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Miroslav Katětov

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A note on semiregular and nearly regular spaces.

By Miroslav Katětov, Praha.

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In the present note relations are analyzed between semiregular[1]) and nearly regular[2]) spaces. A sufficient condition is given for a hereditarily nearly regular space to be regular and examples are constructed showing that the implications: regular → hereditarily semiregular → hereditarily nearly regular cannot be reversed. All spaces considered are Hausdorff spaces.

Definitions. A point x of a space P is called *semiregular,* if for any neighborhood G of x there exists a H such that $a \in \operatorname{Int} \overline{H} \subset G$. If every $x \in P$ is semiregular, the space P is said to be *semiregular.* If every subspace $Q \subset P$ is semiregular, the space P is called *hereditarily semiregular.* A set $Q \subset P$ is said to be *regularly imbedded*[2]) in P if for any closed set $F \subset P$ and any $a \in P - F$ there exists a set $A \subset Q$ such that $F \subset \overline{A} \subset P - a$ (this definition is evidently equivalent with the formally different definition given by Čech and Novák, loc. cit.). If every dense subset $Q \subset P$ is regularly imbedded in P, the space P is called *nearly regular.* The space P is said to be *hereditarily nearly regular* if every subspace $Q \subset P$ is nearly regular.

A regular space is obviously semiregular; since regularity is hereditary, we obtain:

Any regular space is hereditarily semiregular.

Any semiregular space P is nearly regular.

Proof. Let Q be dense in P. If $F \subset P$ is closed, $a \in P - F$, there exists an open $G \subset P$ such that $a \in \operatorname{Int} \overline{G} \subset P - F$. Then $A = Q\overline{P - \overline{G}}$ is closed in Q, $a \in \operatorname{Int} \overline{G} = P - \overline{P - \overline{G}} \subset P - \overline{A}$, $F \subset \overline{P - \overline{G}} \subset \overline{A}$, hence Q is regularly imbedded in P.

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[1]) M. H. Stone, Applications of the Theory of Boolean Rings to General Topology, Trans. Amer. Math. Soc., 41 (1937).

[2]) E. Čech and J. Novák, On regular and combinatorial imbedding, Čas. mat. fys. 72 (1947).

This theorem implies:

Any hereditarily semiregular space is hereditarily nearly regular.

If P is semiregular and Q is dense in P, then Q is semiregular.

Proof. Let $G \subset Q$ be relatively open in Q, $x \in Q$. Let G_0 be open, $G = QG_0$. There exists an open set H_0 such that $x \in \operatorname{Int} \overline{H}_0 \subset$ $\subset G_0$. Setting $H = QH_0$ we have $\overline{H} = \overline{H}_0$, $\overline{Q - Q\overline{H}} = \overline{P - \overline{H}} =$ $= \overline{P - \overline{H}_0}$, $x \in H \subset Q - \overline{Q - Q\overline{H}} = Q \cdot \operatorname{Int} \overline{H}_0 \subset G$. Hence Q is semiregular.

Any Hausdorff space P may be imbedded in a semiregular space R.

Proof. Let R consist of the points x and (x, n) $(x \in P,\ n = 1, 2, \ldots)$. Let the points (x, n) be isolated and each point x_0 possess fundamental neighborhoods $U_{m,G}$ consisting of x and (x, n), $n > m$, $x \in G$, where $m = 1, 2, \ldots$ and G is a neighborhood of x_0. Clearly, R is a Hausdorff space and P is imbedded in R. Every $\overline{U}_{m,G} - U_{m,G}$ contains points $x \in P$ only, and we have $x = \lim (x, n)$, $(x, n) \in R - U_{m,G}$. Hence $\operatorname{Int} \overline{U}_{m,G} \subset U_{m,G}$; therefore R is semiregular.

Let P be hereditarily semiregular. Then every point $x \in P$ possessing a countable family $\{ G_n \}$ of fundamental neighborhoods is a regular point of P.

Proof. Suppose, on the contrary, that x is not regular. Then there exists an open set H such that $x \in H$ and $\overline{G}_n - H \neq 0$ $(n = 1, 2, \ldots)$. Let $a_n \in \overline{G}_n - H$ and denote by A the set of all a_n. Since A is evidently infinite, there exist disjoint open sets B_n such that $x \in P - \overline{B}_n$ and $B_n A \neq 0$ $(n = 1, 2, \ldots)$. Setting $Q = \Sigma B_n G_n$, $S = Q + A + x$ we have $\overline{Q} = S$, $x \in S - \overline{A}$ and, for any $C \subset Q$ such that $\overline{C} \subset A$, $CG_n \neq 0$ $(n = 1, 2, \ldots)$ (since otherwise $CG_n = 0$, $C \subset \sum_{k \neq h} B_k G_k \subset \sum_{k \neq n} B_k$, $CB_n = 0$, $\overline{C}B_n = 0$, $AB_n = 0$), hence $x \in \overline{C}$, which contradicts the regularity of the imbedding $Q \subset S$.

The preceding theorem implies:

A hereditarily nearly regular space satisfying the first countability axiom is regular.

Example 1. P_1 is the plane with an additional point ω. The points (x, y), x irrational, are isolated; the points (x, y), x rational, have their usual neighborhoods. The point ω possesses the fundamental neighborhoods $U_\varphi + \omega$, where U_φ consists of the points (x, y), x irrational, $| y | > \varphi (x)$, φ being an arbitrary real function. Clearly P_1 is a Hausdorff L-space, i. e. for any $M \subset P_1$ and $x \in \overline{M}$ there exist $x_n \in M$ $(n = 1, 2, \ldots)$ such that $x = \lim x_n$.

Consider a U_φ and denote by C_n $(n = 1, 2, \ldots)$ the set of all irrational x such that $\varphi(x) < n$. Then some C_n is of 2. category, hence dense in an interval J; hence for any rational $a \in J$ the points (a, y), $|y| > n$ lie in $\overline{U_\varphi}$. As the closure of the set R of all (x, y), x rational, does not contain ω, it follows that ω is no regular point of P.

Let $\omega \in Q \subset P$. It can be easily shown that there exists a countable set $B \subset Q - R - \omega$ such that (1) $R\overline{QR} \subset \overline{B}$, (2) for any real x the set of all y such that $(x, y) \in B$ is finite or void. Choose φ such that $|y| < \varphi(x)$ for every $(x, y) \in B$. Given a U_ψ, set $\varphi_1(x) = \max(\psi(x), \varphi(x))$, $G = QU_{\varphi_1} + \omega$. Then G is a relative neighborhood of ω in Q, $G \subset U_\psi$, $B\overline{G} = 0$, and, for any $(x, y) \in Q(\overline{G} - G)$, $(x, y) \in R\overline{B}$, hence (x, y) is no interior point (in Q) of $Q\overline{G}$. Hence ω is a semiregular point of Q. All other points being regular Q is semiregular; hence P is hereditarily semiregular.

Example 2.[3]) The space P_2 consists of the points $(\frac{1}{n}, x)$ $(n = 1, 2, \ldots, 0 \leq x \leq 1)$ of the plane (with the usual neighborhoods) and an additional point ω possessing the fundamental neighborhoods $U_m - A + \omega$, where U_m consists of all $(\frac{1}{n}, x) \in P$, $n > m$ $(m = 1, 2, \ldots)$ and A is countable. Clearly P_2 is a Hausdorff space and, for any $G = U_m - A + z$, $\operatorname{Int} \overline{G} = U_m + z$, hence P_2 is not semiregular.

To show that P_2 is hereditarily nearly regular we have to show, for any $Q \subset S \subset P$, $\overline{Q} \supset S$, $F \subset S$, F relatively closed in Q, $a \in S - F$, that a set $B \subset S$ exists such that $\overline{B} \supset F$, $a \in S - \overline{B}$. This is obvious for $a \neq \omega$, since a is regular. For $a = \omega$, we have only to choose a countable $B \subset P_2 - \omega$ such that $\overline{B} \supset F$ which is evidently possible.

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Poznámka o poloregulárních a skoro regulárních prostorech.

(Obsah předešlého článku.)

Hlavním výsledkem článku je věta:

Dědičně skoro regulární prostor, splňující první axiom spočetnosti, je regulární.

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[3]) This example is essentially due to J. Novák (Čech and Novák, l. c., example 3).