P-regularity and a duality theorem of Čech.

Shaun Wylie, Princeton, N. J.

(Received May 23, 1936.)

1. Introduction.

Professor Čech[1]) has proved a duality relation between dimensions p and $n - p$ of a complex under conditions based on his notion of p-regularity. A complex is said to be p-regular if

(I) it is a simple orientable n-circuit,
(II) for $r = p$, star (σ_r) has the n-th Betti number $R_n = 1$,
(III) for $r < p$, star (σ_r) has $R_{n-p+r} = \Theta_{n-p+r-1} = 0$,

i. e. the $(n - p + r)$-th Betti number is zero and there are no torsion coefficients for the dimension $n - p + r - 1$.

The star (σ_r) is the set of simplexes having σ_r as a face. We also make use of link (σ_r), which is the set of simplexes σ such that $\sigma\sigma_r$ is a simplex of K_n, the complex.

We can write (II) and (III) as
(II)′ in link (σ_r), R_{n-p-1} and Θ_{n-p-2} are spherelike.

That is, R_{n-p-1} and Θ_{n-p-2} are identical with the corresponding numbers of an $(n - r - 1)$-sphere. These two statements are equivalent.

In this paper we prove the independence of regularities for different indices by a series of examples, and give an idea of the generality of p- and $(p - 1)$-regular complexes as compared with manifolds. This is significant as it is for such complexes that Čech proves his partial duality theorem.

2. Independence Examples.

Example 1. Showing that p-regularity is not a property invariant under subdivision.

---

[1]) E. Čech, *Multiplications on a complex* (to appear in Annals of Mathematics).

Take a 3-sphere, and mark on it two 1-cells having common endpoints; identify these 1-cells, to give us a single 1-cell E. Now subdivide this point-set in such a way that no vertex lies interior to E; call this K_3. It is easy to see that this is 0-regular; i. e. that in the star of every vertex $R_n = 1$. For the only doubtful vertices are the end-points of E; but the link of such a point is a 2-sphere with two points matched; here $R_{n-1} = 1$, so that in the star $R_n = 1$. Also K_3 is a simple orientable 3-circuit.

It is equally easy to see that the first derived K'_3 is not 0-regular. Take the vertex interior to E; its link is two 2-spheres joined at two separate points, giving $R_{n-1} = 2$; so in the star $R_n = 2$.

Example 2. Showing that p-regularity is independent from all other q-regularities except $q = n - p - 1$.

This is a K_n which is p-regular and $(n - p - 1)$-regular, but not q-regular for any other q.

We first construct an M_{n-1} whose only zero Betti numbers are R_p and R_{n-p-1} and without torsion coefficients. This is due to Bassi.[2]) We consider the product $H_q \times H_{n-q-1}$: (H_r is always an r-sphere); this has only R_{n-1}, R_{n-q-1}, R_q and R_0 different from zero. We now take one such model for every q different from p and $n - p - 1$, and we take the sum of these manifolds. Two manifolds of dimension s are summed by extracting from each an s-cell and matching their boundaries; about sums we have the theorem that the Betti numbers of the sum are the sums of the Betti numbers for every dimension except s and Θ, when the Betti numbers are 1. So, in this M_{n-1} we have constructed, we have only R_{n-p-1} and R_p zero. Now take two models of this M_{n-1} and join them to two points. This is the example. The only irregular points are these two points, and in their links every Betti number is at least 2 except R_{n-p-1} and R_p; sothat this K_n is p- and $(n - p - 1)$-regular but regular for no other index. K_n is obviously a simple orientable n-circuit. It is, of course n-regular; but that is true of every simple orientable n-circuit.

Example 3. Showing that p-regularity is independent of $(n - p - 1)$-regularity.

We could use Example 1: this is a K_3 which is 0-regular but not 2-regular. If it were to be 2-regular, in every link R_0 would have to be spherelike; but in the link of E we see that $R_0 = 2$ and not 1.

We can also give an example of a K_4 which is 2-regular under any subdivision but is not 1-regular however it is subdivided. This K_4 is described as the join of two points to a projective 3-space. The irregular points are the two points; for 2-regularity, in their links we require $R_1 = \Theta_0 = 0$, which is satisfied as their link is

[2]) A. Bassi, *Un Problema Topologico di Esistenza*; Reale Accademia d'Italia, 1935.

projective 3-space. For 1-regularity we would need $R_2 = \Theta_1 = 0$; but Θ_1 is not zero, so that K_4 is not 1-regular. K_4 is a simple circuit and is orientable since projective 3-space is.

If we had taken the join of two points to two models of projective 3-space we would have had a K_4 which is 2-regular but neither 3-, 1- nor 0-regular.

3. The generality of the p- and $(p - 1)$-regular complexes.

This problem can be as easily treated by considering the complexes which are p-regular for every p between q_1 and q_2. We get greater generality by this method and can without difficulty deduce the special cases.

We only consider, from now on, p-regularity which is invariant under subdivision. When we make this assumption, we can strengthen conditions (II) and (III). If σ_r is a simplex of K_n and τ_s is a simplex of the first derived, K'_n and τ_s lies in σ_r, then star (τ_s) in K'_n is homeomorphic to star (σ_r) in K_n. Now we can find such τ_s's for every value of $s \leqq r$; applying the conditions of p-regularity to the stars of these τ_s's, we can use the results as conditions on the star (σ_r). We get that, in star (σ_r), R_{n-p+r} down to R_{n-p} and $\Theta_{n-p+r-1}$ to Θ_{n-p-1} are cell-like; or, in link (σ_r), R_{n-p-1} to $R_{n-p-r-1}$ and Θ_{n-p-2} to $\Theta_{n-p-r-2}$ are spherelike.

If now we have that K_n is p-regular for $p \geqq q_1$ and $\leqq q_2$, we get that in link (σ_r) R_{n-q_1-1} to R_{n-q_2-r-1} and Θ_{n-q_1-2} to Θ_{n-q_2-r-2} are spherelike.

Now it is an elementary matter to verify that, if K_n is p-regular in any subdivision, link (σ_r) is q-regular for all q between p and $p - r - 1$. This follows readily from the fact that

$$\text{link } (\sigma_s) \text{ in link } (\sigma_r) = \text{link } (\sigma_r\sigma_s) \text{ in } K_n.$$

The only condition not fulfilled is that link (σ_r) does not itself have $R_{n-r-1} = 1$; it is a circuit but not a simple circuit. This condition is however unnecessary for the duality theorem, which states that under p- and $(p - 1)$-regularity $R_p = R_{n-p}$ and $\Theta_p = \Theta_{n-p-1}$.

Applying this several times to link (σ_r), we can deduce that in link (σ_r) we have also that R_{q_1-r} up to R_{q_2} and Θ_{q_1-r} up to Θ_{q_2} are spherelike.

We now have a considerable amount of information about the Betti numbers and torsion coefficients of link (σ_r); it is important to see under what conditions we have enough to ensure that link (σ_r) has the homology characters of a sphere. We will have this position if the indices $(n - q_1 - 1)$ to $(n - q_2 - r - 1)$ and q_2 to $(q_1 - r)$ exhaust those from $(n - r - 1)$ to 0; and if also $(n - q_1 -$

$-2)$ to $(n-q_2-r-2)$ and q_2 to (q_1-r) exhaust those from $(n-r-2)$ to 1.

We now separate two cases: $n-q_1-1 \gtrless q_2$; that is $q_1 + q_2 \gtrless n-1$.

(i) $q_1+q_2 \leqq n-1$; then $n-q_1-1 \geqq q_2$, and we will have all the indices represented if and only if $n-q_1-1 \geqq n-r-1$ and $n-q-r-1 \leqq q_2+1$, that is if $r \geqq q_1$ and $\geqq n-2q_2-2$.

(ii) $q_1+q_2 \geqq n-1$; now $n-q_1-1 \leqq q_2$ and we have that the conditions are that $q_2 \geqq n-r-1$ and $q_1-r \leqq n-q_1-1$. We cannot have $q_1-r = n-q_1$, for then we might have $\Theta_{q_1-r-1} = \Theta_{n-q_1-1}$ different from zero. So we get as the conditions — $r \geqq n-q_2-1$ and $\geqq 2q_1-n+1$.

Subdividing (i) into two cases we get

(a) if $q_1+q_2 \leqq n-1$, and $q_1+2q_2 \leqq n-2$ (this of course includes the other), then for $r \geqq n-2q_2-2$, link (σ_r) is spherelike;

(b) if $q_1+q_2 \leqq n-1$, and $q_1+2q_2 \geqq n-2$, we must have $r \geqq q_1$.

Similarly

(c) if $q_1+q_2 \geqq n-1$, and $2q_1+q_2 \leqq 2n-2$, we have $r \geqq n-q_2-1$;

(d) if $2q_1+q_2 \geqq 2n-2$, we need $r \geqq 2q_1-n+1$.

Now if link (σ_r) has the characters of an $(n-r-1)$-sphere, we say that σ_r is regular. A manifold is a complex all of whose simplexes are regular. If a complex has all its simplexes of dimension $\geqq r$ regular, it is called a relative manifold of degree r, and may be written $M_n^{(r)}$. We have shown that if K_n has q_1-regularity up to q_2-regularity, in the range $q_1+2q_2 \leqq n-2$, K_n is an $M_n^{(n-2q_2-2)}$; there are the corresponding statements for other ranges of q_1 and q_2. It is not of course true that every $M_n^{(n-2q_2-2)}$ is q_1-regular to q_2-regular; there are additional local conditions as well as that of being orientable.

It is desirable to give examples of these complexes which are not manifolds of lower degree; that is, complexes of this kind which contain irregular simplexes of dimension one lower than the degree indicated. In three of the four cases this can be done, and in the fourth we only lose one dimension in the degree.

Example 4. A K_n, p-regular from $p=q_1$ to $p=q_2$, where $q_1+2q_2 \leqq n-2$, containing irregular σ_{n-2q_2-3}'s.

Take the topological product of two H_{q_2+1}'s, and join to an H_{n-2q_2-3}; as always H_r is an r-sphere. The simplexes of H_{n-2q_2-3} are irregular and in their stars we have $R_{n-q_2-1} = R_n = 1$; the other R's and Θ's are 0. This gives us that R_{n-q_1+r} down to R_{n-q_2} are cell-like, and all Θ's are zero; so that K_n is q_1- to q_2-regular.

Example 5. $q_1 + 2q_2 \geqq n - 2$, $q_1 + q_2 \leqq n - 1$; K_n containing irregular σ_{q_1-1}'s.

Take an H_n and pick out two non-intersecting H_{q_1-1}'s on it; identify these. The irregular simplexes lie on this H_{q_1-1}, and in their stars $R_n = 2$, $R_{q_1} = 1$; other R's and Θ's $= 0$. Again R_{n-q_1+r} to R_{n-q_1} are cell-like, since we need only consider $r \leqq q_1 - 1$ and we know that $n - q_2 > q_1$.

Example 6. $q_1 + q_2 \geqq n - 1$, $2q_1 + q_2 \leqq 2n - 2$; K_n has irregular σ_{n-q_2-2}'s.

On an H_n identify two non-intersecting H_{n-q_2-2}'s; the irregular simplexes lie on this H_{n-q_2-2} and in their stars $R_n = 2$, $R_{n-q_2-1} = 1$; since $q_1 + q_2 \geqq n - 1$, $n - q_1 + (n - q_2 - 2) \leqq \leqq n - 1$; that is, $n - q_1 + r \leqq n - 1$ for any $r \leqq n - q_2 - 2$. Consequently for simplexes of the irregular set R_{n-q_1+r} to R_{n-q_1} are cell-like.

Example 7. $2q_1 + q_2 \geqq 2n - 2$; K_n has irregular σ_{2q_1-n-1}'s.

Join an H_{2q_1-n-1} to H_{n-q_1}. The verification is as before.

In order to find a best possible result here we would want a K_n with irregular σ_{2q_1-r}'s; if this were found, the link of an irregular σ_{2q_1-n} would be a manifold whose only non-spherelike character would be Θ_{n-q_1-1}, the central torsion coefficients. When $n - q_1 = 2$, the projective 3-space is such a manifold; but a general example is unknown to the author. The construction then of an optimum example hangs entirely on the construction of such an M_{2n-2q_1-1}.

It now remains to point out the special cases. If we take $q_1 = q_2$ we get the simple p-regular complexes. If $q_1 = q_2 - 1$, then the complexes are p- and $(p - 1)$-regular; this is the case of primary interest for the duality theorem, and the examples 4 to 7 show that the range of application of the theorem is considerably broader than the set of manifolds.

If we take $q_1 = 0$ then cases (c) and (d) drop out: in (b), $q_2 \geqq \geqq \frac{1}{2}(n - 2)$, we find that every simplex is regular and the complex is a manifold. This could have been deduced more simply by seeing that the q_1- to q_2-regular complex is also $(n - q_2 - 1)$- to $(n - - q_1 - 2)$-regular; this results immediately from the application of the duality theorem to the links. So if $q_1 = 0$, since an n-circuit is automatically $(n - 1)$-regular if it is 0-regular, we have that the 0- to q-regular complex is also $(n - q - 1)$- to $(n - 1)$-regular: if then $q \geqq \frac{1}{2}(n - 2)$ we get p-regularity for any p, which is the condition for a manifold. Similarly if $q_2 = n - 1$, and $q_1 \leqq \frac{1}{2}(n - 1)$ K_n is a manifold.

The results then are these; —

(I) p-regularity is not invariant under subdivision,

(II) p-regularity is not dependent on any other set of regularities,

(III) if however p-regularity is taken as an invariant property, p- and $(p - 1)$-regularity imply $(n - p - 1)$-regularity,

(IV) if K_n is q_1- to q_2-regular, all regularities being taken as invariant properties, there is an upper bound to the dimension of the irregular set, which can in most cases be shown to be reached,

(V) complexes other than manifolds exist for which Čech's partial duality theorem is applicable.

*

P-regularita a Čechův teorém duality.

(Obsah předešlého článku.)

Poincaréův teorém duality praví, že u n-rozměrné variety M_n p-té a $(n - p)$-té Bettiovo číslo se sobě rovnají a rovněž i p-té a $(n - p - 1)$-ní koeficienty torse. Čech definoval pojem p-regularity komplexu tak, že n-variety splývají s n-komplexy, které jsou p-regulární pro všecka $0 \leqq p \leqq n$; a ukázal, že pro platnost Poincaréova teorému duality při daném p stačí předpokládati p- a $(p - 1)$-regularitu. V tomto článku je vyšetřována vzájemná závislost p-regularity komplexu pro různé hodnoty p. Z diskuse plyne zejména, že teorém duality platí pro komplexy mnohem obecnější než variety.