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On Estermann's proof of a theorem of Minkowski

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All numbers in this note are real. We denote by E_n the n-dimensional space; its points will be denoted by $\mathsf{x} = [x_1, \ldots, x_n]$, $\mathsf{y} = [y_1, \ldots, y_n]$ etc., especially $\mathsf{o} = [0, \ldots, 0]$. If λ, μ are numbers, we put $\lambda\mathsf{x} + \mu\mathsf{y} = [\lambda x_1 + \mu y_1, \ldots, \lambda x_n + \mu y_n]$ etc. k points $\mathsf{x}^1, \ldots, \mathsf{x}^k$ are called independent if $\lambda_1\mathsf{x}^1 + \ldots + \lambda_k\mathsf{x}^k = \mathsf{o}$ implies $\lambda_1 = \ldots = \lambda_k = 0$. Let $M \subset E_n$; we denote by Int M and Fr M the interior and the boundary of M, by αM the set of all points $\alpha\mathsf{x}$ where $\mathsf{x} \in M$, by $M + \mathsf{a}$ the set of all points $\mathsf{x} + \mathsf{a}$, where $\mathsf{x} \in M$ (translation), by $\mathfrak{V}(M)$ the set of all points $\mathsf{x} - \mathsf{y}$ where $\mathsf{x} \in M$, $\mathsf{y} \in M$, by $V(M)$ or $V_n(M)$ the Lebesgue measure of M. (In the following, only measurable sets are considered.) A compact (i. e. closed and bounded) convex set having an interior point will be called a convex body. We say that a set M possesses the center c, if $\mathsf{x} \in M$ implies $2\mathsf{c} - \mathsf{x} \in M$. If M is a convex body, then $\mathfrak{V}(M)$ is also a convex body having the center o. Obviously $\mathfrak{V}(M) = \mathfrak{V}(M + \mathsf{a})$. The set of all points contained either in M_1 or in M_2 ... or in M_k will be denoted by $M_1 \cup M_2 \cup \ldots \cup M_k$ or $\bigcup_{1 \leq i \leq k} M_i$. By K_n we denote the cube $0 \leq x_1 < 1, \ldots, 0 \leq x_n < 1$.

Lattice points are points with integer coordinates. Two points x, y are called congruent, $\mathsf{x} \equiv \mathsf{y}$, if $\mathsf{x} - \mathsf{y}$ is a lattice point, i. e. $x_i \equiv y_i \pmod 1$. If $M \subset E_n$ is a compact set having an interior point, we denote by $\tau_i(M)$ the least number $\tau > 0$ such that τM contains at least i independent lattice points. A theorem of Minkowski says (l. c., p. 211–219): If $M \subset E_n$ is a convex body having the center o, then

$$\tau_1(M)\,\tau_2(M)\ldots\tau_n(M)\,V_n(M) \leqq 2^n. \tag{1}$$

Simpler proofs of this theorem — and, in fact, of a slightly more general theorem — have been given by Davenport (l. c.) and Estermann (l. c.). The reader will find this generalization in this note (Theorem 1(. In this note I shall show that Estermann's method also allows us to give the characterization fo all cases in which the sign of equality occurs in (1) (see Theorem 2 and Remarks 5, 6). This cha acterization is also due to Minkowski (l. c., p. 235—236), but he used another and more comp.icated method. The method of the present note follows very closely Estermann's proof, only with supplementary considerations concerning the appearance of the equality sign in the different inequalities given by Estermann. In order to make this note self-contained, I repeat also some considerations of Estermann.

In the proofs I use some known results concerning geometric properties of convex bodies (Lemma 5 and the inequality $V_n(\mathfrak{B}(S)) > 2^n V(S)$, valid for convex bodies having no center — Remark 3), but I use no results of the geometry of numbers; on the contrary, I give in the proof of Theorem 1 and in the Remarks 2, 3, 4 all auxiliary results of this kind necessary for the proof of Theorem 2.

* * *

If x is a number and ξ the greatest integer $\xi \leqq x$, we put $\mathrm{r}(x) = \mathrm{r}x = x - \xi$; thus $0 \leqq \mathrm{r}x < 1$. If $\mathrm{x} = [x_1, \ldots, x_n]$, we put $\mathrm{r}_k\mathrm{x} = \mathrm{r}_k(\mathrm{x}) = [x_1, \ldots, x_{k-1}, \mathrm{r}x_k, x_{k+1}, \ldots, x_n]$ and $\mathrm{R}_k\mathrm{x} = \mathrm{r}_1\mathrm{r}_2\ldots\mathrm{r}_k\mathrm{x} = [\mathrm{r}x_1, \ldots, \mathrm{r}x_k, x_{k+1}, \ldots, x_n]$. Of course, r_kM and R_kM (M being a set) denote the set of all points $\mathrm{r}_k\mathrm{x}$ or $\mathrm{R}_k\mathrm{x}$, where $\mathrm{x} \in M$.

Lemma 1. *Let $1 \leqq k \leqq n$; let S be a measurable bounded set in E_n. Let C be the set of all points $\mathrm{x} = [x_1, \ldots, x_n] \in S$ such that there is a point $\mathrm{y} = [y_1, \ldots, y_n] \in S$ with $\mathrm{x} \equiv \mathrm{y}$, $\mathrm{x} \neq \mathrm{y}$, $x_{k+1} - y_{k+1} = x_{k+2} - y_{k+2} = \ldots = x_n - y_n = 0$. Then $V(\mathrm{R}_kS) < V(S)$, if $V(C) > 0$, but $V(\mathrm{R}_kS) = V(S)$, if $V(C) = 0$.*

Proof. Let $S_{m_1,\ldots,m_k}$ be the set of all points $[x_1, \ldots, x_n] \in S$ such that $m_i \leqq x_i < m_i + 1$ for $i = 1, \ldots, k$ (m_i integers). Then

$$S = \bigcup_{m_1,\ldots,m_k} S_{m_1,\ldots,m_k},\quad \mathrm{R}_kS = \bigcup_{m_1,\ldots,m_k} \mathrm{R}_kS_{m_1,\ldots,m_k};$$

But $\mathrm{R}_kS_{m_1,\ldots,m_k}$ arises from $S_{m_1,\ldots,m_k}$ by means of a translation, and so

$$V(S) = \Sigma V(S_{m_1,\ldots,m_k}) = \Sigma V(\mathrm{R}_kS_{m_1,\ldots,m_k}) \geqq V(\mathrm{R}_kS),$$

the sign of equality being valid if and only if the common part of every two sets $\mathrm{R}_kS_{m_1,\ldots,m_k}$ has the measure zero. But this condition is equivalent to $V(C) = 0$.

Remark 1. Let M be a bounded set. Obviously every point of $\mathrm{Fr}\, \mathrm{R}_k M$ has either at least one integer coordinate or it belongs to $\mathrm{R}_k \,\mathrm{Fr}\, M$. Thus, if $V(\mathrm{Fr}\, M) = 0$, then also $V(\mathrm{Fr}\, \mathrm{R}_k M) = 0$.

Lemma 2. *If T arises from S by means of a translation, then* $V(\mathrm{R}_k S) = V(\mathrm{R}_k T)$.

Proof. (Exactly the same as in Estermann, Lemma 1.) Decomposing the translation into its components along the axes, we may restrict ourselves to the case $T = t_l^i S$, where t_l^i means the operation replacing the i-th coordinate x_i by $x_i + l$. If $i > k$, then $\mathrm{R}_k T$ arises from $\mathrm{R}_k S$ by means of a translation, and the result is obvious. Next suppose $i \leqq k$; we may also suppose $0 \leqq l < 1$. Put $\mathrm{R}_k S = U$; since $\mathrm{r}(x_i + l) = \mathrm{r}(\mathrm{r}(x_i) + l)$, we have $\mathrm{R}_k(t_l^i S) = \mathrm{R}_k(t_l^i U)$. Let U_1 be the set of all points of U for which $0 \leqq x_i < 1 - l$, and U_2 of those for which $1 - l \leqq x_i < 1$. It follows that $\mathrm{R}_k t_l^i U = t_l^i U_1 \cup t_{l-1}^i U_2$, where the two terms have no points in common. Thus

$$V(\mathrm{R}_k t_l^i S) = V(\mathrm{R}_k t_l^i U) = V(t_l^i U_1) + V(t_{l-1}^i U_2) =$$
$$= V(U_1) + V(U_2) = V(U) = V(\mathrm{R}_k S).$$

Lemma 3. *Let $S \subset E_n$ be a convex body, $\mu > 1$. Then $V(\mathrm{R}_n \mu S) \geqq V(\mathrm{R}_n S)$. The sign of equality is valid if and only if $V(\mathrm{R}_n S) = 1$, i. e. ($\mathrm{R}_n S$ being obviously closed in K_n) if $R_n S = K_n$.*

Proof. Choose a point $\mathbf{a}$ so that $T = S + \mathbf{a}$ contains $\mathbf{o}$ in its interior, so that $T \subset \mathrm{Int}\, \mu T$. But $\mu T = \mu S + \mu \mathbf{a}$, and so $V(\mathrm{R}_n T) = V(\mathrm{R}_n S)$, $V(\mathrm{R}_n \mu T) = V(\mathrm{R}_n \mu S)$. But we have $V(R_n T) \leqq V(\mathrm{R}_n \mu T)$. If $V(\mathrm{R}_n T) < 1$, there is a point $\mathbf{x} \in \mathrm{Fr}\, (\mathrm{R}_n T)$ in the interior of K_n and it is clear (see the Remark 1) that $\mathbf{x} \in \mathrm{Int}\, (\mathrm{R}_n \mu T)$; thus obviously $V(R_n T) < V(\mathrm{R}_n \mu T)$.

In the following two Lemmas a convex body $S \subset E_n$ and an integer k ($0 < k < n$) are given. If $\mathbf{x}_1 = [x_1, \ldots, x_k]$, $\mathbf{x}_2 = [x_{k+1}, \ldots, x_n]$, we shall write $[x_1, x_2, \ldots, x_n] = [\mathbf{x}_1, \mathbf{x}_2]$. We denote by S' the set of all points $\mathbf{x}_2 \in E_{n-k}$ such that there is a point $\mathbf{x}_1 \in E_k$ with $[\mathbf{x}_1, \mathbf{x}_2] \in S$ (the „projection" of S). If $\mathbf{x}_2 \in E_{n-k}$ is given, we denote by $S(\mathbf{x}_2)$ the set of all points $\mathbf{x}_1 \in E_k$ such that $[\mathbf{x}_1, \mathbf{x}_2] \in S$ (the „intersection" of S with a k-dimensional plane).

Lemma 4. *Let k be an integer, $0 < k < n$; let $S \subset E_n$ be a convex body; let $\mu > 1$. Then*

$$V(\mathrm{R}_k \mu S) \geqq \mu^{n-k} V(\mathrm{R}_k S).$$

The sign of equality is valid if and only if

$$\mathrm{R}_k S(\mathbf{x}_2) = K_k \text{ for every } \mathbf{x}_2 \in S'$$

(K_k is of course the cube $0 \leqq x_1 < 1, \ldots, 0 \leqq x_k < 1$).

Proof. $\mathrm{x}_2 \in E_{n-k}$ being given, the set of all points $\mathrm{x}_1 \in E_k$ such that $[\mathrm{x}_1, \mathrm{x}_2] \in \mathrm{R}_k S$ is obviously $\mathrm{R}_k S(\mathrm{x}_2)$ and the set of all points $\mathrm{x}_1 \in E_k$ such that $[\mathrm{x}_1, \mathrm{x}_2] \in \mathrm{R}_k \mu S$ is obviously $\mathrm{R}_k \mu\, S\left(\frac{1}{\mu}\,\mathrm{x}_2\right)$. Thus we have

$$V_n(\mathrm{R}_k S) = \int_{S'} V_k(\mathrm{R}_k\, S(\mathrm{x}_2))\, \mathrm{dx}_2,$$

$$V_n(\mathrm{R}_k \mu S) = \int_{\mu S'} V_k\left(\mathrm{R}_k \mu S\left(\frac{1}{\mu}\,\mathrm{x}_2\right)\right) \mathrm{dx}_2 = \mu^{n-k} \int_{S'} V_k(\mathrm{R}_k \mu\, S(\mathrm{y}_2))\, \mathrm{dy}_2.$$

Comparing these formulae, and using Lemma 3 (with k instead of n) we get the assertion of the Lemma.

Lemma 5. *Let k be an integer, $0 < k < n$; let $S \subset E_n$ be a convex body; suppose that $V(S(\mathrm{x}_2))$ is independent of x_2 for $\mathrm{x}_2 \in S'$. Then there is a convex body $S_0 \subset E_k$ and $k(n-k)$ numbers a_{ij} such that S is the set of all points $[x_1, \ldots, x_n]$, given by the formulae*

$$x_i = \sum_{j=k+1}^{n} a_{ij} x_j + y_i \quad (i = 1, \ldots, k),$$

where $[x_{k+1}, \ldots, x_n]$ runs over S' and $[y_1, \ldots, y_k]$ over S_0.

For the proof see Minkowski, l. c., p. 209—210. Obviously every $S(\mathrm{x}_2)$ arises from S_0 by means of a translation.

* * *

We proceed now to the formulation of the main results. Theorem 1 is the Estermann's theorem, which is, as observed by Estermann, implicitly contained already in Davenport's proof of (1) and which constitutes a slight generalization of Minkowski's inequality (1).

Theorem 1. *Let $S \subset E_n$ be a convex body. Then*

$$\tau_1(\mathfrak{B}(S)) \ldots \tau_n(\mathfrak{B}(S))\; V(S) \leqq 1. \tag{2}$$

Theorem 2 contains the characterization of all convex bodies S for which the sign of equality in (2) is valid. We shall prove Theorem 1 following exactly Estermann's proof, but preparing at the same time the proof of Theorem 2.

S being given, we put $\tau_i(\mathfrak{B}(S)) = \lambda_i$. There are n independent lattice points u^i $(i = 1, \ldots, n)$ so that $\lambda_i\, \mathfrak{B}(S)$ contains the point u^i; thus $\mathrm{u}^i = \mathrm{v}^i - \mathrm{w}^i$, $\mathrm{v}^i \in \lambda_i S$, $\mathrm{w}^i \in \lambda_i S$. Some of the λ_i's may be equal; i. e. there are natural numbers $m, k_1, k_2, \ldots, k_m$ $(k_1 + \ldots + k_m = n)$ and m numbers $0 < \mu_1 < \ldots < \mu_m$ such that, putting

$$l_0 = 0,\ l_i = k_1 + \ldots + k_i \ (i = 1, \ldots, m) \tag{3}$$

we have

$$\lambda_i = \mu_j \text{ for } l_{j-1} < i \leqq l_j;\ l_m = n. \tag{4}$$

In this case we shall say that S belongs to the class $(n; m; k_1, \ldots, k_m; \mu_1, \ldots, \mu_m)$. It is clear that, for $l_{j-1} < i \leqq l_j$, we have $\mathrm{u}^i \in \mathrm{Fr}\, \mu_j \mathfrak{B}(S)$, $\mathrm{v}^i \in \mathrm{Fr}\, \mu_j S$, $\mathrm{w}^i \in \mathrm{Fr}\, \mu_j S$. Applying a homogeneous linear substitution of coordinates with integral coefficients and the determinant unity (such a transformation — we call it U-transformation — changes neither the λ_i's nor $V(S)$) we may attain that the points $\mathrm{u}^i = [u_1^i, \ldots, u_n^i]$ satisfy the conditions

$$u_i^i \neq 0 \text{ for } 1 \leqq i \leqq n, \; u_j^i = 0 \text{ for } 1 \leqq i < j \leqq n. \tag{5}$$

This transformation transforms S into a body S_1 which will be called an „adapted body" (of course, it may happen that S may be adapted in several different ways).

Proof of Theorem 1. Let S be a convex body of the class

$$(n; m; k_1, \ldots, k_m; \mu_1, \ldots, \mu_m);$$

we may suppose S adapted. The inequality (2) may be written

$$\mu_1^{k_1} \mu_2^{k_2} \ldots \mu_m^{k_m} \, V(S) \leqq 1. \tag{6}$$

Put

$$S_k = \mathrm{R}_{l_k} \mu_k S \; (1 \leqq k \leqq m), \quad T_k = \mathrm{R}_{l_k} \mu_{k+1} S \; (1 \leqq k < m),$$

so that

$$S_{k+1} = \mathrm{r}_{l_k+1} \mathrm{r}_{l_k+2} \ldots \mathrm{r}_{l_{k+1}} T_k \; (1 \leqq k < m).$$

Obviously $S_m \subset K_n$, and so

$$V(S_m) \leqq 1. \tag{7}$$

For $0 < \varepsilon < \mu_1$ the body $(\mu_1 - \varepsilon) S$ contains no pair of congruent points, and Lemma 1 and 3 give

$$\begin{aligned} \mu_1^k \, V(S) = V(\mu_1 S) \geqq V(S_1) \geqq V(R_{l_1}(\mu_1 - \varepsilon) S) = \\ = V((\mu_1 - \varepsilon) S) = (\mu_1 - \varepsilon)^n \, V(S), \end{aligned}$$

$$V(S_1) = \mu_1^n \, V(S). \tag{8}$$

Let $1 \leqq j < m$, replace in Lemma 1 the operation $\mathrm{R}_k = \mathrm{r}_1 \mathrm{r}_2 \ldots \mathrm{r}_k$ by $\mathrm{r}_{l_j+1} \ldots \mathrm{r}_{l_{j+1}}$ and apply Lemma 1 to the set $T_j = \mathrm{r}_1 \ldots \mathrm{r}_{l_j} \mu_{j+1} S$. The corresponding set C of Lemma 1 is obviously contained in $\mathrm{r}_1 \ldots \mathrm{r}_{l_j} \cdot \mathrm{Fr}\,(\mu_{j+1} S)$; since $V(\mathrm{Fr}\, \mu_{j+1} S) = 0$, we have (Lemma 1) $V(C) = 0$ and so

$$V(S_{j+1}) = V(\mathrm{r}_{l_j+1} \ldots \mathrm{r}_{l_{j+1}} T_j) = V(T_j). \tag{9}$$

But, following Lemma 4, we have

$$V(T_j) \geqq \left(\frac{\mu_{j+1}}{\mu_j}\right)^{n-l_j} V(S_j) \quad (1 \leqq j < m). \tag{10}$$

From (7), (8), (9), (10) we get

$$1 \geqq V(S_m) \geqq \prod_{j=1}^{m-1} \left(\frac{\mu_{j+1}}{\mu_j}\right)^{l_m - l_j} \cdot \mu_1^{l_m} V(S) = \mu_1^{k_1} \mu_2^{k_2} \ldots \mu_m^{k_m} V(S)$$

and (6) is proved.

Remark 2. If in (2) (i. e. in (6)) the sign of equality is valid, we shall say that S belongs to the (extreme) class $\mathscr{E}(n; m; k_1, \ldots, k_m; \mu_1, \ldots, \mu_m)$. This happens ($S$ being adapted) if and only if in (7) and (10) the sign of equality is valid.

Remark 3. *Suppose that the sign of equality in* (2) *is valid; then S has a center and λS contains to every point of the space a congruent point, if $\lambda \geqq \lambda_n$.*[1])

Proof. I. If $\alpha > 0$, then $\tau_i(\alpha M) = \alpha^{-1}\tau_i(M)$. If S is a convex body, then $V(\mathfrak{B}(S)) > 2^n V(S)$,[2]) except the following case: If S has a center, then $\mathfrak{B}(S)$ arises from $2S$ by means of a translation, and so $V(\mathfrak{B}(S)) = 2^n V(S)$. Because $\mathfrak{B}(S)$ has the center o, we have $\mathfrak{B}(\mathfrak{B}(S)) = 2\mathfrak{B}(S)$. Thus, if S is a convex body having no center, then (2), applied to $\mathfrak{B}(S)$, gives

$$\begin{aligned} 1 &\geqq \tau_1(\mathfrak{B}(\mathfrak{B}(S))) \ldots \tau_n(\mathfrak{B}(\mathfrak{B}(S)))\, V(\mathfrak{B}(S)) \\ &> \tau_1(2\mathfrak{B}(S)) \quad \ldots \tau_n(2\mathfrak{B}(S))\, .\, 2^n V(S) \\ &= \tau_1(\mathfrak{B}(S)) \quad \ldots \tau_n(\mathfrak{B}(S))\, .\, V(S). \end{aligned}$$

II. Following (7) and the Remark 2 we must have $V(S_m) = V(\mathrm{R}_n \lambda_n S) = 1$ and so, following Lemma 3, $V(\mathrm{R}_n \lambda S) = 1$ for $\lambda \geqq \lambda_n$, i. e. $\mathrm{R}_n \lambda S = K_n$.

Remark 4. Especially, (2) implies $\lambda_1^n V(S) \leqq 1$; if the sign of equality is valid, we must have $\lambda_1 = \lambda_2 = \ldots = \lambda_n$. Considering the body $\lambda_1 S$ instead of S, we see: *If S is a convex body, $\tau_1(\mathfrak{B}(S)) = 1$, $V(S) = 1$, then there are n independent lattice points contained in* $\mathrm{Fr}\,\mathfrak{B}(S)$, *$S$ has a center and λS contains, for every $\lambda \geqq 1$, to every point of the space a congruent point.*

Theorem 2. *Let $n, m, k_1, \ldots, k_m$ be positive integers, $k_1 + \ldots + k_m = n$, let $0 < \mu_1 < \mu_2 < \ldots < \mu_m$ and put*

$$l_0 = 0, \; l_i = k_1 + \ldots + k_i \;\; (i = 1, \ldots, m), \tag{11}$$

so that $l_m = n$. Then a convex body $K \subset E_n$ belongs to the class

$$\mathscr{E}(n; m; k_1, \ldots, k_m; \mu_1, \ldots, \mu_m) \tag{12}$$

if and only if it arises by means of a homogeneous linear transformation with integral coefficients and the determinant unity from a convex body S which has the following form: Choose m convex bodies Q_i ($i = 1, \ldots, m$), where

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[1]) These properties are changed neither by a U-transformation (especially by an adaptation) nor by a translation.

[2]) See e. g. Bonnesen-Fenchel, l. c., p. 105.

$$Q_i \subset E_{k_i},\ V(Q_i) = 1,\ \tau_l(\mathfrak{B}(Q_i)) = 1\ (l = 1, \ldots, k_i)$$

(V means V_{k_i}) and real numbers a_{ij}, and let S be the set of all points $[x_1, \ldots, x_n]$ of the form

$$\begin{aligned}
x_i &= \sum_{j=l_1+1}^{n} a_{ij}x_j + y_i \quad \text{for} \quad l_0 < i \leqq l_1, \\
x_i &= \sum_{j=l_2+1}^{n} a_{ij}x_j + y_i \quad \text{for} \quad l_1 < i \leqq l_2, \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
x_i &= \sum_{j=l_{m-1}+1}^{n} a_{ij}x_j + y_i \quad \text{for} \quad l_{m-2} < i \leqq l_{m-1}, \\
x_i &= y_i \qquad\qquad\qquad\qquad \text{for} \quad l_{m-1} < i \leqq l_m = n,
\end{aligned} \tag{13}$$

where the points

$$[y_1, \ldots, y_{l_1}], [y_{l_1+1}, \ldots, y_{l_2}], \ldots, [y_{l_{m-1}+1}, \ldots, y_{l_m}]$$

run — independently of each other — over the sets $\mu_1^{-1}Q_1, \mu_2^{-1}Q_2, \ldots, \mu_m^{-1}Q_m$ respectively.

To prove this theorem, it is sufficient to prove:

I. Every adapted body S of the class (12) has the form described by (13).

II. Every set S given by (13) in the way described above belongs to the class (12).

Proof of I. For $m = 1$, I is a mere tautology. Thus let $m > 1$ and suppose that I is proved if m is replaced by a smaller number. Let S be an adapted body of the class (12). To simplify the notation, put $\mu_1 = \mu$, $\mu_2 = \nu$, $k_1 = k$. We denote again with S' the set of all points $\mathrm{x}_2 \in E_{n-k}$ such that there is a point $\mathrm{x}_1 \in E_k$ with $[\mathrm{x}_1, \mathrm{x}_2] \in S$ and by $S(\mathrm{x}_2)$ (for $\mathrm{x}_2 \in E_{n-k}$) the set of all $\mathrm{x}_1 \in E_k$ with $[\mathrm{x}_1, \mathrm{x}_2] \in S$. Following (10) and the Remark 2, we have

$$V(\mathrm{R}_l \nu S) = \left(\frac{\nu}{\mu}\right)^{n-k} V(\mathrm{R}_k \mu S)$$

and Lemma 4 and 1 give

$$\mu^k\, V(S(\mathrm{x}_2)) \geqq V(\mathrm{R}_k \mu\, S(\mathrm{x}_2)) = 1$$

for all $\mathrm{x}_2 \in S'$. But $(\mu - \varepsilon)\, S(\mathrm{x}_2)$ contains, if $0 < \varepsilon < \mu$, no pair of congruent points, and so, putting $\varrho_i = \tau_i(\mathfrak{B}(S(\mathrm{x}_2)))$ $(i = 1, \ldots, k)$, we have $\varrho_1 > \mu - \varepsilon$ and so

$$(\mu - \varepsilon)^k\, V(S(\mathrm{x}_2)) < \varrho_1^k\, V(S(\mathrm{x}_2)) \leqq \varrho_1 \cdots \varrho_k\, V(S(\mathrm{x}_2)) \leqq 1;$$

thus, finally,

$$\mu^k V(S(\mathsf{x}_2)) = 1,\ \varrho_1 = \varrho_2 = \ldots = \varrho_k = \mu.$$

Thus $V(S(\mathsf{x}_2))$ is independent on x_2.

Following Lemma 5 the body S is the set of all points $[x_1, \ldots, x_n]$ such that

$$x_i = \sum_{j=k+1}^{n} a_{ij}x_j + y_i \quad (i = 1, \ldots, k); \tag{14}$$

here $[x_{k+1}, \ldots, x_n]$ runs over S' and $[y_1, \ldots, y_k]$ over a convex body $\mu^{-1}Q_1$ which of course arises from $S(\mathsf{x}_2)$ by means of a translation, so that

$$Q_1 \subset E_k,\ V(Q_1) = 1,\ \tau_j(\mathfrak{B}(Q_1)) = 1 \text{ for } j = 1, \ldots, k. \tag{15}$$

Evidently

$$V_n(S) = \mu^{-k} V_k(Q_1)\, V_{n-k}(S'). \tag{16}$$

Put $\Lambda_i = \tau_i(\mathfrak{B}(S'))$ for $1 \leqq i \leqq n - k$, $\lambda_i = \tau_i(\mathfrak{B}(S))$ for $1 \leqq i \leqq n$. We shall prove that $\Lambda_i = \lambda_{k+i}$ for $i = 1, \ldots, n - k$.

First: From (5) we see that the points $\bar{\mathsf{u}}^i = [u^i_{k+1}, \ldots u^i_n]$ $(i = k + 1, \ldots, n)$ are independent, $\bar{\mathsf{u}}^i \in \lambda_i \mathfrak{B}(S')$; and so obviously $\Lambda_i \leqq \lambda_{k+i}$. Conversely, let $\mathsf{p}_2^{k+1}, \ldots, \mathsf{p}_2^n$ be independent lattice points, $\mathsf{p}_2^{k+i} \in \operatorname{Fr} \Lambda_i \mathfrak{B}(S')$. Taking a fixed value of i, put $\lambda_i' = \operatorname{Max}(\Lambda_i, \mu)$. We have $\mathsf{p}_2^{k+i} \in \lambda_i' \mathfrak{B}(S')$, so that there are two points $\mathsf{q}_2^{k+i} = [q^{k+i}_{k+1}, \ldots, q^{k+i}_n] \in \lambda_i' S'$, $\mathsf{s}_2^{k+i} = [s^{k+i}_{k+1}, \ldots, s^{k+i}_n] \in \lambda_i' S'$ with $\mathsf{p}_2^{k+i} = \mathsf{q}_2^{k+i} - \mathsf{s}_2^{k+i}$. We shall define $2k$ numbers q_j^{k+i}, s_j^{k+i} $(j = 1, \ldots, k)$ as follows: choose $[y_1, \ldots, y_k] \in \lambda_i' \mu^{-1} Q_1$ and put

$$q_j^{k+i} = \sum_{l=k+1}^{n} a_{jl} q_l^{k+i} + y_j \quad (j = 1, \ldots, k);$$

then put

$$s_j^{k+i} = \sum_{l=k+1}^{n} a_{jl} s_l^{k+i} + y_j' \quad (j = 1, \ldots, k)$$

where $[y', \ldots, y_k'] \in \lambda_i' \mu^{-1} Q_1$ is chosen so that $[s_1^{k+i}, \ldots, s_k^{k+i}] = [q_1^{k+i}, \ldots, q_k^{k+i}]$; this is possible following Remark 4, since $\lambda_i' \mu^{-1} \geqq 1$. Putting

$$\mathsf{q}^{k+i} = [q_1^{k+i}, \ldots, q_n^{k+i}],\ \mathsf{s}^{k+i} = [s_1^{k+i}, \ldots, s_n^{k+i}]$$

we see that $\mathsf{q}^{k+i} \in \lambda_i' S$, $\mathsf{s}^{k+i} \in \lambda_i' S$ and that $\mathsf{u}^1, \ldots, \mathsf{u}^k, \mathsf{q}^{k+1} - \mathsf{s}^{k+1}, \ldots, \mathsf{q}^n - \mathsf{s}^n$ are independent lattice points. Thus $\lambda_i' \geqq \nu > \mu$ and so $\lambda_i' = \Lambda_i$. Since $\mathsf{q}^{k+i} - \mathsf{s}^{k+i} \in \lambda_i' \mathfrak{B}(S) = \Lambda_i \mathfrak{B}(S)$, we see that $\lambda_{k+1} \leqq \Lambda_1$, $\lambda_{k+2} \leqq \Lambda_2$, $\ldots$, $\lambda_n \leqq \Lambda_{n-k}$. Thus

$$\Lambda_i = \lambda_{k+i} \text{ for } i = 1, \ldots, n - k. \tag{17}$$

Using (16), (17), we obtain

$$1 = \lambda_1 \ldots \lambda_n V(S) = \Lambda_1 \Lambda_2 \ldots \Lambda_{n-k} V(S'),$$

i. e. S' is a body of the class

$$\mathscr{E}(n-k_1; m-1; k_2, \ldots, k_m; \mu_2, \ldots, \mu_m);$$

in addition, S' is adapted, as it can be seen from the form of $\bar{u}^{k+1}, \ldots, \bar{u}^n$ (see (5)). Following our supposition, S' is given by the formulae of the form (13), but only for $i > l_1$, i. e. for $i > k$. Combining these formulae with (14) we see that S is given exactly by the complete set of formulae (13).

Proof of II. Let S be given by the formulae (13); we write (13) in the condensed form

$$x_i = \Sigma a_{ij} x_j + y_i \quad (i = 1, \ldots, n).$$

It is obvious that S is a convex body and that $\mu_1^{k_1} \ldots \mu_m^{k_m} V(S) = 1$. It remains to show that, putting $\tau_i = \tau_i(\mathfrak{B}(S))$, we have $\tau_i = \mu_j$, i. e.

$$\tau_i \leqq \mu_j \text{ for } l_{j-1} < i \leqq l_j. \tag{18}$$

$$\tau_i \geqq \mu_j \text{ for } l_{j-1} < i \leqq l_j. \tag{19}$$

First we shall prove: *If* $\bar{x} = [x_{l_{t-1}+1}, \ldots, x_{l_t}]$ *is a lattice point in* $\mathfrak{B}(Q_t)$ (there are k_t independent lattice points in $\mathfrak{B}(Q_t)$) *then there are integers* $x_1, x_2, \ldots, x_{l_{t-1}}$ *such that*

$$[x_1, \ldots, x_{l_t}, 0, \ldots, 0] \in \mu_t \mathfrak{B}(S).$$

(This obviously will prove the inequality (18).)

Proof. We have $\bar{x} = \bar{y}' - \bar{y}''$, where $\bar{y}^{(p)} \in Q_t$ $(p = 1, 2)$; let $y_i^{(p)}$ $(l_{t-1} < i \leqq l_t)$ be the coordinates of $\bar{y}^{(p)}$. We complete these sequences of k_t numbers to sequences of n numbers $y_1^{(p)}, \ldots, y_n^{(p)}$ $(p = 1, 2)$ in the following way:

(i) The numbers $y_i^{(p)}$ with $l_{j-1} < i \leqq l_j$ determine a point in $\mu_j^{-1}\mu_t Q_j$.

(ii) Putting

$$x_i' = \Sigma a_{ih} x_h' + y_i', \quad x_i'' = \Sigma a_{ih} x_h'' + y_i'' \quad (i = 1, \ldots, n), \tag{20}$$

we put $y_i'' = y_i'$ and so $x_i'' = x_i'$ for $i > l_t$ and choose for $i \leqq l_{t-1}$ the numbers y_i' arbitrarily and the numbers y_i'' so that $x_i' \equiv x_i''$ (mod 1); this is possible, since $\mu_j^{-1}\mu_t Q_j$ contains, for $j < t$, to every point of space a congruent point (see the analogous considerations in the proof of (17)).

It is then obvious that the point $[x_1' - x_1'', \ldots, x_n' - x_n'']$ has the required properties.

Next we shall prove:

If $[x_1, \ldots, x_n] = [x_1', \ldots, x_n'] - [x_1'', \ldots, x_n'']$ *is a lattice point,*

$[x_1^{(p)}, \ldots, x_n^{(p)}] \in \lambda S$ $(p = 1, 2)$, *where* $0 < \lambda < \mu_t$, *then* $x_i = 0$ *for* $i > l_{t-1}$. (This obviously will prove the inequalities (19).)

Proof. We have again the formulae (20), but now the numbers $y_i^{(p)}$ $(l_{j-1} < i \leq l_j)$ determine a point in $\lambda \mu_j^{-1} Q_j$. For $i > l_{m-1}$ we have $x_i' - x_i'' = y_i' - y_i'' \equiv 0 \pmod 1$ and so (since $\lambda \mu_m^{-1} < 1$) $y_i' - y_i'' = 0$, $x_i' - x_i'' = 0$. Thus we get for $l_{m-2} < i \leq l_{m-1}$:

$$x_i' - x_i'' = y_i' - y_i'' \equiv 0 \pmod 1$$

and so, if $\lambda \mu_{m-1}^{-1} < 1$, we have $y_i' - y_i'' = 0$, $x_i' - x_i'' = 0$ for $l_{m-1} \geqq i > l_{m-2}$. This consideration may obviously be put forth as long as $i > l_{t-1}$ and we get, for these values of i, the desired equation $x_i' - x_i'' = 0$.

Remark 5. We know that Q_j has a centre (Remark 4) the coordinates of which may be denoted by $\mu_j c_i$ $(l_{j-1} < i \leq l_j)$. It is obvious from (13) that S has the centre the coordinates of which are the numbers $d_1, \ldots, d_n$ given by the equations

$$d_i = \Sigma a_{ij} d_j + c_i \quad (i = 1, \ldots, n).$$

It is obvious (see the explicit form (13)) that $[d_1, \ldots, d_n] = \mathrm{o}$ is equivalent to $[c_1, \ldots, c_n] = \mathrm{o}$.

Remark 6. If a convex body S has the centre o, we have $\mathfrak{B}(S) = 2S$ and so

$$\tau_1(S) \ldots \tau_n(S)\, V(S) = 2^n \tau_1(\mathfrak{B}(S)) \ldots \tau_n(\mathfrak{B}(S))\, V(S).$$

Thus Theorem 1 gives Minkowski's inequality (1) and Theorem 2 (see Remark 5) gives also the characterization of the convex bodies with centre at o for which the sign of equality occurs in (1); it is only necessary to put $c_1 = \ldots = c_n = 0$ in Remark 5.

*

O Estermannově důkazu jedné věty Minkovského.

(Obsah předešlého článku.)

Estermann podal nedávno nový jednoduchý důkaz Minkowského nerovnosti (1), v níž S je konvexní těleso se středem v počátku, $\tau_i(S)$ jsou postupná minima tělesa S. V tomto článku ukazuji, jak lze Estermannovou methodou dokázati další větu Minkowského, jež charakterisuje ona tělesa S, pro něž v (1) platí znamení rovnosti.