Časopis pro pěstování matematiky a fysiky

Alois Urban

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*Časopis pro pěstování matematiky a fysiky*, Vol. 73 (1948), No. 3, 89--92

Persistent URL: http://dml.cz/dmlcz/123121

Časopis pro pěstování matematiky a fysiky, roč. 73 (1948)

# Note on the T. Y. Thomas's Paper: On the Projective Theory of Two Dimensional Riemann Spaces.

By **Alois Urban**, Praha.

(Received December 4, 1947.)

As proved by *T. Y. Thomas*,[1]) there exists for every Riemannian space a contravariant vector density of weight two, namely $\mathfrak{a}a^{\lambda\mu}K_\mu$,[2]) preserved by the projective transformation which changes the geodesics of the considered Riemannian space to these of another one.

In this note there is found the relation of the invariant mentioned above to the curvature tensor of the projective Riemannian space.[3])

Let $\Gamma^\nu_{\lambda\mu}(\xi^\varkappa)$[4]) be the coefficients of a symmetric connection ($\xi^\varkappa$ being the coördinates of an $n$-dimensional space). The coefficients of any symmetric connection with the same paths are given by the projective transformation

$$'\Gamma^{\nu}_{\lambda\mu} = \Gamma^\nu_{\lambda\mu} + 2A^\nu_{(\lambda}p_{\mu)}, \tag{1}$$

where $p_\mu$ is an arbitrary covariant vector.

The quantities

$$\Pi^\nu_{\lambda\mu} = \Pi^\nu_{\mu\lambda} \overset{\text{def.}}{=} \Gamma^\nu_{\lambda\mu} - \frac{2}{n+1} A^\nu_{(\lambda}\Gamma^\varkappa_{\mu)\varkappa} \tag{2}$$

are unaltered under the transformation (1). Let us introduce the

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1) Thomas T. Y.: On the Projective Theory of Two Dimensional Riemann Spaces. (Proc. Nat. Acad. Sci., U. S. A., 31 (1945), 259—261.)

2) Where $a^{\lambda\mu}$ are the contravariant components of the fundamental tensor of the Riemannian space, $\mathfrak{a}$ = det. $(a_{\lambda\mu})$ and $K_\mu$ is the first derivative of the Gaussian curvature.

3) A projective Riemannian space is the totality of Riemannian spaces which yield the same system of geodesics.

4) Small (capital) Greek indices take the values $1, \ldots, n$ $(0, 1, \ldots, n)$.

quantities $\Pi^{\nu}_{\omega\mu\lambda}$ by the relation

$$\Pi^{\nu}_{\omega\mu\lambda} = 2\partial_{[\omega}\Pi^{\nu}_{\mu]\lambda} + 2\Pi^{\nu}_{[\omega|\varkappa|}\Pi^{\varkappa}_{\mu]\lambda}. \quad (3)$$

If we introduce a set of functions $P^{\Omega'}_{\Omega}$ and $P^{\Omega}_{\Omega'}$ defined by

$$P^{o'}_{o} = 1,\ P^{\nu'}_{o} = 0,\ P^{\nu'}_{\lambda} = \frac{\partial \xi^{\nu'}}{\partial \xi^{\lambda}},\ P^{o'}_{\lambda} = -\frac{1}{c(n+1)}\frac{\partial \log \Delta}{\partial \xi^{\lambda}},$$
$$c = \text{const.} \quad (4)$$
$$P^{o}_{o'} = 1,\ P^{\nu}_{o'} = 0,\ P^{\nu}_{\lambda'} = \frac{\partial \xi^{\nu}}{\partial \xi^{\lambda'}},\ P^{o}_{\lambda'} = \frac{1}{c(n+1)}\frac{\partial \log \Delta}{\partial \xi^{\nu}}\frac{\partial \xi^{\nu}}{\partial \xi^{\lambda'}},$$

where

$$\xi^{\nu'} = \xi^{\nu'}(\xi^{\nu}) \quad (5)$$

is a coordinates transformation with $\Delta = \det.\left(\frac{\partial \xi^{\nu'}}{\partial \xi^{\nu}}\right) \neq 0$, then we find that the set of functions

$$\begin{aligned} {}^*\Gamma^{\nu}_{\lambda\mu} &= {}^*\Gamma^{\nu}_{\mu\lambda} = \Pi^{\nu}_{\lambda\mu}, \\ {}^*\Gamma^{\Omega}_{o\Sigma} &= {}^*\Gamma^{\Omega}_{\Sigma o} = c\delta^{\Omega}_{\Sigma}, \\ {}^*\Gamma^{o}_{\lambda\mu} &= {}^*\Gamma^{o}_{\mu\lambda} = \frac{1}{c(n-1)}\Pi^{\nu}_{\lambda\nu\mu} = \frac{1}{c(n-1)}\Pi_{\lambda\mu} \end{aligned} \quad (6)$$

transforms under (5) according to

$${}^*\Gamma^{\Omega'}_{\Lambda'\Sigma'} = {}^*\Gamma^{\Omega}_{\Lambda\Sigma}P^{\Omega'}_{\Omega}P^{\Lambda}_{\Lambda'}P^{\Sigma}_{\Sigma'} + P^{\Omega'}_{\Gamma}\,\partial_{\Lambda'}P^{\Gamma}_{\Sigma'} \quad (7)$$

where

$$\partial_{\Lambda'}P^{\Gamma}_{\Sigma'} = \delta^{\lambda'}_{\Lambda'}\frac{\partial}{\partial \xi^{\lambda'}}P^{\Gamma}_{\Sigma'}.\text{[5]} \quad (8)$$

Because of (7) the functions (6) are the coefficients of a projective connection.

The projective curvature tensor of the ${}^*\Gamma$'s

$$\mathfrak{P}_{\Omega\Sigma\Lambda}{}^{\Phi} = 2\partial_{[\Omega}{}^*\Gamma^{\Phi}_{\Sigma]\Lambda} + 2{}^*\Gamma^{\Phi}_{[\Omega|\Psi|}{}^*\Gamma^{\Psi}_{\Sigma]\Lambda} \quad (9)$$

satisfies the following relations

$$\begin{aligned} \mathfrak{P}_{\Omega\Sigma\Lambda}{}^{\Phi} &= A^{\omega}_{\Omega}A^{\sigma}_{\Sigma}A^{\lambda}_{\Lambda}\mathfrak{P}_{\omega\sigma\lambda}{}^{\Phi}, \\ \mathfrak{P}_{\omega\mu\lambda}{}^{\nu} &= \Pi^{\nu}_{\omega\mu\lambda} + \frac{2}{n-1}A^{\nu}_{[\omega}\Pi_{\mu]\lambda}, \\ \mathfrak{P}_{\omega\mu\lambda}{}^{o} &= \frac{1}{c(n-1)}(2\partial_{[\omega}\Pi_{\mu]\lambda} + 2\Pi_{[\omega|\varkappa|}\Pi^{\varkappa}_{\mu]\lambda}). \end{aligned} \quad (10)$$

The components $\mathfrak{P}_{\omega\mu\lambda}{}^{\Phi}$ may be expressed by means of the curvature

[5]) Cf. V. Hlavatý: Hypersurfaces in a Projective Curved Space (Annals of Mathematics 39 (1938), 728—729.)

tensor of the $\Gamma$'s

$$R_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu} = 2\partial_{[\omega}\Gamma^{\nu}_{\mu]\lambda} + 2\Gamma^{\nu}_{[\omega|\varkappa|}\Gamma^{\varkappa}_{\mu]\lambda},^{6)} \tag{11}$$

namely

$$\begin{aligned} \text{a)}\ \mathfrak{P}_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu} &= P_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu}, \\ \text{b)}\ \mathfrak{P}_{\omega\mu\lambda}^{\cdot\cdot\cdot 0} &= \frac{2}{c}\,(D_{[\omega}P_{\mu]\lambda} + \frac{1}{2(n+1)}\,P_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu}\Gamma^{\varkappa}_{\nu\varkappa}), \end{aligned} \tag{12}$$

where $D_\omega$ denotes the operator of the covariant derivative with regard to $\Gamma$'s and

$$\begin{aligned} P_{\mu\lambda} &= -\frac{1}{n^2-1}\,(nR_{\mu\lambda} + R_{\lambda\mu}),\ R_{\mu\lambda} = R_{\nu\mu\lambda}^{\cdot\cdot\cdot\nu}, \\ P_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu} &= R_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu} - 2P_{[\omega\mu]}A^{\nu}_{\lambda} + 2A^{\nu}_{[\omega}P_{\mu]\lambda} \end{aligned} \tag{13}$$

($P_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu}$ being the *Weyl* curvature tensor). If we use the well known relation

$$D_\nu P_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu} = 2\,(2-n)\,D_{[\omega}P_{\mu]\lambda},^{7)} \tag{14}$$

we may express (12b) in the following form valid for $n > 2$

$$\mathfrak{P}_{\omega\mu\lambda}^{\cdot\cdot\cdot 0} = \frac{1}{c}\left(\frac{1}{2-n}\,D_\nu P_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu} + \frac{1}{n+1}\,P_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu}\Gamma^{\varkappa}_{\nu\varkappa}\right). \tag{15}$$

If $n = 2$, the *Weyl* curvature tensor vanishes identically and therefore

$$\begin{aligned} \text{a)}\ \mathfrak{P}_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu} &= 0, \\ \text{b)}\ \mathfrak{P}_{\omega\mu\lambda}^{\cdot\cdot\cdot 0} &= \frac{2}{c}\,D_{[\omega}P_{\mu]\lambda}. \end{aligned} \tag{16}$$

In the case of the projective twodimensional Riemannian space we have

$$P_{\mu\lambda} = -Ka_{\mu\lambda}, \tag{17}$$

where $a_{\lambda\mu}$ is the fundamental tensor of the Riemannian space and $K$ is the Gaussian curvature. Hence we get from (16b) and (17)

$$\mathfrak{P}_{\omega\mu\lambda}^{\cdot\cdot\cdot 0} = -\frac{2}{c}\,K_{[\omega}a_{\mu]\lambda}. \tag{18}$$

Let us now introduce the bivector density $\mathfrak{E}^{\lambda\mu} = \mathfrak{E}^{[\lambda\mu]}$ of weight $+1$ with the components $\mathfrak{E}^{11} = \mathfrak{E}^{22} = 0$, $\mathfrak{E}^{12} = -\mathfrak{E}^{21} = 1$. By means of (18) we find that the covariant vector density of weight $+1$

$$\mathfrak{M}_\lambda \overset{\text{def.}}{=} K_{[\omega}a_{\mu]\lambda}\mathfrak{E}^{\omega\mu} = K_\omega a_{\mu\lambda}\mathfrak{E}^{\omega\mu} = -\tfrac{1}{2}c\,\mathfrak{P}_{\omega\mu\lambda}^{\cdot\cdot\cdot 0}\mathfrak{E}^{\omega\mu} \tag{19}$$

is a projective invariant.

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[6]) Cf. Schouten J. A.-Struik D. J.: Einführung in die neueren Methoden der Differentialgeometrie, I., p. 110.

[7]) Cf. Eisenhart L. P.: Non-Riemannian Geometry (American Math. Soc., New York, 1927), p. 89.

If we define $\mathfrak{e}_{\lambda\mu}$ by

$$\mathfrak{e}_{\lambda\mu}\mathfrak{E}^{\lambda\nu} = \delta^{\nu}_{\mu} \tag{20}$$

then we have $\mathfrak{e}_{\lambda\mu} = \mathfrak{e}_{[\lambda\mu]}$ and consequently we obtain for the contravariant vector density

$$\mathfrak{N}^{\nu} \overset{\text{def.}}{=} \mathfrak{E}^{\nu\lambda}\mathfrak{M}_{\lambda} \tag{21}$$

the relation

$$\mathfrak{N}^{\varkappa} = \mathfrak{E}^{\varkappa\lambda}K_{\omega}\mathfrak{a}a^{\alpha\beta}\mathfrak{e}_{\mu\alpha}\mathfrak{e}_{\lambda\beta}\mathfrak{E}^{\omega\mu} = \mathfrak{a}K^{\varkappa} \tag{22}$$

which shows that $\mathfrak{N}^{\varkappa}$ as defined by (21) is identical with the vector density introduced by *T. Y. Thomas*.[1])

From (19) and (22) we get

$$\mathfrak{a}K^{\varkappa} = \tfrac{1}{2}c\,\mathfrak{P}_{\omega\mu\lambda}^{\cdot\cdot\cdot\,\nu}\mathfrak{E}^{\omega\mu}\mathfrak{E}^{\lambda\varkappa}, \tag{23}$$

from which follows that $\mathfrak{N}^{\varkappa}$ is a projective invariant. *The equation (23) gives the relation of the invariant $\mathfrak{a}K^{\varkappa}$ introduced by T. Y. Thomas to the components $\mathfrak{P}_{\omega\mu\lambda}^{\cdot\cdot\cdot\,0}$ of the projective curvature tensor $\mathfrak{P}_{\Omega\Sigma\Lambda}^{\cdot\cdot\cdot\,\Phi}$ of the twodimensional projective Riemannian space.*

Praha, Charles University, Department of Geometry.

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## Poznámka k článku T. Y. Thomase: On the Projective Theory of Two Dimensional Riemann Spaces.

(Obsah předešlého článku.)

*T. Y. Thomas* zavedl ve článku uvedeném v nadpise a v poznámce [1]) kontravariantní vektorovou hustotu váhy $+2$, a to $\mathfrak{a}K^{\varkappa}$, a ukázal, že je invariantní vzhledem k projektivní transformaci (1), jestliže $\Gamma^{\nu}_{\lambda\mu}$ a $'\Gamma^{\nu}_{\lambda\mu}$ jsou Christoffelovy symboly druhého druhu dvojrozměrných Riemannových prostorů. V předchozí práci je nalezen rovnicí (23) vztah této vektorové hustoty ke složkám $\mathfrak{P}_{\omega\mu\lambda}^{\cdot\cdot\cdot\,0}$ projektivního tensoru křivosti $\mathfrak{P}_{\Omega\Sigma\Lambda}^{\cdot\cdot\cdot\,\Phi}$ projektivního Riemannova prostoru.