ON THE ZEROS OF THE POLYNOMIALS OF LEGENDRE.

PAUL TURÁN, Budapest.

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1. The polynomials of Legendre[1])

$$P_0(x), P_1(x), \ldots, P_n(x), \ldots$$

can be uniquely defined by the orthogonality-property

$$\int_{-1}^{1} P_n(x) x^\nu \mathrm{d}x = 0 \quad \nu = 0, 1, \ldots, (n-1) \tag{1,1}$$

and by the normalisation

$$P_n(1) = 1, \; n = 0, 1, 2, \ldots \tag{1,2}$$

The explicit representation of Rodriguez

$$P_n(x) = \frac{1}{n!\,2^n} \cdot \frac{\mathrm{d}^n}{\mathrm{d}x^n} (x^2 - 1)^n \tag{1,3}$$

can be easily verified. A classical reasoning, based upon the orthogonality-property (1,1) shows that the zeros of $P_n(x)$ are real, simple and lying in $-1 < x < 1$; the property (1,3) gives that they are symmetrical to $x = 0$. Denoting the zeros of $P_n(x)$ throughout this paper by $x_{\nu n}$ ($\nu = 1, 2, \ldots, n$) we may write

$$x_{\nu n} = \cos\vartheta_{\nu n} \tag{1,4}$$

where the numbers $\vartheta_{\nu n}$, the zeros of the trigonometrical polynomial $P_n(\cos\vartheta)$, satisfy the inequality

$$0 < \vartheta_{1n} < \vartheta_{2n} < \ldots < \vartheta_{nn} < \pi. \tag{1,5}$$

Generally speaking the numbers $\vartheta_{\nu n}$ obey simpler laws than the zeros $x_{\nu n}$.

2. Many efforts have been made to locate the numbers $\vartheta_{\nu n}$ as precisely as possible. BRUNS[2]) proved in 1881

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[1]) For a detailed theory of these polynomials see the book of G. Szegö: Orthogonal polynomials. Amer. Math. Soc. Coll. Publ. 1939.

[2]) Zur Theorie der Kugelfunctionen. Journal für Math. 90 (1881), p. 322—328.

$$\frac{\nu-\frac{1}{2}}{n+\frac{1}{2}}\pi < \vartheta_{\nu n} < \frac{\nu}{n+\frac{1}{2}}\pi \ (1 \leqq \nu \leqq \tfrac{1}{2}n). \tag{2,1}$$

STIELTJES[3]) and A. MARKOFF[4]) improved this to

$$\frac{\nu-\frac{1}{2}}{n}\pi < \vartheta_{\nu n} < \frac{\nu}{n+1}\pi \ (1 \leqq \nu \leqq \tfrac{1}{2}n) \tag{2,2}$$

and finally SZEGÖ[5]) obtained

$$\frac{\nu-\frac{1}{4}}{n+\frac{1}{2}}\pi < \vartheta_{\nu n} < \frac{\nu}{n+1}\pi \ (1 \leqq \nu \leqq \tfrac{1}{2}n). \tag{2,3}$$

Very simple and elegant proofs for all these inequalities have been given by FEJÉR[6]) and SZEGÖ[7]).

3. Another group of results has been initiated by SZEGÖ[8]) who showed that the $\vartheta_{\nu n}$ - values with $1 \leqq \nu \leqq \frac{1}{2}n$ form a *convex* sequence; more exactly that

$$\vartheta_{1n} = \vartheta_{1n} - 0 < \vartheta_{2n} - \vartheta_{1n} < \vartheta_{3n} - \vartheta_{2n} < \cdots < \vartheta_{1+[\frac{1}{2}n],n} - \vartheta_{[\frac{1}{2}n],n}. \tag{3,1}$$

From it follows, as HILLE[9]) showed, that

$$1 - x_{1n} < x_{1n} - x_{2n} < \ldots < x_{[\frac{1}{2}n],n} - x_{1+[\frac{1}{2}n],n}, \tag{3,2}$$

i. e. the zeros $x_{1n}, x_{2n}, \ldots, x_{1+[\frac{1}{2}n],n}$ together with 1 form a convex sequence too; or — somewhat unprecisely — proceeding from left to right the distance of two consecutive positive zeros of $P_n(x)$ decreases monotonically.

4. As to a comparison of the zeros of $P_n(x)$ and $P_{n-1}(x)$ we have the separation-theorem[10]) according to which the zeros of $P_n(x)$ separate the zeros of $P_{n-1}(x)$, i. e.

$$1 > x_{1n} > x_{1,n-1} > x_{2n} > x_{2,n-1} > \cdots > x_{n-1,n-1} > x_{nn} > -1, \tag{4,1}$$

or

$$0 < \vartheta_{1n} < \vartheta_{1,n-1} < \cdots < \vartheta_{n-1,n-1} < \vartheta_{nn} < \pi. \tag{4,2}$$

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[3]) Sur les racines de l'équation $X_n = 0$. Acta Math. **9** (1886), p. 385—400.

[4]) Sur les racines de certaines équations (seconde note). Math. Ann. 1886, p. 177—182.

[5]) Inequalities for the zeros of Legendre-polynomials and related functions. Transact. of the Amer. Math. Soc. **39** (1936), p. 1—17.

[6]) Bestimmung von Grenzen für die Nullstellen des Legendre-polynoms aus der Stieltjesschen Integraldarstellung desselben. Monatshefte für Math. und Phys. **43** (1936) p. 193—209 and Trigonometrische Reihen und Potenzreihen mit mehrfach monotoner Koefficientenfolge. Transact. of the Amer. Math. Soc. **39**, p. 18—59.

[7]) See his paper[5]).

[8]) See his paper[5]).

[9]) Über die Nullstellen der Hermiteschen Polynome. Jahresbericht der deutschen Math. Verein (1933), p. 162—165.

[10]) I don't know whom this theorem is ascribed to. A proof for it one can find in Szegö's book p. 45.

Hence to each zero $x_{\nu, n-1}$ of $P_{n-1}(x)$ we can associate uniquely the zero $x_{\nu n}$ of $P_n(x)$ (as its right neighbour) so that

$$x_{\nu+1, n} < x_{\nu, n-1} < x_{\nu n} \qquad \nu = 1, 2, \ldots, (n-1). \tag{4,3}$$

Now we shall show that the distance of these two associated $x_{\nu n}$ and $x_{\nu, n-1}$ zeros (if they are positive), decreases when proceeding from left to the right. More exactly I shall show the following

Theorem. We have with the above notation of the zeros of $P_{n-1}(x)$ and $P_n(x)$ the inequalities

$$x_{1n} - x_{1, n-1} < x_{2n} - x_{2, n-1} < \cdots < x_{[\frac{1}{2}(n-1)], n} - x_{[\frac{1}{2}(n-1)], n-1}. \tag{4,4}$$

5. I found this theorem in 1941 with a „STIELTJES-type" proof. I communicated this to Prof. SZEGŐ in 1946 in a letter; in his answer[11]) he sketched a proof of STURM-type for the above theorem. As a matter of fact he proved the corresponding inequalities on the circle

$$\vartheta_{1, n-1} - \vartheta_{1n} < \vartheta_{2, n-1} - \vartheta_{2n} < \cdots < \vartheta_{[\frac{1}{2}(n-1)], n-1} - \vartheta_{[\frac{1}{2}(n-1)], n} \tag{5,1}$$

and extended it to the class of ultraspherical polynomials. Though subsequently I observed, my method can furnish a proof also for SZEGŐ's inequalities (5,1) and probably the case of ultraspherical polynomials can be settled too in this way, in this note I confine myself to my original proof of theorem (4,4).

An important part of my proof of theorem (4,4) was the fact that

$$\Delta_n(x) = P_n(x)^2 - P_{n-1}(x)\, P_{n+1}(x)$$

is monotonically decreasing in

$$\frac{1}{2n+1} \leq x \leq 1. \tag{5,2}$$

From this I could easily deduce the interesting inequality

$$\Delta_n(x) \geq 0 \qquad n = 1, 2, \ldots \qquad -1 \leq x \leq 1. \tag{5,3}$$

Many simplifications and extensions of these two results have been given in the mean time. SZEGŐ[12]) gave four very elegant proofs of the inequality (5,3); one of these proofs, the idea of which is due to Pólya, can be extended, as he remarked, to *all* ultraspherical polynomials. O. SZÁSZ[13]) showed that

$$\frac{1 - P_n(x)^2}{(n+1)(2n+1)} \leq \Delta_n(x) \leq \frac{2n+1}{3n(n+1)}, \quad -1 \leq x \leq +1, \tag{5,4}$$

[11]) Dated from May 3. 1946.

[12]) „On an inequality of P. Turán concerning Legendre-polynomials." Bull. of Amer. Math. Soc. Vol. **54** (1948), p. 401—405.

and by a suitable limit process also the interesting inequality

$$J_\mu(x)^2 - J_{\mu-1}(x) J_{\mu+1}(x) > \frac{1}{\mu+1} J_\mu(x)^2, \quad x > 0, \mu > 0 \tag{5,5}$$

where $J_\mu(x)$ denotes the usual Bessel-function. Further interesting generalisations of (5,3) are obtained by FORSYTHE[14]) and by SANSONE[15]). As to the part (5,2) of our assertion BECKENBACH[16]) showed among other interesting results that the polynomials $\Delta_n(x)$ are convex from above for *all* x-values. These results will be extended to determinants of Hankel-type with ultraspherical polynomials as elements in a forthcoming paper of BECKENBACH, SEIDEL and O. SZÁSZ[17]). As to our theorem (4,4) Prof. SZÁSZ obtained [18]) the lower estimation

$$|x_{in} - x_{j,n-1}| > \frac{2}{3n^{\frac{3}{2}}} \frac{1}{(n+1)\sqrt{2n+1}} \tag{5,6}$$

$$i = 1, 2, \ldots, n, \quad j = 1, 2, \ldots, (n-1).$$

In what follows I shall give my original proof for (5,2) and in the Appendix for (5,3).

6. An unexpected possibility for applications of results of type (4,4) is opened by some unpublished theorems of FEJÉR which I mention here with his kind permission; he found his results during the winter-term of the academic year 1928/29. He could characterise *certain sequences* of numbers

$$A = \begin{Bmatrix} \zeta_{11} \\ \zeta_{12}, \zeta_{22} \\ \vdots \\ \zeta_{1k}\, \zeta_{2k} \ldots \zeta_{kk} \\ \ldots\ldots\ldots \end{Bmatrix} \tag{6,1}$$

of the interval $[0,1]$ for which

$$\zeta_{0k} \equiv 1 > \zeta_{1k} > \zeta_{2k} > \ldots > \zeta_{kk} > 0 \equiv \zeta_{k+1,k}, \; k = 1, 2, \ldots \tag{6,2}$$

and the natural separation condition

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13) I know these results only from a lithoprinted copy of a lecture of Prof. SZÁSZ.

14) Certain inequalities concerning to Legendre-polynomials. Bull. of the Amer. Math. Soc. Vol. 55 (1949), p. 66. Prelim. rep.

15) „Su una disuguaglianza di P. Turán relativa ai polinomi di Legendre", Boll. della Unione Mat. Ital. 1949, ser. III, no. 3, p. 221—223 and „Su una disuguaglianza relativa ai polinomi di Legendre", ibid. Ser. III, no. 4, p. 1—3.

16) Convexity theorems for Legendre-polynomials. Prelim. rep. Bull. of the Amer. Math. Soc. Vol. 55, p. 41.

17) „Recurrent determinants of orthogonal polynomials". To appear in Duke Journal.

18) Communicated to me in a letter from 3. June 1949.

$$1 > \zeta_{1k} > \zeta_{1,k-1} > \zeta_{2k} > \dots > \zeta_{\nu k} > \zeta_{\nu,k-1} > \zeta_{\nu+1,k} > \dots \\ \dots > \zeta_{k-1,k-1} > \zeta_{kk} > 0, \tag{6,3}$$

holds and *certain classes of functions*, integrable in [0,1] in Riemann's sense, for which suitable Riemann-sums tend to $\int_0^1 f(t)\,dt$ *monotonically*. The sequences found by him include also the classical equidistant-case

$$\zeta_{\nu k} = 1 - \frac{\nu}{k+1}.$$

$$\nu = 1, 2, \dots, k, \quad k = 1, 2, \dots$$

His class of functions consists generally speaking of functions which are monotonic and convex in [0,1]; he found *necessary and sufficient* conditions for this phenomenon. Wequote exactly only that part of his results which shows the connection with our theorem. Denoting by $l_k(f, A)$ resp. by $r_k(f, A)$ the left- resp. right-Riemann-sums of $f(x)$, i. e.

$$l_k(f, A) = \sum_{\nu=0}^{k} f(\zeta_{\nu+1,k})\,(\zeta_{\nu k} - \zeta_{\nu+1,k}) \tag{6,4}$$

$$r_k(f, A) = \sum_{\nu=0}^{k} f(\zeta_{\nu k})\,(\zeta_{\nu k} - \zeta_{\nu+1,k}) \tag{6,5}$$

and introducing the notation

$$1 - \zeta_{1k} = a_k, \ \zeta_{1k} - \zeta_{1,k-1} = b_{k-1}, \ \zeta_{1,k-1} - \zeta_{2k} = a_{k-1}, \dots, \\ \zeta_{\nu k} - \zeta_{\nu,k-1} = b_{k-\nu}, \ \zeta_{\nu,k-1} - \zeta_{\nu+1,k} = a_{k-\nu}, \dots, \\ \zeta_{k-1,k} - \zeta_{k-1,k-1} = b_1, \ \zeta_{k-1,k-1} - \zeta_{kk} = a_1, \ \zeta_{kk} = \zeta_{kk} - 0 = b_0 \tag{6,6}$$

he proved that in the special case

$$b_0 \geqq b_1 \geqq \dots \geqq b_{k-1} \tag{6,7}$$

his necessary and sufficient condition is fulfilled and thus the left Riemann-sums $l_k(f, A)$ tend to $\int_0^1 f(t)\,dt$ monotonically $\left\{\begin{matrix}\text{increasingly}\\ \text{decreasingly}\end{matrix}\right\}$ if $f(x)$ is in [0,1] positive, monotonically $\left\{\begin{matrix}\text{increasing}\\ \text{decreasing}\end{matrix}\right\}$ and convex from $\left\{\begin{matrix}\text{above}\\ \text{below}\end{matrix}\right\}$, further the right-Riemann-sums $r_k(f, A)$ tend to $\int_0^1 f(t)\,dt$ monotonically $\left\{\begin{matrix}\text{increasingly}\\ \text{decreasingly}\end{matrix}\right\}$ if $f(x)$ is in [0,1] positive, monotonically $\left\{\begin{matrix}\text{decreasing}\\ \text{increasing}\end{matrix}\right\}$ and convex from $\left\{\begin{matrix}\text{below}\\ \text{above}\end{matrix}\right\}$. Here and later in the brackets either *always* the upper expression or *always* the lower one is understood of course. And if

$$a_1 \leqq a_2 \leqq \dots \leqq a_k, \tag{6,8}$$

then the sums $l_k(f, A)$ tend to $\int_0^1 f(t)\,dt$ monotonically $\begin{Bmatrix}\text{increasingly}\\ \text{decreasingly}\end{Bmatrix}$ if $f(x)$ is in $[0,1]$ positive, monotonically $\begin{Bmatrix}\text{decreasing}\\ \text{increasing}\end{Bmatrix}$ and convex from $\begin{Bmatrix}\text{above}\\ \text{below}\end{Bmatrix}$ and the sums $r_k(f, A)$ tend to $\int_0^1 f(t)\,dt$ monotonically $\begin{Bmatrix}\text{increasingly}\\ \text{decreasingly}\end{Bmatrix}$ if $f(x)$ is in $[0,1]$ positive, monotonically $\begin{Bmatrix}\text{decreasing}\\ \text{increasing}\end{Bmatrix}$ and convex from $\begin{Bmatrix}\text{above}\\ \text{below}\end{Bmatrix}$. As a glance to (6,7) shows, our theorem means that taking for A the system P of points, the k^{th} row of which consists of the non-negative zeros of the k^{th} Legendre-polynomial (replacing in the ν^{th} row, if ν is odd, the 0 by an arbitrary positive number which is $> \zeta_{\frac{1}{2}(\nu+1),\nu+1}$ and $< \zeta_{\frac{1}{2}(\nu-1),\nu-1}$) the condition

$$b_2 \geqq b_3 \geqq \dots \geqq b_{k-1}$$

is fulfilled, i. e. the sufficiency-criterion (6,7) of Fejér is „almost" fulfilled for this important P-system. We shall not discuss here the question how to modify the construction of P in order to obtain an A-system satisfying the criterion (6,7) of FEJÉR.

7. Before turning to the proof of our theorem we need some lemmas.

Lemma I. Let $0 \leqq \vartheta < 1$ and $y_1 > y_2 > \dots$ all those values in $0 \leqq x < 1$ for which

$$|P_n(y_\nu)| = \vartheta. \tag{7,1}$$

Then

$$|P_n'(y_1)| > |P_n'(y_2)| > \dots \tag{7,2}$$

Proof. It is known according to a theorem of SONINE[19]) the polynomial

$$n(n+1)P_n(x)^2 + (1-x^2)\,P_n'(x)^2$$

is for $0 \leqq x \leqq 1$ an increasing function of x. Replacing x by $y_1, y_2, \dots$ we obtain that numbers

$$n(n+1)\vartheta^2 + (1-y_\nu^2)\,P_n'(y_\nu)^2$$

and also the numbers

$$(1-y_\nu^2)P_n'(y_\nu)^2,$$

form for $\nu = 1, 2, \dots$ a decreasing sequence of numbers. Since the numbers $(1-y_\nu^2)$ form for $\nu = 1, 2, \dots$ an increasing sequence, our assertion obviously follows.

8. We consider the $x_{\nu n}$ resp. $x_{\nu, n-1}$ *non negative* zeros of $P_n(x)$ resp. $P_{n-1}(x)$. The zeros η_ν of $P_n'(x)$ satisfy obviously the inequality

[19]) See e. g. Szegö's book[1]), p. 160—161.

$$x_{\nu+1,n} < \eta_\nu < x_{\nu n}. \tag{8,1}$$

We assert moreover the following

Lemma II. With the above notation we have for $x_{\nu+1\,n} \geqq 0$ the inequality

$$x_{\nu+1,n} < \eta_\nu < x_{\nu,n-1} < x_{\nu n}.$$

Proof: We have obviously

$$sgP_n(x) = (-1)^\nu \tag{8,2}$$

for

$$x_{\nu+1,n} < x < x_{\nu n} \quad (\nu = 1, 2, \ldots, n-1), \tag{8,3}$$

and

$$sgP_{n-1}(x) = (-1)^\nu \tag{8,4}$$

for

$$x_{\nu+1,n-1} < x < x_{\nu,n-1} \quad (\nu = 1, 2, \ldots, n-2). \tag{8,5}$$

Using the identity[20])

$$(1-x^2)P_n'(x) = -nxP_n(x) + nP_{n-1}(x),$$

we obtain since $x_{\nu,n-1} > 0$

$$sgP_n'(x_{\nu,n-1}) = sg(1-x^2_{\nu,n-1})P_n'(x_{\nu,n-1}) = -sgnx_{\nu,n-1}P(x_{\nu,n-1}) =$$
$$= -sgP_n(x_{\nu,n-1}) = (-1)^{\nu+1}$$

using (8,2), (8,3); further

$$sgP_n'(x_{\nu+1,n}) = sg(1-x^2_{\nu+1,n})P_n'(x_{\nu+1,n}) = sgP_{n-1}(x_{\nu+1,n}) = (-1)^\nu$$

using (8,4) — (8,5). Hence the interval

$$(0 \leqq)\, x_{\nu+1,n} < x < x_{\nu,n-1} \tag{8,6}$$

contains *at least* one zero of $P_n'(x)$. Since Rolle's theorem gives that each of the intervals (8,3) contain exactly one zero of $P_n'(x)$, it follows a fortiori that each of the intervals (8,6) contain exactly one η_ν. Q. e. d.

9. Further we need

Lemma III. The polynomial

$$\Delta_n(x) = P_n(x)^2 - P_{n-1}(x)P_{n+1}(x)$$

decreases for $\dfrac{1}{2n+1} \leqq x \leqq 1$ monotonically.[21])

Proof: Starting from the well-known recurrence-formula

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x) \tag{9,1}$$

[20]) See e. g. SZEGÖ'S book p. 83, formula (4. 7. 27).

[21]) A simpler proof of this lemma follows from the identity

$$\frac{n(n+1)}{2}\frac{d^2\Delta_n(x)}{dx^2} = -P_n'(x)^2$$

communicated to me by Prof. O. Szász in a letter of June 3. 1949.

we obtain

$$(n+1)\Delta_n(x) = (n+1)P_n(x)^2 - (2n+1)xP_n(x)P_{n-1}(x) + + nP_{n-1}(x)^2 \tag{9,2}$$

$$(n+1)\Delta_n' = 2(n+1)P_nP_n' - (2n+1)P_nP_{n-1} - - (2n+1)xP_n'P_{n-1} - (2n+1)xP_nP_{n-1}' + 2nP_{n-1}P_{n-1}'.$$

Since[22])

$$P_n(x) = \frac{x}{n} P_n'(x) - \frac{1}{n} P_{n-1}'(x)$$

$$P_{n-1}(x) = \frac{1}{n} P_n'(x) - \frac{x}{n} P_{n-1}'(x)$$

we obtain

$$(n+1)\Delta_n(x) = -\frac{n+1}{n^2} P_n'(x)^2 - \frac{n+1}{n^2} P_{n-1}'(x)^2 + + \frac{1+(2n+1)x^2}{n^2} P_n'(x)P_{n-1}'(x).$$

This is a negativ definit quadratic form of $P_n'(x)$ and $P_{n-1}'(x)$, if the determinant is $\leqq 0$. But this is apart from a positive factor

$$[1+(2n+1)x^2]^2 - 4(n+1)^2x^2 = = (1-x)[1-(2n+1)x][1+2(n+1)x+(2n+1)x^2] \leqq 0$$

for

$$\frac{1}{2n+1} \leqq x \leqq 1. \quad \text{Q. e. d.}$$

10. Now we prove our last

Lemma IV. We have for all positive $x_{\nu,\,n-1}$ zeros

$$|P_n(x_{1,\,n-1})| < |P_n(x_{2,\,n-1})| < \ldots < |P_n(x_{[\frac{1}{2}(n-1)],\,n-1})|. \tag{10,1}$$

Proof. Replacing in Lemma III. x by $x_{\nu,\,n-1}$ our assertion follows, if we show that

$$x_{[\frac{1}{2}(n-1)],\,n-1} = \cos\vartheta_{[\frac{1}{2}(n-1)],\,n-1} \geqq \frac{1}{2n+1}. \tag{10,2}$$

But using e. g. Bruns's inequality (2,1) we have

$$x_{[\frac{1}{2}(n-1)],\,n-1} > \cos\frac{[\frac{1}{2}(n-1)]\pi}{n-\frac{1}{2}} \geqq \cos\frac{n-1}{2n-1}\pi = \sin\frac{\pi}{4n-2} > > \frac{1}{2n-1} > \frac{1}{2n+1}. \quad \text{Q. e. d.}$$

[22]) See SZEGÖ's book p. 84.

Remark. Using instead of Lemma III. the inequality of footnote[21]) we had obtained the chain (10,1) for all *non negative* $x_{\nu,n-1}$ zeros instead of the *positive* ones.

11. We can now prove our theorem formulated as (4,4). Let be (see Lemma II.)

$$0 < x_{\nu+1,n} < \eta_\nu < x_{\nu,n-1} < x_{\nu n} < \eta_{\nu-1} < x_{\nu-1,n-1} < x_{\nu-1,n}. \quad (11,1)$$

We have to consider the graph of the function

$$y = |P_n(x)|. \quad (11,2)$$

in the intervals

$$x_{\nu,n-1} \leqq x \leqq x_{\nu n}, \quad (11,3)$$

resp.

$$x_{\nu-1,n-1} \leqq x \leqq x_{\nu-1,n}. \quad (11,4)$$

Our Lemma II. shows that in these intervals $P'_n(x) \neq 0$, i. e. the function (11,2) decreases monotonically and they have in the respective intervals well-determined monotonically decreasing inverzes say

$$x = g_\nu(y),\ g_\nu(0) = x_{\nu n},\quad 0 \leqq y \leqq |P_n(x_{\nu,n-1})|,$$

resp.

$$x = g_{\nu-1}(y),\ g_{\nu-1}(0) = x_{\nu-1,n},\quad 0 \leqq y \leqq |P_n(x_{\nu-1,n-1})|.$$

Lemma IV. means that $g_\nu(y)$ is defined in a larger interval than $g_{\nu-1}(y)$ and Lemma I. gives that for

$$0 \leqq y \leqq |P_n(x_{\nu-1,n-1})|$$

we have

$$|g'_{\nu-1}(y)| < |g'_\nu(y)|.$$

Hence we have

$$x_{\nu-1\,n} - x_{\nu-1,n-1} = \left|\int_0^{|P_n(x_{\nu-1,n-1})|} g'_{\nu-1}(y)dy\right| = \int_0^{|P_n(x_{\nu-1,n-1})|} |g'_{\nu-1}(y)|dy <$$

$$< \int_0^{|P_n(x_{\nu-1,n-1})|} |g'_\nu(y)|\,dy < \int_0^{|P_n(x_{\nu,n-1})|} |g'_\nu(y)|\,dy =$$

$$= \left|\int_0^{|P_n(x_{\nu,n-1})|} g'_\nu(y)\,dy\right| = x_{\nu n} - x_{\nu,n-1} \text{ Q. e. d.}$$

Appendix.

12. Here I give my original proof for the inequality

$$\Delta_n(x) = P_n^2(x) - P_{n-1}(x)P_{n+1}(x) \geqq 0.$$

$$n = 1, 2, \ldots \quad -1 \leqq x \leqq 1.$$

Since $\Delta_n(x)$ is even, it is sufficient to consider $0 \leqq x \leqq 1$. Using the formula (9,2) we have

$$(n+1)\Delta_n(x) = (n+1)P_n(x)^2 - (2n+1)xP_n(x)P_{n-1}(x) + nP_{n-1}(x)^2.$$

This is a positive definit quadratic form if its determinant is $\leqq 0$. This is

$$(2n+1)^2x^2 - 4n(n+1) \leqq 0,$$

if

$$0 \leqq x \leqq \frac{2\sqrt{n(n+1)}}{2n+1}. \tag{12,1}$$

On the other hand, since

$$\Delta_n(1) = 0,$$

Lemma III. gives immediately the positivity for

$$\frac{1}{2n+1} \leqq x \leqq 1. \tag{12,2}$$

But the intervals (12,1) and (12,2) cover for $n \geqq 1$ the whole interval $0 \leqq x \leqq 1$. Q. e. d.

Added in proof. Using Fejér's criterion (6,7) and instead of the method of this paper the much shorter one of Szegö, mentioned at the beginning of *5.*, the vague statement at the end of *6.* can be made exact in different forms. Here I mention only one theorem in this direction.

If for $k = 1, 2, \ldots$ the k^{th} row of the matrix A in (6,1) is formed by the k positive zeros of the $(2k+1)^{th}$ Legendre-polynomial $P_{2k+1}(x)$ of (1,1) and $f(x)$ is in $[0,1]$ monotonically decreasing and convex from below, then the left-Riemann-sums $l_n(f, A)$ of (6,4) belonging to this matrix A tend to $\int_0^1 f(t)dt$ *monotonically decreasingly* for $n = 1, 2, \ldots$.

All the proofs including of course Fejér's proof for his criterion will be given in a joint paper.

O nulových bodech Legendrových polynomů.

(Obsah předešlého článku.)

Označme $x_{1n} > x_{2n} > \ldots > x_{nn}$ nulové body Legendrova polynomu $P_n(x)$. Hlavním cílem článku je důkaz nerovností

$$x_{1,n} - x_{1,n-1} < x_{2,n} - x_{2,n-1} < \ldots < x_{[\frac{1}{2}(n-1)],n} - x_{[\frac{1}{2}(n-1)],n-1}.$$

Jest vytčena také možnost, použíti těchto nerovností v souvislosti s některými výsledky Fejérovými. V dodatku je podán důkaz nerovnosti

$$P_n^2(x) - P_{n-1}(x)P_{n+1}(x) \geqq 0 \text{ pro } -1 \leqq x \leqq 1.$$