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Multiple Decoding Scheme and Bounds on the Probability of Error and Erasure over a Multiple Channel

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The output sequences may be partitioned into different subsets and different decoders may be operating on these subsets. This is the idea of multiple decoding. The paper contains ***multiple decoding schemes*** for multiple access channels by modification of *Forney's maximum likelihood* ***decoding*** *scheme with erasures*. Exponential bounds on the probability of error and erasure are obtained.

1. INTRODUCTION

Given a discrete channel with input $X = (x_1, \ldots, x_K)$ and output $Y = (y_1, \ldots, y_J)$, with transition matrix $\{P(y_j/x_k)\}$, $k = 1, \ldots, K$, $j = 1, \ldots, J$, Gallager [3] obtained an upper bound on the probability of decoding error in the form

$$P_e \leqq \exp\left[-(N - \varrho R + \max_{\boldsymbol{p}} E_0(\varrho, \boldsymbol{p}))\right], \quad 0 \leqq \varrho \leqq 1, \tag{1.1}$$

where P_e denotes the minimum probability of decoding error, N the length of code words, R the rate of transmission, $\boldsymbol{p} = \{p(x_k)\}$, $k = 1, \ldots, K$ the input probability distribution and

$$E_0(\varrho, \boldsymbol{p}) = -\ln \sum_{j=1}^{J} \left(\sum_{k=1}^{K} P(x_k)\, P^{1/(1+\varrho)}(y_j/x_k)\right)^{1+\varrho} \tag{1.2}$$

To obtain this bound on the probability of decoding error for a code of size M, Gallager [3] used maximum likelihood decoding scheme. Forney [2] generalized this scheme to one called "maximum likelihood decoding with decoding erasures" so that having received an N-sequence $\boldsymbol{y}$ it is read as an input N-sequence $\boldsymbol{x}_{\hat{m}}$ iff

$$P(\boldsymbol{y}/\boldsymbol{x}_{\hat{m}}) > e^{\beta}\, P(\boldsymbol{y}/\boldsymbol{x}_m), \quad \text{for} \quad \text{all}\; m \neq \hat{m}, \quad 1 \leqq \hat{m} \leqq M, \tag{1.3}$$

where β is a nonnegative number and if no code word satisfies it, decoding erasure is inferred.

Using above scheme, Forney [2] obtained bounds on the probability of erasure and probability of decoding error.

From a practical and tactical point of view it is sometimes necessary to partition the set of output sequences into disjoint subsets and forward these for decoding to different decoders of different efficiencies just as the incoming mail is classified and handled at different parts of an establishment. Sharma and Gurdial [5] have handled this problem of "multiple decoding" for a discrete memoryless channel by partitioning the set of received sequences in two disjoint classes A_1 and A_2 and then defining different decoding schemes in terms of two values of the parameter β.

In recent years there has been interest in channels having multiple inputs/outputs (Cover [1]). For performance of multiple access channels the bounds over probability of error etc. have been found by maximum likelihood decoding scheme (Liao [4]). However there is no modification of these for erasures under the generalized maximum likelihood scheme (Forney [2]). In this paper we propose to undertake the problem of multiple decoding in a multiple access discrete channel. As a particular case it will cover the maximum likelihood decoding with erasure that could be applied to such channels.

2. MULTIPLE DECODING SCHEME FOR MULTIPLE ACCESS CHANNELS

Consider a multiple access channel with m independent sources in which each source sends a message i^l chosen from a finite set $\{1, \ldots, M^l\}$, $l = 1, \ldots, m$. We assume that the messages chosen from the respective sets are equally likely, the sources are time synchronous and the messages of different sources are statistically independent. Also the encoders are independent and they map $i^1, \ldots, i^m$ into transmitting sequences $\boldsymbol{x}^1_{i^1}, \ldots, \boldsymbol{x}^m_{i^m}$ each with a block length N such that $\boldsymbol{x}^l_{i^l} = (x^l(1), \ldots \ldots, x^l(N))_{i^l}$ is a function of i^l only and $x^l(t)$, $t \in \{1, \ldots, N\}$ takes values from a finite alphabet $\{1, \ldots, J^l\}$, $l = 1, \ldots, m$. We assume that the encoders are also time synchronous, each encoder sending one digit each unit of time. On the other hand the output corresponding to $\boldsymbol{x}^1_{i^1}, \ldots, \boldsymbol{x}^m_{i^m}$ is denoted by $\boldsymbol{y} = (y(1), \ldots, y(N))$ where $y(t)$ takes a value from the finite alphabet $\{1. \ldots, J\}$. Further the channel is characterized by the conditional probabilities $P(\boldsymbol{y}/\boldsymbol{x}^1, \ldots, \boldsymbol{x}^m)$ and we confine to the memoryless case where

$$P(\boldsymbol{y}/\boldsymbol{x}^1, \ldots, \boldsymbol{x}^m) = \prod_{t=1}^{N} P(y(t)/x^1(t), \ldots, x^m(t)) .$$

Taking a priori probability vector $\boldsymbol{Q}_k$ of a sequence of length k with source independence the bound over the probability of error for such a multiple access channel has been obtained by Liao [4]. In obtaining bound on the probability of decoding error Liao [4] used maximum likelihood decoding scheme according to which a received sequence y is read as $(x^1_{i^1}, \ldots, x^m_{i^m})$ if

$$(2.1)\quad P(y/x^1_{i^1}, \ldots, x^m_{i^m}) > P(y/x^1_{\hat{\imath}^1}, \ldots, x^m_{\hat{\imath}^m}), \quad \text{for all} \quad (i^1, \ldots, i^m) \neq (\hat{\imath}^1, \ldots, \hat{\imath}^m), \\ 1 \leqq \hat{\imath}^l \leqq M^l, \quad l = 1, \ldots, m.$$

For purposes of mathematical analysis it shall be sufficient to confine to the case $m = 2$.

As a first step it may be pointed out that modification of the scheme (2.1) on the lines of Forney [2] will be to decode a received sequence y into $(x^1_{i^1}, x^2_{i^2})$ iff

$$(2.2)\quad P(y/x^1_{i^1}, x^2_{i^2}) > e^{\beta} P(y/x^1_{\hat{\imath}^1}, x^2_{\hat{\imath}^2}), \quad \text{for all} \quad (i^1, i^2) \neq (\hat{\imath}^1, \hat{\imath}^2), \\ 1 \leqq \hat{\imath}^l \leqq M^l, \quad l = 1, 2$$

and to infer erasure if no pair $(x^1_{i^1}, x^2_{i^2})$ satisfies it, where β is a non-negative number.

Further (refer Liao [4]) if i^1, i^2 are the messages transmitted, then on the basis of scheme in (2.2) there can be decoding error if anyone of the following three disjoint events occurs:

$$(2.3)\quad P(y/x^1_{\hat{\imath}^1}, x^2_{i^2}) > e^{\beta} P(y/x^1_{i^1}, x^2_{i^2}), \quad \text{for some} \quad \hat{\imath}^1 \neq i^1$$

$$(2.4)\quad P(y/x^1_{i^1}, x^2_{\hat{\imath}^2}) > e^{\beta} P(y/x^1_{i^1}, x^2_{i^2}), \quad \text{for some} \quad \hat{\imath}^2 \neq i^2$$

$$(2.5)\quad P(y/x^1_{\hat{\imath}^1}, x^2_{\hat{\imath}^2}) > e^{\beta} P(y/x^1_{i^1}, x^2_{i^2}), \quad \text{for some} \quad \hat{\imath}^1 \neq i^1 \quad \text{and} \quad \hat{\imath}^2 \neq i^2.$$

The results under this scheme can be obtained directly from a more general situation in which the set of output sequences is partitioned into certain number of disjoint subsets which are subjected to different decoding schemes.

For purposes of study it is again sufficient to consider partitioning of the set of output sequences into two disjoint subsets A_1 and A_2 of received sequences so that $A_1 \cup A_2$ is the set of all received sequences and to adopt for decoding what may be termed as a "double decoding scheme" defined as follows:

A received sequence $y \in A_i$ is decoded into the input pair sequence $(x^1_{i^1}, x^2_{i^2})$ iff

$$(2.6)\quad P(y/x^1_{i^1}, x^2_{i^2}) > e^{\beta_i} P(y/x^1_{\hat{\imath}^1}, x^2_{\hat{\imath}^2}), \quad i = 1, 2, \quad \text{for all} \quad (\hat{\imath}^1, \hat{\imath}^2) \neq (i^1, i^2), \\ 1 \leqq \hat{\imath}^l \leqq M^l, \quad l = 1, 2,$$

where β_i, $i = 1, 2$ are two fixed nonnegative numbers and if no input pair $(x^1_{i^1}, x^2_{i^2})$ satisfies (2.6) for $y \in A_i$, then an erasure is inferred. (This may be followed by a repeat order.)

The scheme in (2.6) for an i ($i = 1, 2$) can now be partitioned into three disjoint error events on the lines (2.2) was partitioned into (2.3), (2.4) and (2.5).

Further a little consideration will show (cf. Forney [2]) that the event of erasure when $(x^1_{i^1}, x^2_{i^2})$ is sent and y is received is covered in the set determined by condition

$$(2.7)\quad P(y/x^1_{i^1}, x^2_{i^2}) \leqq e^{\beta_i} P(y/x^1_{\hat{\imath}^1}, x^2_{\hat{\imath}^2}), \quad \text{for some} \quad (\hat{\imath}^1, \hat{\imath}^2) \neq (i^1, i^2), \\ i = 1, 2, \quad 1 \leqq \hat{\imath}^l \leqq M^l, \quad l = 1, 2.$$

And considering the three ways in which $(\hat{\imath}^1, \hat{\imath}^2) \neq (i^1, i^2)$ the scheme (2.7) can also be partitioned into three events which together cover the erasure event.

3. AN UPPER BOUND ON THE PROBABILITY OF DECODING ERROR AND ERASURE

We shall keep the notations very near to those adopted by Forney [2], so let $P_{(i^1,i^2)x}$ and $P_{(i^1,i^2)e}$ denote respectively the probabilities of erasure and decoding error associated with a transmitted pair sequence $(x^1_{i^1}, x^2_{i^2})$. An erasure will be made if for a received sequence $y \in A_i$ no input pair sequence satisfies (2.6), $i = 1, 2$. On the other hand a decoding error will be made if $y \in A_i$ is received such that (2.6) is satisfied for some $(\hat{\imath}^1, \hat{\imath}^2) \neq (i^1, i^2)$ in any of three possible ways. We then have

$$P_{(i^1,i^2)x} = \sum_{i=1}^{2} \sum_{y \in A_i} P(y/x^1_{i^1}, x^2_{i^2})\, \varphi_{i(i^1,i^2)x}(y) \tag{3.1}$$

and

$$P_{(i^1,i^2)e} = \sum_{i=1}^{2} \sum_{y \in A_i} P(y/x^1_{i^1}, x^2_{i^2})\, \psi_{i(i^1,i^2)e}(y) \tag{3.2}$$

where

$$\varphi_{i(i^1,i^2)x}(y) = \begin{cases} 1 & \text{if equation (2.6) remains unsatisfied for all } (i^1, i^2)\,, \\ 0 & \text{otherwise} \end{cases} \tag{3.3}$$

and

$$\psi_{i(i^1,i^2)e}(y) = \begin{cases} 1 & \text{if equation (2.6) is satisfied for some } (\hat{\imath}^1, \hat{\imath}^2) \neq (i^1, i^2)\,, \\ 0 & \text{otherwise}\,. \end{cases} \tag{3.4}$$

The output sequences, whose maximum number is J^N, are partitioned into two disjoint subsets A_1 and A_2. It is always possible to work on the basis of proportion in which these are divided. Thus let $\lambda_1 J^N$ and $\lambda_2 J^N$ with $\lambda_1 + \lambda_2 = 1$, $0 \leqq \lambda_1 \leqq 1$, be the number of received sequences in A_1 and A_2 respectively. A decoder, which operates on sequences in A_i, $i = 1, 2$, has to decode each received sequence $y \in A_i$ into a pair of input sequences $(x^1_{\hat{\imath}^1}, x^2_{\hat{\imath}^2})$ and as such it possesses a pair of decoding rates (cf. Liao [4]) which we denote by (R_{1i}, R_{2i}), $i = 1, 2$, such that

$$M^i \leqq (\lambda_j J^N)^{R_{ji}}\,, \quad i = 1, 2\,, \quad j = 1, 2\,. \tag{3.5}$$

In a following theorem, we shall obtain estimates as upper bounds over the probabilities of erasure and error for a multiple (two sources) access channel having multiple (double) decoding. These are obtained under random coding argument. Before coming to the theorem, we define a few terms which will make our presentation precise:

Let

$$(3.6)\qquad E_{0i}^{1/2}(R_{i1}) = \max_{Q_{k'},0\leqq \varrho_i{}^1\leqq 1} \left[-\varrho_i^1 R_{i1} \ln J + E_{0i}^{1/2}(\varrho_i^1, Q_{k'})\right],$$

$$E_{0i}^{2/1}(R_{i2}) = \max_{Q_{k'},0\leqq \varrho_i{}^2\leqq 1} \left[-\varrho_i^2 R_{i2} \ln J + E_{0i}^{2/1}(\varrho_i^2, Q_{k'})\right],$$

$$E_{0i}^{1,2}(R_{i1} + R_{i2}) = \max_{Q_{k'},0\leqq \varrho_i\leqq 1} \left[-\varrho_i(R_{i1} + R_{12}) \ln J + E_{0i}^{1,2}(\varrho_i, Q_{k'})\right],$$

where

$$(3.7)\qquad E_{0i}^{1/2}(\varrho_i^1, Q_{k'}) = -\frac{1}{K'} \ln \sum_{y\in A_i} \sum_{x^2} Q^2(x^2) \left[\sum_{x^1} Q^1(x^1)\, P^{1/(1+\varrho_i{}^1)}(y/x^1, x^2)\right]^{1+\varrho_i{}^1},$$

$$E_{0i}^{2/1}(\varrho_i^2, Q_{k'}) = -\frac{1}{K'} \ln \sum_{y\in A_i} \sum_{x^1} Q^1(x^1) \left[\sum_{x^2} Q^2(x^2)\, P^{1/(1+\varrho_i{}^2)}(y/x^1, x^2)\right]^{1+\varrho_i{}^2},$$

$$E_{0i}^{1,2}(\varrho_i, Q_{k'}) = -\frac{1}{K'} \ln \sum_{y\in A_i} \left[\sum_{x^1} \sum_{x^2} Q^1(x^1)\, Q^2(x^2)\, P^{1/(1+\varrho_i}(y/x^1, x^2)\right]^{1+\varrho_i}.$$

Theorem. For a multiple (two sources) access discrete memoryless channel there exists a code of length N and a multiple (double) parameter β (β_1 and β_2) which works on disjoint subsets of output sequences (containing a fraction λ_1 and λ_2; $\lambda_1 + \lambda_2 = 1$, $0 \leqq \lambda_1 \leqq 1$ of maximum number of output sequences respectively) such that the average erasure probability $Pr(\mathrm{x})$ and the average error probability $Pr(\mathrm{E})$ satisfy

$$(3.8)\qquad Pr(\mathrm{X}) < \left[\sum_{i=1}^{2} (\lambda_i)^{\varrho_i{}^1(R_{i1})R_{i1}}\, K^{1/2}(\beta_i)\right] \bar{F}^{1/2}_{(i^1,i^2)} +$$

$$+ \left[\sum_{i=1}^{2} (\lambda_i)^{\varrho_i{}^2(R_{i2})R_{i2}}\, K^{2/1}(\beta_i)\right] \bar{F}^{2/1}_{(i^1,i^2)} + \left[\sum_{i=1}^{2} (\lambda_i)^{(\varrho_i(R_{i1}+R_{i2})(R_{i1}+R_{i2}))}\, K^{1,2}(\beta_i)\right] \bar{F}^{1,2}_{(i^1,i^2)}$$

and

$$(3.9)\qquad Pr(\mathrm{E}) < \left[\sum_{i=1}^{2} (\lambda_i)^{\varrho_i{}^1(R_{i1})R_{i1}} (K^{1/2}(\beta_i))^{-1}\right] \bar{F}^{1/2}_{(i^1,i^2)} +$$

$$+ \left[\sum_{i=1}^{2} (\lambda_i)^{\varrho_i{}^2(R_{i2})R_{i2}} (K^{2/1}(\beta_i))^{-1}\right] \bar{F}^{2/1}_{(i^1,i^2)} +$$

$$+ \left[\sum_{i=1}^{2} (\lambda_i)^{\varrho_i(R_{i1}+R_{i2})(R_{i1}+R_{i2})} (K^{1,2}(\beta_i))^{-1}\right] \bar{F}^{1,2}_{(i^1,i^2)},$$

where

$$(3.10)\qquad K^{1/2}(\beta_i) = \exp\left[\frac{\beta_i\, \varrho_i^1(R_{i1})}{1 + \varrho_i^1(R_{i1})}\right], \quad K^{2/1}(\beta_i) = \exp\left[\frac{\beta_i\, \varrho_i^2(R_{i2})}{1 + \varrho_i^2(R_{i2})}\right],$$

$$K^{1,2}(\beta_i) = \exp\left[\frac{\beta_i\, \varrho_i(R_{i1} + R_{i2})}{1 + \varrho_i(R_{i1} + R_{i2})}\right],$$

$$\bar{F}^{1/2}_{(i^1,i^2)} = \max\left[e^{-NE_{01}(R_{11})},\quad e^{-NE_{02}(R_{21})}\right], \tag{3.11}$$
$$\bar{F}^{2/1}_{(i^1,i^2)} = \max\left[e^{-NE_{01}(R_{12})},\quad e^{-NE_{02}(R_{22})}\right],$$
$$\bar{F}^{1,2}_{(i^1,i^2)} = \max\left[e^{-NE_{01}(R_{11}+R_{12})},\quad e^{-NE_{02}(R_{21}+R_{22})}\right],$$

$\varrho_i^1(R_{i1})$ be the ϱ_i^1 which maximizes $E_{0i}^{1/2}(R_{i1})$, similarly for others.

Proof. Having sent $(x^1_{i^1}, x^2_{i^2})$ the expressions for $P_{(i^1,i^2)x}$ and $P_{(i^1,i^2)e}$ are as determined in (3.1) and (3.2).

Now consider the following characteristic functions

$$\varphi'_{i(i^1,i^2)x}(y) = \begin{cases} 1 & \text{if } P(y/x^1_{i^1}, x^2_{i^2}) \leqq e^{\beta_i} P(y/x^1_{\hat{i}^1}, x^2_{\hat{i}^2}), \\ & \text{some } (\hat{i}^1, \hat{i}^2) \neq (i^1, i^2),\ y \in A_i, \\ 0 & \text{otherwise} \end{cases} \tag{3.12}$$

and

$$\psi'_{i(i^1,i^2)e}(y) = \begin{cases} 1 & \text{if } P(y/x^1_{i^1}, x^2_{i^2}) \geqq e^{\beta_i} P(y/x^1_{\hat{i}^1}, x^2_{\hat{i}^2}), \\ & \text{some } (\hat{i}^1, \hat{i}^2) \neq (i^1, i^2),\ y \in A_i, \\ 0 & \text{otherwise}, \end{cases} \tag{3.13}$$

$i = 1, 2$.

It can be seen (cf. Forney [2]) that $\varphi'_{i(i^1,i^2)x}(y) = 1$ whenever $\varphi_{i(i^1,i^2)x}(y) = 1$ and $\psi'_{i(i^1,i^2)e}(y) = 1$ whenever $\psi_{i(i^1,i^2)e}(y) = 1$. Thus the events of erasure and error are contained respectively in the supports of these functions. Further considering the three ways in which $(\hat{i}^1, \hat{i}^2) \neq (i^1, i^2)$ (refer (2.6)) we may write for $i = 1, 2$:

$$\varphi'_{i(i^1,i^2)x}(y) = \varphi'^{1/2}_{i(i^1,i^2)x}(y) + \varphi'^{2/1}_{i(i^1,i^2)x}(y) + \varphi'^{1,2}_{i(i^1,i^2)x}(y) \tag{3.14}$$

and

$$\psi'_{i(i^1,i^2)e}(y) = \psi'^{1/2}_{i(i^1,i^2)e}(y) + \psi'^{2/1}_{i(i^1,i^2)e}(y) + \psi'^{1,2}_{i(i_1,i_2)e}(y), \tag{3.15}$$

where

$$\varphi'^{1/2}_{i(i^1,i^2)x}(y) = \begin{cases} 1 & \text{if } P(y/x^1_{i^1}, x^2_{i^2}) \leqq e^{\beta_i} P(y/x^1_{\hat{i}^1}, x^2_{i^2}), \\ & \text{some } \hat{i}^1 \neq i^1,\ y \in A_i, \\ 0 & \text{otherwise}, \end{cases} \tag{3.16}$$

$$\varphi'^{2/1}_{i(i^1,i^2)x}(y) = \begin{cases} 1 & \text{if } P(y/x^1_{i^1}, x^2_{i^2}) \leqq e^{\beta_i} P(y/x^1_{i^1}, x^2_{\hat{i}^2}), \\ & \text{some } \hat{i}^2 \neq i^2,\ y \in A_i, \\ 0 & \text{otherwise}, \end{cases} \tag{3.17}$$

$$\varphi'^{1,2}_{i(i^1,i^2)x}(y) = \begin{cases} 1 & \text{if } P(y/x^1_{i^1}, x^2_{i^2}) \leqq e^{\beta_i} P(y/x^1_{\hat{i}^1}, x^2_{\hat{i}^2}), \\ & \text{some } \hat{i}^1 \neq i^1 \text{ and } \hat{i}^2 \neq i^2, y \in A_i, \\ 0 & \text{otherwise}, \end{cases} \tag{3.18}$$

$$\psi'^{1/2}_{i(i^1,i^2)e}(y) = \begin{cases} 1 & \text{if } P(y/x^1_{i^1}, x^2_{i^2}) \geqq e^{\beta_i} P(y/x^1_{\hat{i}^1}, x^2_{i^2}), \\ & \text{some } \hat{i}^1 \neq i^1, y \in A_i, \\ 0 & \text{otherwise}, \end{cases} \tag{3.19}$$

$$\psi'^{2/1}_{i(i^1,i^2)e}(y) = \begin{cases} 1 & \text{if } P(y/x^1_{i^1}, x^2_{i^2}) \geqq e^{\beta_i} P(y/x^1_{i^1}, x^2_{\hat{i}^2}), \\ & \text{some } \hat{i}^2 \neq i^2, y \in A_i, \\ 0 & \text{otherwise}, \end{cases} \tag{3.20}$$

$$\psi'^{1,2}_{i(i^1,i^2)e}(y) = \begin{cases} 1 & \text{if } P(y/x^1_{i^1}, x^2_{i^2}) \geqq e^{\beta_i} P(y/x^1_{\hat{i}^1}, x^2_{\hat{i}^2}), \\ & \text{some } \hat{i}^1 \neq i^1 \text{ and } \hat{i}^2 \neq i^2, y \in A_i, \\ 0 & \text{otherwise}. \end{cases} \tag{3.21}$$

In the notations adopted above and earlier the incorrect sources are indicated by the numerator and the correct sources by the denominator of the superscript.

By applying the random coding argument, the expectation of the probability of erasure, denoted by a bar above, over the ensemble of all possible transmitted sequences is upperbounded by

$$\begin{aligned} \bar{P}_{(i^1,i^2)x} \leqq \sum_{i=1}^{2} \sum_{y \in A_i} \sum_{x^1{}_{i^1}} \sum_{x^2{}_{i^2}} P(y, x^1_{i^1}, x^2_{i^2}) \big[& Pr[P(y/x^1_{i^1}, x^2_{i^2}) \leqq e^{\beta_i} P(y/x^1_{\hat{i}^1}, x^2_{i^2}), \\ & \text{some } \hat{i}^1 \neq i^1/i^1, i^2, x^1_{i^1}, x^2_{i^2}, y] + \\ + Pr[P(y/x^1_{i^1}, x^2_{i^2}) \leqq e^{\beta_i} & P(y/x^1_{i^1}, x^2_{\hat{i}^2}), \text{ some } \hat{i}^2 \neq i^2/i^1, i^2, x^1_{i^1}, x^2_{i^2}, y]) + \\ + Pr[P(y/x^1_{i^1}, x^2_{i^2}) \leqq e^{\beta_i} & P(y/x^1_{\hat{i}^1}, x^2_{\hat{i}^2}), \text{ some } \hat{i}^1 \neq i^1 \text{ and} \\ & \hat{i}^2 \neq i^2/i^1, i^2, x^1_{i^1}, x^2_{i^2}, y]\big]. \end{aligned} \tag{3.22}$$

And the expectation of the probability of decoding error $\bar{P}_{(i^1,i^2)e}$ over the ensemble of all possible transmitted sequences is upperbounded by

$$\begin{aligned} \bar{P}_{(i^1,i^2)e} \leqq \sum_{i=1}^{2} \sum_{y \in A_i} \sum_{x^1{}_{i^1}} \sum_{x^2{}_{i^2}} P(y, x^1_{i^1}, x^2_{i^2}) \big[& Pr[P(y/x^1_{i^1}, x^2_{i^2}) \geqq e^{\beta_i} P(y/x^1_{\hat{i}^1}, x^2_{i^2}), \\ & \text{some } \hat{i}^1 \neq i^1/i^1, i^2, x^1_{i^1}, x^2_{i^2}, y] + \\ + Pr[P(y/x^1_{i^1}, x^2_{i^2}) \geqq e^{\beta_i} & P(y/x^1_{i^1}, x^2_{\hat{i}^2}), \text{ some } \hat{i}^2 \neq i^2/i^1, i^2, x^1_{i^1}, x^2_{i^2}, y] + \\ + Pr[P(y, x^1_{i^1}, x^2_{i^2}) \geqq e^{\beta_i} & P(y/x^1_{\hat{i}^1}, x^2_{\hat{i}^2}), \text{ some } \hat{i}^1 \neq i^1 \text{ and } \hat{i}^2 \neq i^2/i^1, i^2, \\ & x^1_{i^1}, x^2_{i^2}, y]\big]. \end{aligned} \tag{3.23}$$

Now let us assume that the sequences of length N are time independent block by block, each block of length K' and identically distributed. Let the a priori probability of the sequence of length K' of ith block be denoted by $Q^l(x^l(i))$, $l = 1, 2$, then

$$Q^l(x^l) = \prod_{i=1}^{n} Q^l(x^l(i)), \quad l = 1, 2, \tag{3.24}$$

so that if corresponding output and input sequences of length K' are also denoted by $\boldsymbol{y}$, $\boldsymbol{x}^1$, $\boldsymbol{x}^2$ then using (3.24) and following the same steps as in Liao [4] we have

$$\begin{aligned}\bar{P}_{(i^1,i^2)x} \leqq{}& n \sum_{i=1}^{2} (M^1 - 1)^{\varrho_i^1} e^{\beta_i \varrho_i^1/(1+\varrho_i^1)} \sum_{y \in A_i} \sum_{x^2} Q^2(x^2) \times \\ &\times [\sum_{x^1} Q^1(x^1) P^{1/(1+\varrho_i^1)}(y/x^1, x^2)]^{1+\varrho_i^1} + n \sum_{i=1}^{2} (M^2 - 1)^{\varrho_i^2} e^{\beta_i \varrho_i^2/(1+\varrho_i^2)} \times \\ &\times \sum_{y \in A_i} \sum_{x^1} Q^1(x^1) [\sum_{x^2} Q^2(x^2) P^{1(1+\varrho_i^2)}(y/x^1, x^2)]^{1+\varrho_i^2} + \\ &+ n \sum_{i=1}^{2} [(M^1 - 1)(M^2 - 1)]^{\varrho_i} e^{\beta_i \varrho_i/(1+\varrho_i)} \times \\ &\times \sum_{y \in A_i} [\sum_{x^1} \sum_{x^2} Q^1(x^1) Q^2(x^2) P^{1/(1+\varrho_i)}(y/x^1, x^2)]^{1+\varrho_i}\end{aligned} \tag{3.25}$$

and

$$\begin{aligned}\bar{P}_{(i^1,i^2)e} \leqq{}& n \sum_{i=1}^{2} (M^1 - 1)^{\varrho_i^1} e^{-\beta_i \varrho_i^1/(1+\varrho_i^1)} \times \\ &\times \sum_{y \in A_i} \sum_{x^2} [\sum_{x^1} Q^1(x^1) P^{1/(1+\varrho_i^1)}(y/x^1, x^2)]^{1+\varrho_i^1} + \\ &+ n \sum_{i=1}^{2} (M^2 - 1)^{\varrho_i^2} e^{-\beta_i \varrho_i^2/(1+\varrho_i^2)} \sum_{y \in A_i} \sum_{x^1} Q^1(x^1) [\sum_{x^2} Q^2(x^2) P^{1/(1+\varrho_i^2)}(y/x^1, x^2)]^{1+\varrho_i^2} + \\ &+ n \sum_{i=1}^{2} [(M^1 - 1)(M^2 - 1)]^{\varrho_i} e^{-\beta_i \varrho_i/(1+\varrho_i)} \times \\ &\times \sum_{y \in A_i} [\sum_{x^1} \sum_{x^2} Q^1(x^1) Q^2(x^2) P^{1/(1+\varrho_i)}(y/x^1, x^2)]^{1+\varrho_i}.\end{aligned} \tag{3.26}$$

These in view of (3.5), (3.6), (3.10) and (3.7), give

$$\begin{aligned}\bar{P}_{(i^1,i^2)x} \leqq{}& \sum_{i=1}^{2} (\lambda_i)^{\varrho_i^1(R_{i1})R_{i1}} K^{1/2}(\beta_i)] \exp[-N E_{0i}^{1/2} R_{i1})] + \\ &+ [\sum_{i=1}^{2} (\lambda_i)^{\varrho_i^2(R_{i2})R_{i2}} K^{2/1}(\beta_i)] \exp[-N E_{0i}^{2/1}(R_{i2})] + \\ &+ [\sum_{i=1}^{2} (\lambda_i)^{(\varrho_i(R_{i1}+R_{i2}))(R_{i1}+R_{i2})} K^{1,2}(\beta_i)] \exp[-N E_{0i}^{1,2}(R_{i1} + R_{i2})]\end{aligned} \tag{3.27}$$

and

$$\bar{P}_{(i^1, i^2)e} \leqq [\sum_{i=1}^{2} (\lambda_i)^{\varrho_i^1(R_{i1}) R_{i1}} (K^{1/2}(\beta_i))^{-1}] \exp[-N E_{0i}^{1/2}(R_{i1})] + \tag{3.28}$$

$$+ [\sum_{i=1}^{2} (\lambda_i)^{\varrho_i^2(R_{i2}) R_{i2}} (K^{2/1}(\beta_i))^{-1}] \exp[-N E_{0i}^{2/1}(R_{i2})] +$$

$$+ [\sum_{i=1}^{2} (\lambda_i)^{\varrho_i(R_{i1}+R_{i2})(R_{i1}+R_{i2})} (K^{1\;2}(\beta_i))^{-1}] \exp[-N E_{0i}^{1,2}(R_{i1} + R_{i2})] .$$

The results (3.8) and (3.9) of the theorem now follow from (3.27) and (3.28) with the help of (3.11).

Remarks. It is clear from the definition of $E_{0i}^{1/2}(\varrho_i^1, \boldsymbol{Q}_{k'})$ that the function $E_{0i}^{1/2}(\varrho_i^1, \boldsymbol{Q}_{k'})$ is non-negative for $\varrho_i^1 \geqq 0$, $i = 1, 2$. Similarly for $E_{0i}^{2/1}(\varrho_i^2, \boldsymbol{Q}_{k'})$ and $E_{0i}^{1,2}(\varrho_i, \boldsymbol{Q}_{k'})$ for $i = 1, 2$, are also non-negative.

Particular Cases

Case 1. When $\beta_1 = \beta_2 = 0$, then the double decoding scheme reduces to ordinary maximum likelihood decoding scheme and the bound on the probability of error for $\varrho_1^1 = \varrho_2^1 = \varrho^1$, $\varrho_1^2 = \varrho_2^2 = \varrho^2$ and $\varrho_1 = \varrho_2 = \varrho$ reduces to bound given in Liao [4].

Case 2. When one of the λ_i say $\lambda_2 = 0$ then the partitioning is absent and the bounds given by (3.8) and (3.9) give

$$Pr(\mathrm{X}) < K^{1/2}(\beta_1)\, \bar{F}_{(i^1, i^2)}^{1/2} + K^{2/1}(\beta_1)\, \bar{F}_{(i^1, i^2)}^{2/1} + K^{1,2}(\beta_1)\, \bar{F}_{(i^1, i^2)}^{1,2} \tag{3.29}$$

and

$$Pr(\mathrm{E}) < (K^{1/2}(\beta_1))^{-1}\, \bar{F}_{(i^1, i^2)}^{1/2} + (K^{2/1}(\beta_1))^{-1}\, \bar{F}_{(i^1, i^2)}^{2/1} + (K^{1,2}(\beta_1))^{-1}\, \bar{F}_{(i^1, i^2)}^{1,2} . \tag{3.30}$$

These are in fact results for multiple access channels corresponding to a modified maximum likelihood decoding scheme with erasures depending on parameter β_1.

Case 3. When β_1 and β_2 are both different from zero and $J^1 = J^2 = J$. Denoting the classical rate of the lth source by R^l, we have

$$M^l = (J^N)^{R^l}, \quad l = 1, 2 \tag{3.31}$$

so that (3.31) together with (3.5) yields

$$R^1 \geqq R_{i1} ,$$
$$R^2 \geqq R_{i2} , \quad i = 1, 2 .$$

Thus the components of the decoding rate pairs do not exceed the components of classical rate pair.

Case 4. For classical channels i.e. one with a single source the bounds given by (3.9) and (3.10) reduce to bounds obtained by Sharma and Gurdial [5] for such channels under double decoding scheme with erasures.

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