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On the successive minima of arbitrary sets

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[2] V. Jarník and V. Knichal, K hlavní větě geometrie čísel, Rozpravy II. tř. Čes. Akademie 53 (1943), No. 43 (Czech; a French summary will appear in the Bulletin International).

[3] C. A. Rogers, A note on a theorem of Blichfeldt, Nederl. Akad. Wetensch. 49, 930—935 = Indagationes Mathem. 8, 589—594 (1946).

[4] C. A. Rogers, The Successive Minima of Measurable Sets, submitted to the London Math. Soc. — I am very obliged to Mr. Rogers for having sent me a copy of his manuscript before its publication.

All numbers in this note are real. Let $n > 1$ be an integer; let R_n be the n-dimensional space of all points $\mathbf{x} = [x_1, \ldots, x_n]$. We use the standard notation: $\alpha\mathbf{x} + \beta\mathbf{y} = [\alpha x_1 + \beta y_1, \ldots, \alpha x_n + \beta y_n]$, $\mathbf{o} = [0, \ldots, 0]$; k points $\mathbf{x}^1, \ldots, \mathbf{x}^k$ are called independent if the equation $\alpha_1\mathbf{x}^1 + \ldots + \alpha_k\mathbf{x}^k = \mathbf{o}$ is satisfied only for $\alpha_1 = \ldots = \alpha_k = 0$. If $M \subset R_n$, then αM denotes the set of all points $\alpha\mathbf{x}$, where $\mathbf{x} \in M$. By $\mathfrak{W}(M)$ we denote the set of all points $\frac{1}{2}(\mathbf{x} - \mathbf{y})$ where $\mathbf{x} \in M$, $\mathbf{y} \in M$; obviously $\mathfrak{W}(\alpha M) = \alpha\mathfrak{W}(M)$. By $L(M)$ and $J(M)$ we denote the inner Lebesgue or Jordan measure of M.[1])

With every set $M \subset R_n$ we shall associate, in four different ways, a sequence of n „successive minima":

(i) Let λ_i $(1 \leq i \leq n)$ be the lower bound of all numbers $\alpha > 0$ such that the union $\bigcup\limits_{0<\beta\leq\alpha} \beta M$ contains at least i independent lattice points (i. e. points with integer co-ordinates).[2])

[1]) By definition, $L(M)$ is the upper bound of the Lebesgue measures of all measurable subsets of M (or, what amounts to the same, of all closed bounded subsets of M).

[2]) If there is no such α, we put $\lambda_i = +\infty$; an analogous convention holds in the following cases.

(ii) Let μ_i $(1 \leqq i \leqq n)$ be the lower bound of all numbers $\alpha > 0$ such that αM contains at least i independent lattice points.

(iii) Let ν_i $(1 \leqq i \leqq n)$ be the least number $\alpha \geqq 0$ such that every set βM with $\beta > \alpha$ contains at least i independent lattice points.

(iv) Let π_i $(1 \leqq i \leqq n)$ be the lower bound of all numbers $\alpha > 0$ such that the common part $\bigcap_{\beta \geqq \alpha} \beta M$ contains at least i independent lattice points.

We have obviously $\lambda_i \leqq \lambda_{i+1}$, $\mu_i \leqq \mu_{i+1}$, $\nu_i \leqq \nu_{i+1}$, $\pi_i \leqq \pi_{i+1}$, $0 \leqq \lambda_i \leqq \mu_i \leqq \nu_i \leqq \pi_i \leqq +\infty$. If necessary, we write $\lambda_i(M)$ instead of λ_i etc.[3])

I proved the following theorem [1]: If $n > 1$, $0 < J(M) < +\infty$, then

$$\lambda_1 \lambda_2 \ldots \lambda_n J(M) \leqq 2^{2n-1}, \text{ where } \lambda_i = \lambda_i(\mathfrak{W}(M)). \tag{1}$$

Krichal [2] improved this result by replacing $J(M)$ by $L(M)$[4]) and 2^{2n-1} by $2^{2n-\frac{3}{2}}$. Finally, Rogers [4] succeeded in proving the following sharper theorem: If $n > 1$, $0 < L(M) < +\infty$, then

$$\mu_1 \ldots \mu_n L(M) \leqq 2^{\frac{1}{2}(3n-1)}, \text{ where } \mu_i = \mu_i(\mathfrak{W}(M)). \tag{2}$$

He also proved that (if $0 < L(M) < +\infty$)

$$(\mu_1 \ldots \mu_n L(M))^{1-\frac{1}{n}} (\nu_1 \ldots \nu_n L(M))^{\frac{1}{n}} \leqq 2^{2n-1}. \tag{3}$$

These results suggest the question whether there exists a finite upper bound for the product

$$\nu_1 \ldots \nu_n L(M), \text{ where } \nu_i = \nu_i(\mathfrak{W}(M)). \tag{4}$$

The answer is negative; in fact, we shall prove the following

Theorem 1. *To every integer $n > 1$ and to every $T > 0$ there is a set $M \subset R_n$ (which is the union of a finite number of parallelepipeds) such that the product* (4) *is greater than T.*[5])

More generally we shals prove

Theorem 2. *If $T > 0$ and if i, j, n are integers, $1 \leqq i < j \leqq n$, then there is a set $M \subset R_n$ (which is the union of a finite number of parallelepipeds) such that*

[3]) If M is a cube, then evidently $0 < \lambda_i(M) < +\infty$. If $M_1 \subset M_2$, then $\lambda_i(M_2) \leqq \lambda_i(M_1)$. Hence: if M is bounded, then $\lambda_i(M) > 0$; if M has an inner point, then $\lambda_i(M) < +\infty$. Analogous remarks apply to the μ_i 's, ν_i 's, π_i 's.

[4]) It follows from [1]) [3]) that, if a theorem of this kind is true for closed bounded sets, it is true also for arbitrary sets.

[5]). From (3) we see that, if (4) is very large, the product (2) is very small. Following [3]), the numbers λ_k, ν_k, π_k in Theorems 1, 2, 3, 4 are finite and positive.

$$\lambda_1\lambda_2 \dots \lambda_{i-1}\nu_i\lambda_{i+1} \dots \lambda_{j-1}\nu_j\lambda_{j+1} \dots \lambda_n L(M) > T, \qquad (5)$$
$$\textit{where } \lambda_k = \lambda_k(\mathfrak{W}(M)),\ \nu_k = \nu_k(\mathfrak{W}(M)).$$

This generalization is perhaps not without interest, if we compare it with the following theorem of Rogers [4]: If $0 < L(M) < +\infty$ then

$$\mu_1 \dots \mu_{i-1}\nu_i\mu_{i+1} \dots \mu_n L(M) \leqq 2^{2n-1}. \qquad (6)$$

Further results have been obtained by iterating the operation $\mathfrak{W}$. Put $\mathfrak{W}^0(M) = M$, $\mathfrak{W}^p(M) = \mathfrak{W}(\mathfrak{W}^{p-1}(M))$ for $p = 1, 2, \dots$ The following facts are almost obvious:

(a) $\mathfrak{W}(M)$ is symmetrical about $\mathbf{o}$, i. e. if $\mathbf{x} \in \mathfrak{W}(M)$, then $-\mathbf{x} \in \mathfrak{W}(M)$.

(b) If M is symmetrical about $\mathbf{o}$, then $M \subset \mathfrak{W}(M)$ and so $\lambda_i(M) \geqq \lambda_i(\mathfrak{W}(M))$, $\mu_i(M) \geqq \mu_i(\mathfrak{W}(M))$ etc.

(c) It follows from (a) and (b) that $\lambda_i(\mathfrak{W}^{p-1}(M)) \geqq \lambda_i(\mathfrak{W}^p(M))$ etc. for $p = 2, 3, \dots$

In [2], I proved the following theorem: If $0 < L(M) < +\infty$, then there is an integer $p_0 > 0$ such that

$$L(M)\prod_{i=1}^{n}\pi_i(\mathfrak{W}^p(M)) \leqq 2^n \qquad (7)$$

for every integer $p > p_0$.[6])

This inequality suggests the question whether the number p_0 may be chosen as function of n only, i. e. independently of M. The answer is negative, and even more can be proved: If $n > 1$ and $p \geqq 0$ are arbitrary but fixed integers, there exists no finite upper bound, neither for the left side of (7), nor for the product

$$L(M)\prod_{i=1}^{n}\nu_i(\mathfrak{W}^p(M)). \qquad (8)$$

Still more generally we shall prove the following

Theorem 3. *Let $T > 0$; let n, i, j, p be integers, $1 \leqq i < j \leqq n$, $p \geqq 0$. Then there is a set $M \subset R_n$ (which is the union of a finite number of parallelepipeds) such that*

$$\lambda_1\lambda_2 \dots \lambda_{i-1}\nu_i\lambda_{i+1} \dots \lambda_{j-1}\nu_j\lambda_{j+1} \dots \lambda_n L(M) > T, \qquad (9)$$

where

$$\lambda_k = \lambda_k(\mathfrak{W}^p(M)),\ \nu_k = \nu_k(\mathfrak{W}^p(M)),\ \pi_k = \pi_k(\mathfrak{W}^p(M)). \qquad (10)$$

So much the more, the products in (7), (8) *are greater than T.*

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6) If M is a convex body, symmetrical about $\mathbf{o}$, then $\mathfrak{W}^p(M) = M$ and (7) reduces to a well known theorem of Minkowski. On the contrary, it has been proved by Knichal [2] (and for $n = 2$ also by Rogers [3]) that the constant 2^{2n-1} in (1) cannot be replaced by 2^n, if $n > 1$.

It is obvious that Theorems 1, 2 follow from Theorem 3.

Theorem 2 is a countrepart to (6); but there is another theorem of a similar character, concerning the π_i 's:

Theorem 4.[7] *Let i, n, p be integers, $1 \leqq i \leqq n$, $p \geqq 0$, $T > 0$. Then there is a set $M \subset R_n$ (which is the union of a finite number of parallelepipeds) so that we have, using the notation* (10),

$$\lambda_1 \ldots \lambda_{i-1}\pi_i\lambda_{i+1} \ldots \lambda_n L(M) > T. \tag{10bis}$$

Proof of Theorem 3 for $n = 2$. Here $i = 1$, $j = 2$. Let $p \geqq 0$ (p integer), $T > 0$ be given. We choose four numbers a, t, φ, N as follows:

$$a \text{ integer}, \ a > 10 \,.\, 2^p \,.\, T; \ 2^p t(a!) = \tfrac{1}{2};$$

$$0 < \varphi < \frac{1}{10 \,.\, 2^p \,.\, a \,.\, (a!)}; \ N \text{ integer}, \ 2^p\varphi N > 1. \tag{11}$$

Then we define $M' \subset R_2$ as the set of all points $[x, y]$ with the following property: There is an integer m such that

$$|y| \leqq \tfrac{1}{2}, \ |x - ty - m| \leqq \varphi, \ |m| \leqq N.$$

Obviously the set $\gamma\mathfrak{W}^p(M')$ (where $\gamma > 0$) is defined in an analogous way by the conditions

$$|y| \leqq \tfrac{1}{2}\gamma, \ |x - ty - 2^{-p}m\gamma| \leqq \varphi\gamma, \ |m| \leqq 2^p N \tag{12}$$

(m integer). We have $L(M') = 2\varphi(2N + 1)$. Put

$$\nu'_k = \nu_k(\mathfrak{W}^p(M')), \ \lambda'_k = \lambda_k(\mathfrak{W}^p(M')) \ (k = 1, 2). \tag{13}$$

We shall prove

$$\frac{1}{80 \,.\, 2^p N\varphi} \leqq \nu'_1 \leqq \frac{1}{2^p N\varphi}, \ \nu'_2 \geqq 2a; \tag{14}$$

this will give the required result

$$\nu'_1\nu'_2 \, L(M') > \frac{a}{10 \,.\, 2^p} > T. \tag{15}$$

In order to prove (14), we observe first: Corresponding to every $\alpha > (2^p N\varphi)^{-1}$ there is a pair of integers m, x other than 0, 0 and such that

$$|x - 2^{-p}m\alpha| \leqq 2^{-p}N^{-1} < \varphi\alpha, \ |m| \leqq 2^p N.$$

$x = 0$ would imply $|m| < 2^p\varphi < 1$ (see (11)) and so $x = m = 0$, which is impossible. Hence $x \neq 0$ and, by (12), $[x, 0] \in \alpha\mathfrak{W}^p(M')$, whence $\nu'_1 \leqq (2^p N\varphi)^{-1}$.

Next let us observe that there is an α such that

[7]) This theorem is almost obvious, as will be seen from its proof (here, n can be equal to 1).

$$\frac{1}{80\,.\,2^p\,.\,N\varphi} < \alpha < \frac{1}{40\,.\,2^p\,.\,N\varphi} \tag{16}$$

and such that there exists no pair of integers m, x satisfying the following conditions:

$$0 < |m| \leqq 2^p N, \left|\alpha - \frac{2^p x}{m}\right| \leqq \frac{1}{40N\,|m|}, |x| \leqq \frac{2\,|m|}{40\,.\,2^{2p}\,.\,N\varphi}. \tag{17}$$

For the measure of the set of all numbers $\alpha > 0$ to which there is a pair of integers m, x satisfying (17) is at most[8])

$$\frac{1}{40N} + \sum_{|m|=1}^{2^pN} \frac{2}{40N\,|m|} \cdot \frac{4\,|m|}{40\,.\,2^{2p}\,.\,N\varphi} =$$

$$= \frac{1}{40N} + \frac{2}{5} \cdot \frac{1}{40\,.\,2^p\,.\,N\varphi} < \frac{1}{80\,.\,2^p\,.\,N\varphi}$$

$$\left(\text{since}\ \frac{1}{10} \cdot \frac{1}{2^p\varphi} > 1\right).$$

Let us suppose (per absurdum) that $\nu'_1 < (80\,.\,2^p\,.\,N\varphi)^{-1}$. Let α be an arbitrary number satisfying (16). Following the definition of ν'_1 there must be a lattice point $[x, y] \in \alpha\,\mathfrak{W}^p(M')$ other than [0, 0]. Since (see (12)) $|y| \leqq \frac{1}{2}\alpha < (80\,.\,2^p\,.\,N\varphi)^{-1} < 1$ (see (11)), we have $y = 0$ and so $x \neq 0$, and there is (see (12)) an integer m such that

$$|m| \leqq 2^p N, \ |x - 2^{-p} m\alpha| \leqq \varphi\alpha.$$

Since $\varphi\alpha < 1$, we have $m \neq 0$ and so

$$\left|\alpha - \frac{2^p x}{m}\right| \leqq \frac{2^p\varphi\alpha}{|m|} < \frac{1}{40N\,|m|},$$

$$|x| \leqq 2^{-p}|m|\,\alpha + \varphi\alpha < 2\,.\,2^{-p}|m|\,\alpha < \frac{2\,|m|}{40\,.\,2^{2p}\,.\,N\varphi}.$$

In other words, to every α of the interval (16) there are two integers m, x satisfying (17). But this is a contradiction, and so

$$\nu'_1 \geqq (80\,.\,2^p\,.\,N\varphi)^{-1}.$$

Finally, let us suppose that $\nu'_2 < 2a$, so that there must be a lattice point $[x, y] \in 2a\,\mathfrak{W}^p(M')$ with $y \neq 0$ and so (see (12))

$$|y| \leqq a, \ |2^{-p}\,.\,q - ty| \leqq 2a\varphi, \tag{18}$$

where q is an integer. Thus $(a!)\,.\,y^{-1}$ is an integer; multiplying (18) by $(a!)\,.\,y^{-1}\,.\,2^p$ and comparing with (11) we get (X being an integer)

[8]) For, if $\alpha > 0$, then (17) implies: it is either $\alpha \leqq (40N)^{-1}$ or $x \neq 0$.

$$|X - 2^p . a! . t| = |X - \tfrac{1}{2}| \leqq 2a\varphi . a! . 2^p < \tfrac{1}{5},$$

which is a contradiction, and (14) is proved.

Proof of Theorem 3 in the general case. Let $T > 0$ and the integers p, i, j, n ($p \geqq 0$, $1 \leqq i < j \leqq n$) be given. Let $M' \subset R_2$ be the same set as in the preceding proof. Using the notation (13), we have $\nu'_1 \nu'_2 L(M') > T$. Further: *If* $0 < \alpha < 2$ *and* $[x, y]$ *is a lattice point of* $\alpha \mathfrak{W}^p(M')$, *we have* $|y| \leqq \frac{1}{2}\alpha < 1$ *and so* $y = 0$. *Hence* $\lambda'_2 \geqq 2 > \dfrac{1}{2^p N \varphi} \geqq \nu'_1$ (see (11), (14)). Following[3]), we have $\lambda'_1 > 0$, $\nu'_2 < +\infty$.

Now choose three numbers ξ, η, ζ such that

$$0 < \xi < \lambda'_1 \leqq \nu'_1 < \eta < 2 \leqq \lambda'_2 \leqq \nu'_2 < \zeta < +\infty \tag{19}$$

and let $M \subset R_n$ be the set of all points $[x_1, \ldots, x_n]$ which satisfy the conditions

$$|x_b| \leqq \frac{1}{\xi} \text{ for } 1 \leqq b < i, \quad |x_c| \leqq \frac{1}{\eta} \text{ for } i < c < j,$$

$$|x_d| \leqq \frac{1}{\zeta} \text{ for } j < d \leqq n, [x_i, x_j] \in M'.$$

If $\alpha > 0$, then $\alpha \mathfrak{W}^p(M)$ consists obviously of all points $[x_1, \ldots, x_n]$ such that

$$|x_b| \leqq \frac{\alpha}{\xi}, \ |x_c| \leqq \frac{\alpha}{\eta}, \ |x_d| \leqq \frac{\alpha}{\zeta}, \ [x_i, x_j] \in \alpha \mathfrak{W}^p(M').$$

Let $[x_1, \ldots, x_n]$ be a lattice point contained in $\alpha \mathfrak{W}^p(M)$. Then we have (see (19)):

If $0 < \alpha < \xi$, then $x_1 = x_2 = \ldots = x_n = 0$.
If $0 < \alpha < \eta$, then $x_{i+1} = x_{i+2} = \ldots = x_n = 0$.
If $0 < \alpha < \zeta$, then $x_{j+1} = x_{j+2} = \ldots = x_n = 0$.

It follows that (using the notation (10))

$$\lambda_1 = \ldots = \lambda_{i-1} = \xi, \ \nu_i = \nu'_1, \ \lambda_{i+1} = \ldots = \lambda_{j-1} = \eta, \ \nu_j = \nu'_2, \ \lambda_{j+1} = \ldots = \lambda_n = \zeta$$

and so (compare the definition of M)

$$\lambda_1 \ldots \lambda_{i-1} \nu_i \lambda_{i+1} \ldots \lambda_{j-1} \nu_j \lambda_{j+1} \ldots \lambda_n L(M) = 2^{n-2} \nu'_1 \nu'_2 L(M') > T.$$

Proof of Theorem 4. We may suppose that T is an integer, $T \geqq 2^{p+1}$. Let $M \subset R_n$ be the set of all points $\mathbf{x} = [x_1, \ldots, x_n]$ such that there is an integer m so that

$$|x_j| \leqq 2T \text{ for } j < i, \ |x_k| \leqq \frac{1}{2T} \text{ for } k > i,$$

$$|x_i - m| \leqq \frac{1}{2T}, \ |m| \leqq T.$$

Put $M_p = \mathfrak{W}_p(M)$, $\lambda_j = \lambda_j(M_p)$, $\pi_j = \pi_j(M_p)$ $(j = 1, \ldots, n)$. Then αM_p is defined by the inequalities (if $\alpha > 0$)

$$|x_j| \leqq 2T\alpha \ (j < i), \ |x_k| \leqq \alpha(2T)^{-1} \ (k > i), \quad |x_i - \alpha m \,.\, 2^{-p}| \leqq \alpha(2T)^{-1}, \ |m| \leqq 2^pT, \tag{20}$$

m integer.

If $0 < \alpha < (2T)^{-1}$ and $\mathsf{x} \in \alpha M_p$, then $|x_j| < 1$ for $j \neq i$ and $|x_i| \leqq \alpha\,(T + (2T)^{-1}) < 1$, and so $\lambda_1 \geqq (2T)^{-1}$. Further, if $0 < < \alpha < 2T$ and $\mathsf{x} \in \alpha M_p$, then $|x_k| < 1$ for $k > i$ and so $\lambda_{i+1} \geqq 2T$. Finally, let us suppose that $\pi_i < T$. Then there must be a lattice point

$$\mathsf{y} = [y_1, \ldots, y_n] \in \bigcap_{\beta \geqq T} \beta M_p$$

with $|y_i| + |y_{i+1}| + \ldots + |y_n| > 0$. Since $\mathsf{y} \in TM_p$, we have $y_k = 0$ for $k > i$ and so $y_i \neq 0$. Put $\alpha = T\,|y_i| \geqq T$. We must have $\mathsf{y} \in \alpha M_p$. But $|y_i - 0| > \frac{1}{2}\,|y_i| = \alpha(2T)^{-1}$, and for $|m| \geqq 1$ we have

$$|y_i - \alpha m \,.\, 2^{-p}| = |y_i| \,.\, |\pm 1 - Tm \,.\, 2^{-p}| \geqq |y_i| \,.\, T \,.\, 2^{-p-1} = = \alpha \,.\, 2^{-p-1} > \alpha(2T)^{-1}.$$

Thus we obtain (see (20)) y *non* $\in \alpha M_p$ — contradiction, and so $\pi_i \geqq T$, $\lambda_j \geqq (2T)^{-1}$ for $j < i$, $\lambda_k \geqq 2T$ for $k > i$. Calculating $L(M)$, we obtain (10^{bis}).

*

O postupných minimech libovolných množin.

(Obsah předešlého článku.)

Budiž M bodová množina v n-rozměrném prostoru; $\mathfrak{W}(M)$ budiž množina všech bodů $\frac{1}{2}\,(\mathsf{x} - \mathsf{y})$, kde x, y leží v M. Je-li M konvexní těleso o středu v počátku, mající objem $L(M)$, a jsou-li $\lambda_1, \ldots, \lambda_n$ postupná minima (ve smyslu Minkowského) množiny $\mathfrak{W}(M)$ (jež jest ovšem v tomto speciálním případě prostě rovna M), je podle Minkowského

$$\lambda_1\lambda_2 \ldots \lambda_n\, L(M) \leqq 2^n. \tag{21}$$

Pro obecné množiny M byla čísla λ_i definována dosud čtyřmi různými způsoby (jež v Minkowského případě splývají; viz [1], [2], [3], [4]). Pro dvě z těchto definicí platí nerovnost obdobná k (21), ale s větší konstantou vpravo. Autor ukazuje naopak, že pro zbývající dvě definice *není* levá strana v (21) omezená (Theorem 1). Theorem 2 a 3 obsahují další zobecnění tohoto výsledku. Další doplněk jest obsažen v Theoremu 4.