*)$ which concludes the proof of the Theorem. □

(Received December 31, 1993.)

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