Kybernetika

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Kybernetika, Vol. 27 (1991), No. 6, 522--534

Persistent URL: http://dml.cz/dmlcz/124294

KYBERNETIKA — VOLUME 27 (1991), NUMBER 6

SEPARATION THEOREMS FOR SETS IN PRODUCT SPACES AND EQUIVALENT ASSERTIONS

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This paper is devoted to several separation and extension theorems in product spaces $X \times Z$ where Z is partially ordered. To give general assertions suitable for using in vector optimization we replace the order relation "$\leqq$" by "$\ngtr$". We show that all propositions are equivalent.

1. INTRODUCTION

In [13] several separation theorems for sets in product spaces with order structure were formulated by the author. These theorems differ from the assertions in [1], [9], [10], [14], [15], [16] because the order relation "$\leqq$" (smaller or equal) was replaced by the relation "$\ngtr$" (not greater). By this one can first neglect the validity of the least upper bound property of the order relation. Second we weaken the demand of comparability of the elements and so our assertions are more suitable to characterize Pareto optimal solutions of vector optimization problems.

In the present paper we will show analogously to the well-known results of convex analysis and the above mentioned papers that the separation theorems published in [13] are equivalent to concrete extension theorems in sense of Hahn-Banach and Krein-Rutman and to concrete propositions about the solutions of convex inequality systems and convex optimization problems.

Throughout this paper X and Z are real vector spaces, Z is partially ordered by a convex cone $Z_+ \subseteq Z$ which does not contain a proper subspace. For $z_1, z_2 \in Z$ we write

$$z_1 \leqq z_2 \quad \text{iff} \quad z_2 - z_1 \in Z_+ ,$$

$$z_1 < z_2 \quad \text{iff} \quad z_2 - z_1 \in Z_+ \setminus \{0\} ,$$

$$z_1 \ngtr z_2 \quad \text{iff} \quad z_1 - z_2 \notin Z_+ \setminus \{0\} ,$$

$$z_1 \nless\ngtr z_2 \quad \text{iff} \quad z_1 - z_2 \notin [Z_+ \cup (-Z_+)] \setminus \{0\} .$$

For two subsets $A, B \subseteq Z$ we define

$$A \prec B \quad \text{iff} \quad z_1 \prec z_2 \quad \forall z_1 \in A \quad \forall z_2 \in B,$$

where $\prec$ may be an above mentioned relation.

The elementary operations as convex hull, convex cone hull, algebraic closure, algebraic interior (core) and relative algebraic interior (intrinsic core) of a set M will be denoted by conv M, cone M, lin M, cor M and icr M respectively.

If $X_0 \subseteq X$ is a linear subspace, then $\mathscr{L}(X_0, Z)$ is the vector space of all linear mappings $l: X_0 \to Z$. Especially, $X^* := \mathscr{L}(X, \mathbb{R})$ and $Z^* := \mathscr{L}(Z, \mathbb{R})$ are the algebraic dual spaces of X and Z. Let $M \subseteq X \times Z$. We set

$$M(x) := \{z \in Z \mid (x, z) \in M\},$$
$$\mathscr{P}_Z(M) := \{z \in Z \mid \exists x \in X : (x, z) \in M\},$$
$$\mathscr{P}_X(M) := \{x \in X \mid \exists z \in Z: (x, z) \in M\}.$$

If M is convex, then obviously the following inclusions hold

$$\operatorname{cor} M \neq \emptyset \;\Rightarrow\; \mathscr{P}_X(\operatorname{cor} M) = \operatorname{cor} \mathscr{P}_X(M); \tag{1.1}$$

$$(\operatorname{cor} M)(x) \neq \emptyset \;\Rightarrow\; \operatorname{cor} M(x) = (\operatorname{cor} M)(x); \tag{1.2}$$

$$\operatorname{cor} \mathscr{P}_X(M) \neq \emptyset, \quad \operatorname{cor} M(x) \neq \emptyset \quad \forall x \in \operatorname{cor} \mathscr{P}_X(M) \;\Rightarrow\; \operatorname{cor} M \neq \emptyset. \tag{1.3}$$

Let $D(f), D(g) \subseteq X$ be convex sets. A mapping $f: D(f) \to Z$ is said to be convex (sublinear) iff the set

$$\operatorname{epi} f := \{(x, z) \in X \times Z \mid f(x) \leqq z\}$$

is convex (a convex cone). $g: D(g) \to Z$ is said to be concave (superlinear) iff the set

$$\operatorname{hypo} g := \{(x, z) \in X \times Z \mid g(x) \geqq z\}$$

is convex (a convex cone).

For the ordering cone $Z_+ \subseteq Z$ we define the algebraic dual cone and its quasi-interior by

$$Z_+^* := \{z^* \in Z^* \mid z^*(z) \geqq 0 \;\forall z \in Z_+\},$$
$$q\text{-int}\, Z_+^* := \{z^* \in Z^* \mid z^*(z) > 0 \;\forall z \in Z_+ \setminus \{0\}\}.$$

By means of these cones we get the following relations

$$z_1 \leqq z_2 \;\Rightarrow\; z^*(z_1) \leqq z^*(z_2) \quad \forall z^* \in Z_+^*, \tag{1.4}$$

$$z_1 < z_2 \;\Rightarrow\; z^*(z_1) < z^*(z_2) \quad \forall z^* \in q\text{-int}\, Z_+^* \tag{1.5}$$

or equivalently

$$\exists z^* \in q\text{-int}\, Z_+^* : z^*(z_1) \leqq z^*(z_2) \;\Rightarrow\; z_1 \not> z_2. \tag{1.6}$$

We notice that generally the converse implications are not true. For this we must introduce a topology in Z (cf. [13]).

2. SEPARATION OF SETS

Two sets $A, B \subseteq X \times Z$ are called (cf. [13])

- *separable by an affine mapping*, if there exist a linear mapping $l \in \mathscr{L}(X, Z)$ and an element $z_0 \in Z$ such that

$$z_1 - l(x_1) \nless z_0 \nless z_2 - l(x_2) \tag{2.1}$$

for all $(x_1, z_1) \in A$ and all $(x_2, z_2) \in B$.

- *strictly separate by an affine mapping*, if there exist a linear mapping $l \in \mathscr{L}(X, Z)$, an element $\alpha \in \mathbb{R}$ and a linear functional $z^* \in q\text{-int } Z^*_+$ such that

$$z^*(z_1 - l(x_1)) \geqq \alpha \geqq z^*(z_2 - l(x_2)) \tag{2.2}$$

for all $(x_1, z_1) \in A$ and $(x_2, z_2) \in B$.

It is obvious that A and B are separable by an affine mapping if they are strictly separable by this mapping. Really, if (2.2) is fulfilled then since $z^* \neq 0$ we can find a $z_0 \in Z$ with $z^*(z_0) = \alpha$. According to (1.6) we get (2.1).

To give an equivalent description of the strict separability of two sets we will say a hyperplane

$$H = \{(x, z) \in X \times Z \mid x^*(x) + z^*(z) = \alpha\}$$

(here $x^* \in X^*$, $z^* \in Z^*$, $\alpha \in \mathbb{R}$) is *nonvertical* iff $z^* \in q\text{-int } Z^*_+$. In [13] nonvertical affine manifolds were inquired. Especially, it was showed that a hyperplane is nonvertical if and only if $\mathscr{P}_X(H) = X$ and $z_1 \nless \ngtr z_2$ for all $z_1, z_2 \in H(x)$ and all fixed $x \in X$.

Proposition 2.1. Two sets $A, B \subseteq X \times Z$ are strictly separable by an affine mapping if and only if there exists a nonvertical hyperplane $H \subset X \times Z$ which separates A and B in the sense of convex analysis.

Proof. 1) Let $H = \{(x, z) \in X \times Z \mid x^*(x) + z^*(z) = \alpha\}$ be a nonvertical hyperplane which separates A and B in classical sense, i.e.

$$x^*(x_1) + z^*(z_1) \geqq \alpha \geqq x^*(x_2) + z^*(z_2) \tag{2.3}$$

for all $(x_1, z_1) \in A$ and all $(x_2, z_2) \in B$. Since $z^* \in q\text{-int } Z^*_+$ we get $\mathscr{P}_X(H) = X$ and we can find a linear mapping $l \in \mathscr{L}(X, Z)$ and an element $z_0 \in Z$ such that

$$l(x) + z_0 \in H(x)$$

for all $x \in X$. Moreover, with $L := \{z \in Z \mid z^*(z) = 0\}$ we get

$$H(x) = l(x) + z_0 + L$$

for all $x \in X$ (cf. [13]). Thus, it holds for all $x \in X$

$$x^*(x) + z^*(l(x) + z_0) = \alpha$$

and that means $x^* = -z^* \circ l$, $z^*(z_0) = \alpha$. From (2.3) we get (2.2).

2) Conversely let A and B be strictly separable by an affine mapping, i.e. let (2.2)

be fulfilled with $l \in \mathscr{L}(X, Z)$, $\alpha \in \mathbb{R}$ and $z^* \in q\text{-int}\, Z_+^*$. Obviously the set

$$H := \{(x, z) \in X \times Z \mid z^*(z - l(x)) = \alpha\}$$

is a nonvertical hyperplane since $z^* \circ l \in X^*$ and $z^* \in q\text{-int}\, Z_+^*$. With $x^* := -z^* \circ l$ (2.2) and (2.3) are equivalent. □

By this assertion we get the possibility to use the well-known separation theorems from convex analysis for our considerations. Especially, we can state that two sets A and B are strictly separable by an affine mapping if and only if the sets $A - B$ and $\{(0, 0)\}$ have this property. This important fact will be used in the following.

All the sets are assumed to be convex. More general assertions can be derived by means of the convex hull of the sets (cf. [13]).

Theorem 2.2. Let $A, B \subseteq X \times Z$ be convex sets with the following properties:

(1) there exists a linear functional $z^* \in q\text{-int}\, Z_+^*$ with

$$z^*(z_1) \geqq z^*(z_2)$$

for all $z_1 \in A(x)$, $z_2 \in B(x)$ and any fixed $x \in \mathscr{P}_X(A) \cap \mathscr{P}_X(B)$,

(2) $0 \in \operatorname{icr}(\mathscr{P}_X(A - B))$.

Then there exist a linear mapping $l \in \mathscr{L}(X, Z)$ and a real number $\alpha \in \mathbb{R}$ such that

$$z^*(z_1 - l(x_1)) \geqq \alpha \geqq z^*(z_2 - l(x_2))$$

for all $(x_1, z_1) \in A$ and all $(x_2, z_2) \in B$, i.e. A and B are strictly separable.

Proof. First we shall assume that $\operatorname{cor} \mathscr{P}_X(A - B) \neq \emptyset$ and $0 \in \operatorname{cor} \mathscr{P}_X(A - B)$. We set

$$C := A - B + (\{0\} \times \{z \in Z \mid z^*(z)) > 0\}),$$
$$D := \{0\} \times \{z \in Z \mid z^*(z) = 0\}.$$

Since $\operatorname{cor} \mathscr{P}_X(C) \neq \emptyset$ and $\operatorname{cor} C(x) \neq \emptyset$ for all $x \in \mathscr{P}_X(C)$, according to (1.3) we get $\operatorname{cor} C \neq \emptyset$. Further, with (1) we have $C \cap D = \emptyset$ and so there exists a hyperplane H which separates C and D in the classical sense. Moreover, since D is an affine manifold the separating hyperplane H can be chosen such that $D \subseteq H$. With (2) we get $0 \in \operatorname{cor} \mathscr{P}_X(H)$, but this is equivalent to $\mathscr{P}_X(H) = X$ and the hyperplane H can be represented in the form

$$H = \{(x, z) \in X \times Z \mid x^*(x) + z^*(z) = 0\}$$

with $x^* \in X^*$. That means especially

$$(x^*(x_1) + z^*(z_1)) - (x^*(x_2) + z^*(z_2)) \geqq 0$$

for all $(x_1, z_1) \in A$ and all $(x_2, z_2) \in B$ and we can find a real number $\alpha \in \mathbb{R}$ such that the hyperplane

$$\tilde{H} := \{(x, z) \in X \times Z \mid x^*(x) + z^*(z) = \alpha\}$$

separates the sets A and B. Since $z^* \in q\text{-int}\, Z_+^*$ both hyperplanes are nonvertical. Proposition 2.1 completes the proof.

Now let cor $\mathscr{P}_X(A - B) = \emptyset$, i.e. the linear hull of $\mathscr{P}_X(A - B)$ is a proper linear subspace X_0 of X. In this case we regard the sets C and D as subsets of $X_0 \times Z$. Thus, we get the existence of a nonvertical hyperplane H_0 in $X_0 \times Z$ which separates C and D. If X_1 is the algebraically complementary subspace of X_0 then every $x \in X$ has an unique representation in the form $x = x_0 + x_1$, where $x_0 \in X_0$ and $x_1 \in X_1$. Now, H_0 can be extended to a hyperplane in $X \times Z$ according to

$$H(x) = H(x_0 + x_1) := H_0(x_0), \quad x \in X .$$

The nonvertical hyperplane H separates C and D and the proof can be finished analogously as above. □

The first consequence of Theorem 2.2 is the following Sandwich assertion.

Theorem 2.3. If f: $D(f) \subseteq X \to Z$ is a convex mapping and g: $D(g) \subseteq X \to Z$ is a concave mapping with the following properties:

(1) there exists a linear functional $z^* \in q\text{-int}\, Z_+^*$ with

$$z^*(f(x)) \geqq z^*(g(x))$$

for all $x \in D(f) \cap D(g)$,

(2) $0 \in \operatorname{icr}(D(f) - D(g))$,

then there exist a linear mapping $l \in \mathscr{L}(X, Z)$ and an element $z_0 \in Z$ such that

$$z^*(f(x)) \geqq z^*(l(x) + z_0) \quad \text{for all} \quad x \in D(f) ,$$
$$z^*(g(x)) \leqq z^*(l(x) + z_0) \quad \text{for all} \quad x \in D(g) .$$

Proof. Let $A := \operatorname{epi} f$ and $B := \operatorname{hypo} g$. Then we can derive the result immediately from Theorem 2.2. □

Concerning the classical separation concept it is well-known that an affine manifold can be separated from a convex mapping (if possible) in such a way that the separating hyperplane contains the affine manifold. This property we have used previously in the proof of Proposition 2.2. To formulate an analogous result for our separation concept we need

Lemma 2.4. Let h: $Y_0 \subseteq X \to Z$ be an affine mapping, $l \in \mathscr{L}(X, Z)$, $z_0 \in Z$ and $z^* \in q\text{-int}\, Z_+^*$ with

$$z^*(h(x)) = z^*(l(x) + z_0)$$

for all $x \in Y_0$. Then there exists an affine extension $\tilde{h}: X \to Z$ such that

$$z^*(\tilde{h}(x)) = z^*(l(x) + z_0)$$

for all $x \in X$.

Proof. Obviously $Y_0 \subseteq Y$ is an affine manifold. Let $\bar{x} \in Y_0$, $X_0 := Y_0 - \bar{x}$ and let X_1 be the algebraically complementary subspace of X_0. Since any $x \in X$ has the unique representation of the form $x = \bar{x} + x_0 + x_1$, $x_0 \in X_0$, $x_1 \in X_1$ we can set

$$\tilde{h}(x) = \tilde{h}(\bar{x} + x_0 + x_1) := h(\bar{x} + x_0) + l(x_1).$$

On the one hand we get

$$\tilde{h}(x) = h(x) \quad \text{for all} \quad x \in Y_0 = \bar{x} + X_0 .$$

On the other hand we have

$$\begin{aligned} z^*(\tilde{h}(x)) &= z^*(\tilde{h}(\bar{x} + x_0 + x_1)) \\ &= z^*(h(\bar{x} + x_0) + l(x_1)) \\ &= z^*(l(\bar{x} + x_0) + z_0) + z^*(l(x_1)) \\ &= z^*(l(\bar{x} + x_0 + x_1) + z_0) = z^*(l(x) + z_0)) \end{aligned}$$

for all $x \in X$. □

Now we can formulate the following Mazur-Bourgin assertion.

Theorem 2.5. Let $A \subseteq X \times Z$ be a convex set, $B \subseteq X \times Z$ the graph of an affine mapping h: $Y_0 \subseteq X \to Z$ with the following properties:

(1) there exists a linear functional $z^* \in q\text{-int}\, Z_+^*$ with

$$z^*(z) \geqq z^*(h(x))$$

for all $z \in A(x)$ and any fixed $x \in \mathscr{P}_X(A) \cap Y_0$,

(2) $0 \in \operatorname{icr}(\mathscr{P}_X(A) - Y_0)$.

Then there exist a linear mapping $l \in \mathscr{L}(X, Z)$ and an element $z_0 \in Z$ such that

$$\begin{aligned} z^*(z) &\geqq z^*(l(x) + z_0) \quad \text{for all} \quad (x, z) \in A , \\ h(x) &= l(x) + z_0 \quad \text{for all} \quad x \in Y_0 . \end{aligned}$$

Proof. According to Theorem 2.2 there exist a linear mapping $\hat{l} \in \mathscr{L}(X, Z)$ and an element $\hat{z}_0 \in Z$ such that

$$\begin{aligned} z^*(z) &\geqq z^*(\hat{l}(x) + \hat{z}_0) \quad \text{for all} \quad (x, z) \in A , \\ z^*(z) &\leqq z^*(\hat{l}(x) + \hat{z}_0) \quad \text{for all} \quad (x, z) \in B . \end{aligned}$$

Since B is an affine manifold, without loss of generality we can assume that in the second relation even the equality holds, that means

$$z^*(h(x)) = z^*(\hat{l}(x) + \hat{z}_0) \quad \text{for all} \quad x \in Y_0 .$$

Using Lemma 2.4 there exists an affine extension $\tilde{h}$: $X \to Z$ of h with

$$z^*(\tilde{h}(x)) = z^*(\hat{l}(x) + \hat{z}_0) \quad \text{for all} \quad x \in X .$$

In the representation $\tilde{h}(x) = l(x) + z_0$, $x \in X$, the linear mapping $l \in \mathscr{L}(X, Z)$ and the element $z_0 \in Z$ fulfil the assertion. □

3. EXTENSION OF LINEAR MAPPINGS

Analogous to the results in convex analysis in this section we shall formulate extension assertions for linear mappings in sense of Hahn-Banach and Krein-Rutman by means of the separation Theorem 2.2. Moreover, we shall show that all assertions are equivalent, meaning that one assertion can be derived by means of the other.

First we give a Hahn-Banach extension theorem.

Theorem 3.1. Let f: $D(f) \subseteq X \to Z$ be a sublinear mapping, $X_0 \subseteq X$ a linear subspace and $l_0 \in \mathscr{L}(X_0, Z)$ with the following properties:

(1) there exists a linear functional $z^* \in q\text{-int } Z^*_+$ with

$$z^*(f(x)) \geqq z^*(l_0(x))$$

for all $x \in D(f) \cap X_0$,

(2) $0 \in \operatorname{icr}(D(f) - X_0)$.

Then there exists a linear mapping $l \in \mathscr{L}(X, Z)$ such that

$$z^*(f(x)) \geqq z^*(l(x)) \quad \text{for all} \quad x \in D(f),$$
$$l_0(x) \quad = l(x) \qquad \text{for all} \quad x \in X_0 .$$

Proof. The statement follows immediately from Theorem 2.5 or from Theorem 2.3 by consideration of Lemma 2.4. □

We note that assumption (2) of Theorem 3.1 is equivalent to the demand $D(f) - X_0$ is a linear subspace of X.

Now we can derive an extension theorem for nonnegative linear mappings in the sense of Krein-Rutman.

Theorem 3.2. Let $X_+ \subseteq X$ be a convex cone, $X_0 \subseteq X$ a linear subspace and $l_1 \in \mathscr{L}(X_0, Z)$ with the following properties:

(1) there exists a linear functional $z^* \in q\text{-int } Z^*_+$ with

$$z^*(l_1(x)) \geqq 0 \quad \text{for all} \quad x \in X_+ \cap X_0 ,$$

(2) $0 \in \operatorname{icr}(X_+ - X_0)$.

Then there exists a linear mapping $l \in \mathscr{L}(X, Z)$ such that

$$z^*(l(x)) \geqq 0 \qquad \text{for all} \quad x \in X_+ ,$$
$$l(x) \quad = l_1(x) \qquad \text{for all} \quad x \in X_0 .$$

Proof. We set $D(f) := X_+$ and $f(x) := 0$ for all $x \in D(f)$. Further, let $l_0(x) := -l_1(x)$ for all $x \in X_0$. Then the assumptions of the Hahn-Banach theorem are fulfilled and so there exists a linear mapping $-l \in \mathscr{L}(X, Z)$ such that

$$z^*(-l(x)) \leqq z^*(f(x)) = 0 \quad \text{for all} \quad x \in X_+$$

and

$$-l(x) \quad = l_0(x) = -l_1(x) \quad \text{for all} \quad x \in X_0 .$$ □

Remark. The convex cone $X_+ \subseteq X$ generates a quasi-order in X and the assumption (1) of Theorem 3.2 means that the linear mapping $l_1 \in \mathscr{L}(X_0, Z)$ is nonnegative on X_0 (with respect to this quasiorder restricted to X_0). Really, according to (1.6), the condition (1) implicates

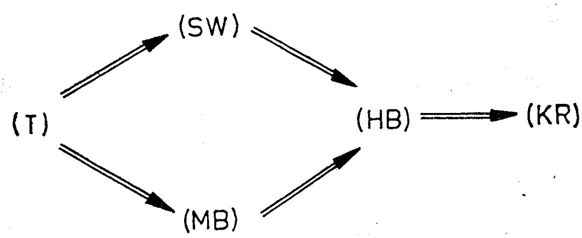
$$l_1(x) \nless 0 \quad \text{for all} \quad x \in X_+ \cap X_0 .$$

Thus, Theorem 3.2 demonstrates that under validity of the formulated assumptions a nonnegative linear mapping can be extended to be nonnegative on X.

Now, we will show that all theorems formulated in Section 2 and 3 are equivalent in such sense that one theorem can be derived by means the other. For this we shall use the following notations:

(T) ... separation theorem (Theorem 2.2),
(SW) ... sandwich theorem (Theorem 2.3),
(MB) ... Mazur-Bourgin theorem (Theorem 2.5),
(HB) ... Hahn-Banach theorem (Theorem 3.1),
(KR) ... Krein-Rutman theorem (Theorem 3.2).

Until now we have demonstrated the following implications:



Now we will show that we can prove (T) by means of (KR), hence all the assertions are equivalent.

For this let A and B be two convex subsets of $X \times Z$ and let the assumptions of Theorem 2.2 be fulfilled. We define a convex cone $(X \times Z)_+ \subseteq X \times Z$ by

$$(X \times Z)_+ := \operatorname{cone}(A - B),$$

a linear subspace $(X \times Z)_0 \subseteq X \times Z$ by

$$(X \times Z)_0 := \{0\} \times Z$$

and a linear mapping $l_1 \in \mathscr{L}((X \times Z)_0, Z)$ by

$$l_1(0, z) := z , \quad z \in Z .$$

By the assumptions of Theorem 2.2 we get

$$z^*(l_1(x, z)) \geqq 0$$

for all $(x, z) \in (X \times Z)_+ \cap (X \times Z)_0 = \{0\} \times [\operatorname{cone}(A - B)](0)$ and

$$(0, 0) \in \operatorname{icr}((X \times Z)_+ - (X \times Z)_0) = \operatorname{icr}((\operatorname{cone} \mathscr{P}_X(A - B)) \times Z) .$$

Thus, the assumptions of Theorem 3.2 are fulfilled and there exists a linear mapping $\tilde{l} \in \mathscr{L}(X \times Z, Z)$ such that

$$z^*(\tilde{l}(x, z)) \geqq 0 \quad \text{for all} \quad (x, z) \in (X \times Z)_+ ,$$
$$\tilde{l}(x, z) = l_1(x, z) = z \quad \text{for all} \quad (x, z) \in (X \times Z)_0 .$$

According to the linearity of $\tilde{l}$ from the last equality we get

$$\begin{aligned} \tilde{l}(x, z) &= \tilde{l}(0, z) + \tilde{l}(x, 0) \\ &= z + \tilde{l}(x, 0) \\ &= z - l(x) \end{aligned}$$

where $l(\cdot) := -l(\cdot, 0) \in \mathscr{L}(X, Z)$. Thus, we have

$$z^*(z - l(x)) \geqq 0$$

for all $(x, z) \in (X \times Z)_+ \supseteq A - B$ and we can find a real number $\alpha \in \mathbb{R}$ such that

$$z^*(z_1 - l(x_1)) \geqq \alpha \geqq z^*(z_2 - l(x_2))$$

for all $(x_1, z_1) \in A$ and all $(x_2, z_2) \in B$.

4. INEQUALITY SYSTEMS AND OPTIMIZATION PROBLEMS

In this section we will formulate two assertions concerning the solution of convex inequality systems and convex optimization problems respectively. Analogously to the last results we show that these theorems are equivalent to the separation theorem (Theorem 2.2). At first a generalized Farkas-Minkowski assertion is given.

Theorem 4.1. Let Y be a vector space quasiordered by a convex cone $Y_+ \subseteq Y$. Further let $g\colon D(g) \subseteq X \to Y$ and $f\colon D(f) \subseteq X \to Z$ be convex mappings with the following properties:

(1) there exists a linear functional $z^* \in q\text{-int } Z_+^*$ with

$$z^*(f(x)) \geqq 0$$

for all $x \in D(f) \cap D(g)$, $g(x) \leqq 0$,

(2) $0 \in \operatorname{icr} \{y \in Y \mid \exists x \in D(f) \cap D(g)\colon y \geqq g(x)\}$.

Then there exists a linear mapping $l \in \mathscr{L}(Y, Z)$ such that

$$z^*(f(x) + l(g(x))) \geqq 0 \quad \text{for all} \quad x \in D(f) \cap D(g)$$

and $z^*(l(y)) \geqq 0$ for all $y \in Y_+$.

Proof. For $A, B \subseteq Y \times Z$ set

$$A := \{(y, z) \in Y \times Z \mid \exists x \in D(f) \cap D(g)\colon y \geqq g(x), z \geqq f(x)\}$$
$$B := \{(0, 0)\} .$$

Both sets are convex and the assumptions of Theorem 2.2 are fulfilled. Thus, there exists a linear mapping $-l \in \mathscr{L}(Y, Z)$ such that

$$z^*(f(x) + z + l(g(x) + y)) \geqq 0 \tag{4.1}$$

for all $x \in D(f) \cap D(g)$, $z \in Z_+$ and $y \in Y_+$. Especially (set $z = 0$, $y = 0$) we have

$$z^*(f(x) + l(g(x))) \geqq 0$$

for all $x \in D(f) \cap D(g)$.

To show the nonnegativity of l let $y \in Y_+$. In (4.1) we set $z = 0$ and for a fixed $x \in D(f) \cap D(g)$ and any $\lambda > 0$ we get

$$z^*(f(x) + l(g(x)) + l(\lambda y)) \geqq 0$$

or equivalently

$$z^*(l(y)) \geqq -\lambda^{-1} z^*(f(x) + l(g(x))) .$$

But this means that $z^*(l(y)) \geqq 0$ and the theorem is proved. □

Now we present a Kuhn-Tucker assertion for convex vector optimization problems. Let us regard the following problem:

(*P*) $\quad f(x) \Rightarrow \min \quad \text{s.t.} \quad x \in S$

$\quad$ where $\quad S := \{x \in X \mid g(x) \leqq 0\}$.

Here analogously to Theorem 4.1 $f\colon D(f) \subseteq X \to Z$ and $g\colon D(g) \subseteq X \to Y$ are convex mappings. A feasible vector $x_0 \in S$ is said to be an *efficient solution* of (***P***) iff

$$f(x) \nless f(x_0) \quad \text{for all} \quad x \in S \cap D(f) .$$

Moreover, $x_0 \in S$ is called a *properly efficient solution* (cf. [67]) if there exists a linear functional $z^* \in q\text{-int}\, Z_+^*$ such that

$$z^*(f(x)) \geqq z^*(f(x_0)) \quad \text{for all} \quad x \in S \cap D(f) .$$

Obviously, by (1.6) any properly efficient solution of (***P***) is also an efficient solution. We give the following necessary optimality condition.

Theorem 4.2. In the optimization problem (***P***) we assume that

(1) $x_0 \in S$ is a properly efficient solution, i.e. there exists a linear functional $z^* \in q\text{-int}\, Z_+^*$ such that

$$z^*(f(x)) \geqq z^*(f(x_0))$$

for all $x \in D(f) \cap D(g)$, $\quad g(x) \leqq 0$, ,

(2) $0 \in \operatorname{icr} \{y \in Y \mid \exists x \in D(f) \cap D(g)\colon y \geqq g(x)\}$.

Then there exists a linear mapping $l \in \mathscr{L}(Y, Z)$ such that

$$z^*(f(x) + l(g(x))) \geqq z^*(f(x_0)) \quad \text{for all} \quad x \in D(f) \cap D(g) , \tag{4.2}$$

$$z^*(l(y)) \geqq 0 \quad \text{for all} \quad y \in Y_+ , \tag{4.3}$$

$$l(g(x_0)) = 0 . \tag{4.4}$$

Proof. We have

$$z^*(f(x) - f(x_0)) \geqq 0$$

for all $x \in D(f) \cap D(g)$ with $g(x) \leqq 0$. Using Theorem 4.1 we get the existence of a linear mapping $\bar{l} \in \mathscr{L}(Y, Z)$ such that the multiplicator inequality (4.2) and the nonnegativity condition (4.3) are fulfilled. But in general the complementary slack condition fails. However, with (4.2) and (4.3) together we get

$$z^*(\bar{l}(g(x_0))) = 0 .$$

According to Lemma 2.4 we can find a linear mapping $l \in \mathscr{L}(Y, Z)$ such that

$$l(g(x_0)) = 0$$

and

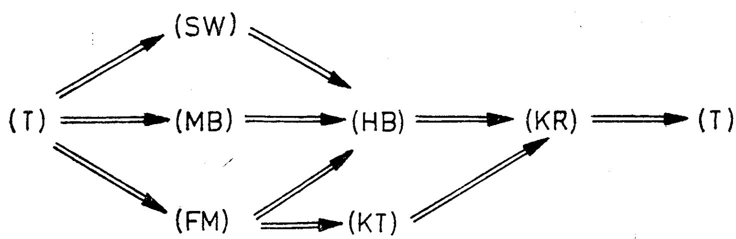
$$z^*(l(y)) = z^*(\bar{l}(y))$$

for all $y \in Y$. This mapping has the desired properties. □

Finally we still demonstrate that Theorem 4.1 and Theorem 4.2 are equivalent to the separation theorem. For this it is sufficient to show that the Krein-Rutman theorem can be regarded as a special assertion of the Kuhn-Tucker theorem. When we denote

(FM) ... Farkas-Minkowski theorem (Theorem 4.1),
(KT) ... Kuhn-Tucker theorem (Theorem 4.2),

then together with the remarks of the last section we get the following implications



Now, let the assumptions of Theorem 3.2 be fulfilled. We set

$$Y := X\,, \quad Y_+ := X_+\,, \quad D(f) := X_0\,, \quad D(g) := X\,,$$
$$f(x) := l_1(x) \quad \text{for} \quad x \in X_0\,,$$
$$g(x) := -x \quad \text{for} \quad x \in X\,,$$
$$x_0 := 0\,.$$

Thus, we have

$$z^*(f(x)) = z^*(l_1(x)) \geqq 0 = z^*(f(x_0))$$

for all $x \in X_+ \cap X_0 = \{x \in D(f) \cap D(g) \mid g(x) = -x \leqq 0\}$ and

$$0 \in \operatorname{icr}(X_+ - X_0) = \operatorname{icr}\{g(x) + y \mid x \in X_0,\, y \in Y_+ = X_+\}\,.$$

The assumptions of Theorem 4.2 are fulfilled and we get a linear mapping $\tilde{l} \in \mathscr{L}(X, Z)$ such that

$$z^*(f(x) + \tilde{l}(g(x))) = z^*(l_1(x) - \tilde{l}(x)) \geqq 0$$

for all $x \in X_0$ and

$$z^*(\tilde{l}(x)) \geqq 0$$

for all $x \in X_+$. Since all the mappings are linear in the first relation even equality holds according to

$$z^*(l_1(x)) = z^*(\tilde{l}(x))$$

for all $x \in X_0$. Using Lemma 2.4 again we get a suitable linear mapping $l \in \mathscr{L}(X, Z)$ with

$$l_1(x) = l(x)$$

for all $x \in X_0$ and

$$z^*(l(x)) = z^*(\tilde{l}(x)) \geqq 0$$

for all $x \in X_+$.

(Received September 29, 1989.)

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