Josef Korous

On a generalization of Fourier series

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Josef Korous, Praha.

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1. Preliminary.

Before we proceed to formulate the problem which will be discussed, we need several definitions.

First of all, I introduce the following sets of real numbers:

1. $\{l_\nu\}$ denotes the set of real numbers l_ν $(\nu = 0, \pm 1, \pm 2, \ldots)$, satisfying the following conditions:

$$l_\nu < l_{\nu+1}, \quad l_{-1} < 0 \leqq l_0, \quad l_\nu = \nu + a + \lambda_\nu,$$

where a*) is a fixed real number and

$$\limsup_{\nu = \pm\infty} |\lambda_\nu| < \tfrac{1}{12}. \tag{1,1}$$

The aggregate of all possible $\{l_\nu\}$ is denoted by $A_1(a)$.

2. $A_2(a) \subset A_1(a)$ is the aggregate of all $\{l_\nu\}$ satisfying

$$\lambda_\nu = O(\log^{-1} |\nu|) \tag{1,2}$$

for $|\nu| > 1$ and

$$\sum_{\nu=-\infty}^{\infty}{}' \left|\frac{\lambda_\nu}{\nu}\right| < \infty.\text{**)} \tag{1,3}$$

3. $A_3(a) \subset A_2(a)$ contains all $\{l_\nu\}$ satisfying

$$\lambda_\nu = o(\log^{-1} |\nu|) \text{ for } \nu = \pm \infty. \tag{1,4}$$

4. To $A_4(a) \subset A_3(a)$ belong all $\{l_\nu\}$ with

$$\lambda_\nu = O(|\nu|^{-1} \log^{-1} |\nu|) \tag{1,5}$$

for $|\nu| > 1$ and

$$\sum_{\nu=-\infty}^{\infty} |\lambda_\nu| < \infty. \tag{1,6}$$

*) For our purposes only $a = 0$ and $a = \pm \frac{1}{4}$ will be needed.

**) The prime indicates throughout this paper that $\nu = 0$ should be excluded.

For brevity, we put $A_i(0) = A_i$, where $i = 1, 2, 3, 4$.

To every $\{l_\nu\}$ we associate an integral function of a complex variable

$$l(z) = (z - l_0) \prod_{\nu=-\infty}^{\infty}{}' \left(1 - \frac{z}{l_\nu}\right) e^{\frac{z}{\nu}}. \qquad (1,7)$$

We denote by $L_i(a)$ the class of all $l(z)$ belonging to $\{l_\nu\} \in A_i(a)$ and write simply L_i instead of $L_i(0)$.

Further, we put for $l(z) \in L_1$

$$k(z) = l(z) \operatorname{cotg} \pi z + \varrho(z), \qquad (1,8)$$

where

$$\varrho(z) = \frac{z}{\pi} \sum_{\nu=-\infty}^{\infty}{}' \frac{l(\nu)}{\nu(\nu - z)} - \frac{l(0)}{\pi z} + b,\text{*}) \qquad (1,9)$$

b being an arbitrary real constant.

If $l(z) \in L_2$, we define also

$$\varrho(z) = \frac{1}{\pi} \sum_{\nu=-\infty}^{\infty} \frac{l(\nu)}{\nu - z}, \qquad (1,10)$$

and denote by P the class of all such $\varrho(z)$.

2. The problem.

Let α be a real number and $f(x)$ a real function defined in $[\alpha, \alpha + \pi]$**) and such that $|f(x)|$ is integrable (in the sense of Lebesgue) over this interval. (2,1)

Our object is to investigate expansions of such functions in series

$$\lim_{n=\infty} \sum_{\nu=-n}^{n} (a_\nu \cos l_\nu x + b_\nu \sin l_\nu x)\text{†}) \qquad (2,2)$$

for $x \in [\alpha, \alpha + \pi]$ and $\{l_\nu\} \in A_1$.

*) The convergence of this series and of the series (1,10) will be made evident in the proof of lemma 4.

**) $[a, b]$ is a closed, (a, b) an open interval.

†) Mr. Walsh occupied himself in a paper entitled „A generalization of the Fourier cosine series" (Am. M. S. Transactions 22) with a similar series. His series is (with our notations) $\sum_{\nu=0}^{\infty} a_\nu \cos l_\nu x$ for $x \in [0, \pi]$ under the following simultaneous assumptions;

$$\sum_{n=1}^{\infty} n^2 \lambda_n^2 < \infty \text{ and } l_0^2 + 4 \sum_{n=1}^{\infty} \lambda_n^2 < \frac{1}{\pi}.$$

With the aid of the theory of functions of an infinite number of variables Mr. Walsh proves the equiconvergence of his series with that of Fourier. (A cosine series is obtained from (2,2), if e. g. $\alpha = -\frac{1}{2}\pi$ and $f(t)$ is an even function.)

The coefficients a_ν and b_ν are given by the following formulas

$$a_\nu = \tfrac{1}{2}k(l_\nu)\{l'(l_\nu)\}^{-1}\int_{\alpha}^{\alpha+\pi} f(t)\cos l_\nu t\,dt,$$

$$b_\nu = \tfrac{1}{2}k(l_\nu)\{l'(l_\nu)\}^{-1}\int_{\alpha}^{\alpha+\pi} f(t)\sin l_\nu t\,dt, \tag{2,3}$$

where $l(z)$ belongs to the same $\{l_\nu\}$.

It is seen at once that the Fourier series of the function $f(t)$ in $[\alpha, \alpha+\pi]$ is a special case of (2,2). This Fourier series is

$$\lim_{n=\infty}\sum_{\nu=-n}^{n}(a'_\nu \cos \nu x + b'_\nu \sin \nu x), \tag{2,4}$$

where

$$a'_\nu = \frac{1}{2\pi}\int_{\alpha}^{\alpha+\pi} f(t)\cos \nu t\,dt,$$

$$b'_\nu = \frac{1}{2\pi}\int_{\alpha}^{\alpha+\pi} f(t)\sin \nu t\,dt, \tag{2,5}$$

and tends to zero in $(\alpha-\pi, \alpha)$.

Further, we observe that the coefficients (2,3) are not determined uniquely, for their formulas contain an arbitrary constant b.*)

Put

$$s_n(x; f) = \sum_{\nu=-n}^{n}(a_\nu \cos l_\nu x + b_\nu \sin l_\nu x)$$

with coefficients (2,3) and

$$S_n(x; f) = \sum_{\nu=-n}^{n}(a'_\nu \cos \nu x + b'_\nu \sin \nu x)$$

with coefficients (2,5), n being a positive integer.

Putting

$$k_n(x, t) = \tfrac{1}{2}\sum_{\nu=-n}^{n} k(l_\nu)\,\{l'(l_\nu)\}^{-1}\cos[l_\nu(x-t)]$$

and

$$K_n(x, t) = \frac{1}{2\pi}\sum_{\nu=-n}^{n}\cos[\nu(x-t)],$$

it is obvious that

$$s_n(x; f) = \int_{\alpha}^{\alpha+\pi} f(t)\,k_n(x, t)\,dt$$

*) Consequently the coefficients (2,5) are not unique. Replacing them by $(1+(-1)^\nu b)\,a'_\nu$ and $(1+(-1)^\nu b)\,b'_\nu$ respectively, we obtain a series equiconvergent with (2,4) in $(\alpha, \alpha+\pi)$, but not in $(\alpha-\pi, \alpha)$ as we may easily convince ourselves by methods used in the theory of Fourier series.

and

$$S_n(x; f) = \int_{\alpha}^{\alpha+\pi} f(t)\, K_n(x, t)\, dt.$$

If (R) denotes the circumference of the circle $|z| = R$, it follows by the theorem of residues for almost all values of n*)

$$k_n(x, t) = \frac{1}{4\pi i} \int_{(n+\frac{1}{2})} k(z)\, l^{-1}(z) \cos[(x-t)z]\, dz.$$

Putting

$$\varrho_n(x, t) = \frac{1}{4\pi i} \int_{(n+\frac{1}{2})} \varrho(z)\, l^{-1}(z) \cos[(x-t)z]\, dz \tag{2,6}$$

and replacing $k(z)$ by (1,8), we obtain

$$k_n(x, t) = \frac{1}{4\pi i} \int_{(n+\frac{1}{2})} \operatorname{cotg} \pi z \cos[(x-t)z]\, dz + \varrho_n(x, t) = K_n(x, t) + \varrho_n(x, t),$$

so that

$$s_n(x; f) - S_n(x; f) = \int_{\alpha}^{\alpha+\pi} f(t)\, \varrho_n(x, t)\, dt. \tag{2,7}$$

We observe that if the last integral tends to zero for $n \to \infty$, the series (2,2) and (2,4) are equiconvergent so that the question of the convergence of (2,2) reduces to that of the corresponding Fourier series. In the next chapter we give some sufficient conditions for this equiconvergence, while in chapter 4 an analogue of the Riemann-Lebesgue theorem on coefficients (2,3) is established.

3. Theorem on convergence.

In this chapter I establish the following theorem:

(A) *When $f(x)$ is of bounded variation in $[\alpha, \alpha+\pi]$, then for $x \in (\alpha, \alpha+\pi)$*

$$s_n(x; f) - S_n(x; f) \to 0 \text{ for } n \to \infty \tag{3,1}$$

uniformly in $(\alpha + \eta, \alpha + \pi - \eta)$, where η is an arbitrary fixed real number in $(0, 1)$.

The sum of (2,2) is therefore $\frac{1}{2}[f(x-0) + f(x+0)]$, as results from the theory of Fourier series.

(B) *When $\{l_\nu\} \in A_2$, then (3,1) holds for any function (2,1) uniformly in $(\alpha + \eta, \alpha + \pi - \eta)$.*

(C) *When $\{l_\nu\} \in A_2$ and $\varrho(z) \in P$, (3,1) is true also for $x = \alpha$*

*) In consequence of (1,1) it is obvious that no zero of $l(z)$ coincides with $n + \frac{1}{2}$, if n is large enough.

provided that $f(t)$ is of bounded variation in $(\alpha + \pi - \eta, \alpha + \pi)$, and for $x = \alpha + \pi$, if $f(t)$ possesses the same property in $(\alpha, \alpha + \eta)$.)*

(D) *When $\{l_\nu\} \in A_4$ and $\varrho(z) \in P$, (3,1) is satisfied uniformly in $[\alpha, \alpha + \pi]$.*

Before proceeding to the proof of this theorem, I introduce some notations which will be employed in what follows (also in the next chapter) and prove some preliminary lemmas.

Notations.

1. β is any fixed number in $(6 \limsup_{\nu=\pm\infty} |\lambda_\nu|, \frac{1}{2})$.

2. η is any number in $(0, 1)$.

3. $\varphi \in [-\pi, \pi]$ is the argument of a complex variable $z = re^{i\varphi}$ where $r > 0$.

4. c_i $(i = 1, 2, \ldots)$ are positive constants independent of r as well as of $\varphi \in [-\pi, \pi]$. They may depend on β. The numbering is independent in every lemma.

5. n denotes in this chapter a positive integer, in the next chapter any integer.

6. k_i are positive constants independent of n and of q (see Lemma 5 e. s.).

7. $M(a) = (-\infty, \infty) - \sum_{\nu=-\infty}^{\infty} [(\nu + a - \delta, \nu + a + \delta) + (\nu - a - \delta, \nu - a + \delta) + (l_\nu - \delta, l_\nu + \delta) + (-l_\nu - \delta, -l_\nu + \delta)]$, where δ is an arbitrary constant in $(0, \frac{1}{24})$. $M = M(0)$.

Lemma 1. *Suppose $l(z) \in L_1$ and $r \in (1, \infty) M$. If*

$$\lambda(z) = l(z) \operatorname{cosec} \pi z,$$

then

$$c_1 r^{-\beta} < |\lambda(re^{i\varphi})| < c_2 r^{\beta}. \tag{3,5}$$

Proof. Since by (1,1)

$$|\lambda_\nu| < \lambda < \tfrac{1}{12} \tag{3,6}$$

for almost all values of ν, we have

$$\left| \frac{\lambda_\nu}{r - |\nu|} \right| < 1 \tag{3,7}$$

for all values of $r \in (r_0, \infty) M$, where $r_0 > 0$ is properly chosen, and all values of ν with one possible exception.

*) These conditions concerning $f(t)$ are sufficient, their necessity is not asserted.

Accordingly, for the same values of r and ν

$$\left|\log\left|1-\frac{\lambda_\nu}{re^{i\varphi}-\nu}\right|\right|<\left|\frac{\lambda_\nu}{r-|\nu|}\right|+c_3\frac{1}{(r-|\nu|)^2}. \qquad (3,8)$$

(3,7) may be false for a $|\nu_0| \in (r-\frac{1}{12}, r+\frac{1}{12})$. But in this case

$$\left|\log\left|1-\frac{\lambda_{\nu_0}}{re^{i\varphi}-\nu_0}\right|\right|<|\log|r-\nu_0||+|\log|r-l_{\nu_0}||<2\log\frac{1}{\delta}. \qquad (3,9)$$

Further, putting

$$\varrho_1(z)=\prod_{|\nu|\geqq r^2}\left(1-\frac{z}{l_\nu}\right)e^{\frac{z}{\nu}},$$

we deduce easily

$$|\log|\varrho_1(re^{i\varphi})||<c_4 r\sum_{|\nu|\geqq r^2}\frac{|\lambda_\nu|}{|l_\nu\nu|}+c_5 r^2\sum_{|\nu|\geqq r^2}\frac{1}{l_\nu^2}<c_6, \qquad (3,10)$$

and by a similar argument

$$|\log|\varrho_2(re^{i\varphi})||<c_7, \qquad (3,11)$$

where

$$\varrho_2(z)=\prod_{|\nu|\geqq r^2}\left(1-\frac{z}{\nu}\right)e^{\frac{z}{\nu}}.$$

We may now write

$$\lambda(z)=\frac{1}{\pi}\varrho_1(z)\varrho_2^{-1}(z)\prod_{|\nu|<r^2}{}'\frac{\nu}{l_\nu}\cdot\prod_{|\nu|<r^2}\left(1-\frac{\lambda_\nu}{z-\nu}\right). \qquad (3,12)$$

Taking the real parts of logarithms of both sides and utilising (3,6)—(3,11), it follows

$$|\log|\lambda(re^{i\varphi})||<c_8+\sum_{|\nu|<r^2}{}'\log\left(1+\left|\frac{\lambda_\nu}{\nu}\right|\right)+$$

$$+\sum_{\nu\in N_1}\frac{|\lambda_\nu|}{\sqrt{\nu^2+r^2}}+\sum_{\nu\in N_2}\frac{|\lambda_\nu|}{r-|\nu|}+\sum_{\nu\in N_3}\left|\frac{\lambda_\nu}{r-|\nu|}\right|.$$

In this formula $N_1=(-r^2,-1)$ when $\varphi\in[-\frac{1}{2}\pi,\frac{1}{2}\pi]$ and $N_1=$ $=(1,r^2)$ for the remaining values of φ. Similarly $N_2=(0,\frac{1}{2}r)$ or $(-\frac{1}{2}r,0)$ and $N_3=(-r^2,r^2)-N_1-N_2$.

Hence we deduce by employing (3,6)

$$|\log|\lambda(re^{i\varphi})||<c_9+\lambda\log r^2+\lambda(\log r^2-\log r)+$$
$$+c_{10}+\lambda[\log\tfrac{1}{2}r+\log(r^2-r)]<c_{11}+6\lambda\log r,$$

or

$$|\log|\lambda(re^{i\varphi})||<c_{12}+\beta\log r,$$

which is equivalent to (3,5).

Lemma 2. *If* $g(z) \in L_2(a)$ *and*

$$\gamma(z) = g(z) \operatorname{cosec}[\pi(z - a)],$$

then for $r \in M(a)$ *and* $|a| < 1$

$$c_1 < |\gamma(re^{i\varphi})| < c_2. \tag{3,13}$$

Proof. Let $g_\nu = \nu + a + \gamma_\nu$ be the zeros of $g(z)$.

Using the well-known formula

$$\sin \pi (z - a) = \frac{\sin \pi a}{a} (z - a) \prod_{\nu=-\infty}^{\infty}{}' \left(1 - \frac{z}{\nu + a}\right) e^{\frac{z}{\nu}},$$

$\frac{\sin \pi a}{a}$ being replaced by π when $a = 0$, and proceeding on the lines of the previous proof, we obtain

$$\frac{\sin \pi a}{a} \gamma(z) = \prod_{\nu=-\infty}^{\infty}{}' \frac{\nu + a}{g_\nu} \prod_{\nu=-\infty}^{\infty} \left(1 - \frac{\gamma_\nu}{z - \nu - a}\right). \tag{3,14}$$

The convergence of both products is evident from (1,3).

From (1,2) it can be easily seen that taking $r \in M(a)$ large enough, we have

$$\left|\frac{\gamma_\nu}{r - |\nu + a|}\right| < \tfrac{1}{2},$$

and consequently

$$\left|\log\left|1 - \frac{\gamma_\nu}{re^{i\varphi} - \nu - a}\right|\right| < 2\left|\frac{\gamma_\nu}{r - |\nu + a|}\right|$$

for all values of ν.

Accordingly,

$$|\log|\gamma(re^{i\varphi})|| < c_3 + 2\left[\sum_{|\nu| \leqq \sqrt{r}} + \sum_{\sqrt{r} < |\nu| < 2r} + \sum_{|\nu| \geqq 2r} \left|\frac{\gamma_\nu}{r - |\nu + a|}\right|\right] <$$

$$< c_3 + c_4 \sqrt{r}\, r^{\beta - 1} + c_5 \max_{\sqrt{r} < |\nu| < 2r} |\gamma_\nu| \log r + c_6 \sum_{|\nu| \geqq 2r} \left|\frac{\gamma_\nu}{\nu}\right|,$$

and the result follows from (1,2) and (1,3).

Lemma 3. *When* u *is real and* $|u| > 1$, *we have*

$$l(u) = O(|u|^\beta), \tag{3,15}$$

$$g(u) = O(1). \tag{3,16}$$

If m *is an integer,*

$$|g(m + a)| \leqq k\, |\gamma_m|, \tag{3,17}$$

k being a positive constant independent of m.

Proof. (3,15) holds by (3,5) for the above $u \in M$.

Putting

$$\lambda_m(u) = \frac{u - m}{u - l_m} \lambda(u)$$

and omitting in (3,12) the term $1 - \frac{\lambda_m}{z - m}$, we deduce by repeating with this modification the analysis in the proof of Lemma 1

$$\lambda_m(u) = O(|u|^\beta)$$

also for $u \in M + (m - \delta, m + 1 - \delta)$.

Hence for these values of u

$$l(u) = (u - l_m) \lambda_m(u) \frac{\sin \pi u}{u - m} = O(1) O(|u|^\beta) = O(|u|^\beta), \quad (3,18)$$

and the left-hand side being independent of m, (3,18) holds for all real values of u.

The proof of (3,16) follows on the same lines by modifying the proof of Lemma 2. We have

$$g(u) = (u - g_m) \gamma_m(u) \frac{\sin [\pi(u - a)]}{u - m - a},$$

where

$$\gamma_m(u) = O(1) \text{ for } u \in (m + a - \delta, m + a + \delta).$$

Hence

$$|g(m + a)| = |\pi \gamma_m \gamma_m(m + a)|,$$

and (3,17) follows immediately.

Lemma 4. *Suppose* $r \in (1, \infty) M$. *Then*

I. $|\varrho(re^{i\varphi})| < c_1 r^\beta$. (3,19)

II. *When* $\{l_\nu\} \in A_2$, *then*

$$|\varrho(re^{i\varphi})| < c_2. \quad (3,20)$$

III. *When* $\{l_\nu\} \in A_3$ *and* $\varrho(z) \in P$, *then*

$$\varrho(re^{i\varphi}) = o(1) \text{ for } r \to \infty \quad (3,21)$$

uniformly in $[-\pi, \pi]$.

IV. *If* $\{l_\nu\} \in A_4$ *and* $\varrho(z) \in P$, *then*

$$|\varrho(re^{i\varphi})| < c_3 r^{-1}. \quad (3,22)$$

Proof. I. Putting $z = re^{i\varphi}$, we have by (3,15) for $r \in M$ and $\nu \neq 0$

$$\left| \frac{z\, l(\nu)}{(\nu - z)\, \nu} \right| < \frac{c_4 r\, |\nu|^\beta}{|\nu(r - |\nu|)|}, \quad (3,23)$$

whence the convergence of (1,9) follows immediately.*)

*) The constants c are in this proof independent also of ν.

The last expression being $O(|\nu|^{\beta-1})$, $O(r^{\beta}\,|r-|\nu||^{-1})$ and $O(r\,|\nu|^{\beta-2})$ for $|\nu| \leq \frac{1}{2}r$, $\frac{1}{2}r < |\nu| < 2r$ and $|\nu| \geq 2r$ respectively, we obtain on carrying out the summation of (3,23) with respect to $\nu \in (-\infty, \infty)$

$$|\varrho(re^{i\varphi})| < c_5 r^{\beta} \log r < c_6 r^{\beta'},$$

where $\beta < \beta' < \frac{1}{2}$.

II. Suppose $\varrho(z) \in P$.*) Then by (3,17) for $r \in M$

$$\left|\frac{l(\nu)}{\nu - z}\right| < c_5 \frac{|\lambda_\nu|}{|r - |\nu||}.$$

The convergence of (1,10) is now evident by (1,3).

Further, we have

$$|\varrho(re^{i\varphi})| < c_7 \left[\sum_{|\nu| \leq \sqrt{r}} + \sum_{\sqrt{r} < |\nu| < \frac{r}{2}} + \sum_{\frac{r}{2} \leq |\nu| \leq 2r} + \sum_{|\nu| > 2r} \left|\frac{\lambda_\nu}{r - |\nu|}\right|\right]. \tag{3,24}$$

The first sum gives $O(\sqrt{r}r^{-1})$, the second and the third ones are $O(\max_{\sqrt{r} < |\nu| \leq 2r} |\lambda_\nu| \log r)$, while the last one is $o(1)$ for $r \to \infty$ by (1,3).

Hence it can be easily seen that (3,21) and also (3,20) in the case when $\varrho(z) \in P$ are a consequence of (1,4) and (1,2) respectively. (3,20) is established also for the case when $\varrho(z)$ does not belong to P, for such a $\varrho(z)$ differs from a $\varrho(z) \in P$ only by a constant provided that $\{l_\nu\} \in A_2$.

When $\{l_\nu\} \in A_4$, the first two sums and the last one in (3,24) are less than $c_7 r^{-1} \sum_{\nu=-\infty}^{\infty} |\lambda_\nu|$, while the third sum is by (1,5) $O(r^{-1} \log^{-1} r \log r) = O(r^{-1})$, and (3,22) is proved.

Lemma 5. *Let $q \in [-\pi, \pi]$ be a constant independent of n and $l = \max_{|z|=n+\frac{1}{2}} |\lambda^{-1}(z)|$. Then for almost all values of n*

$$I_n(q) = \int_{(n+\frac{1}{2})} |e^{iqz} l^{-1}(z)\, dz| < k_1 nl, \tag{3,25}$$

and if $|q| < \pi$,

$$I_n(q) < \frac{k_2 l}{\pi - |q|}. \tag{3,26}$$

Proof. Putting $R = n + \frac{1}{2}$ and taking n large enough,**) we obtain for $z = Re^{i\varphi}$

$$|l^{-1}(z)| < l\,|\operatorname{cosec} \pi z| = 2le_1\,[1 + e_1^2 - 2e_1 \cos(2\pi R \cos\varphi)]^{-\frac{1}{2}} < k_3 le_1,$$

where $e_1 = e^{-\pi R |\sin\varphi|}$.

*) Consequently $\{l_\nu\} \in A_2$.

**) Notice the footnote on page 11.

Since $\sin\varphi > 2\pi^{-1}\varphi$ for $\varphi \in (0, \frac{1}{2}\pi)$, we have

$$l^{-1}I_n(q) < k_3R\int_0^{2\pi} \exp(-qR\sin\varphi - \pi R|\sin\varphi|)\,d\varphi <$$

$$< k_4R\int_0^{\frac{1}{2}\pi} \exp[2R\pi^{-1}(|q| - \pi)\varphi]\,d\varphi \leqq$$

$$\leqq k_4R\int_0^{\frac{1}{2}\pi} d\varphi < k_5 n. \tag{3,27}$$

When $|q| < \pi$, (3,27) yields

$$l^{-1}I_n(q) < k_4R\int_0^{\frac{1}{2}\pi} \exp[2R\pi^{-1}(|q| - \pi)\varphi]\,d\varphi =$$

$$= \frac{k_4\pi R}{2R(\pi - |q|)} < \frac{k_6}{\pi - |q|},$$

which completes the proof.

Lemma 6.

$$\text{(i)}\quad \varrho_n(x, t) = O(1) \text{ for } n \to \infty \tag{3,28}$$

uniformly with respect to $x \in (\alpha + \eta, \alpha + \pi - \eta)$ *as well as to* $t \in [\alpha, \alpha + \pi]$.

(ii) *If, moreover,* $\{l_\nu\} \in A_4$ *and* $\varrho(z) \in P$ *then*

$$\varrho_n(x, t) = O(1) \tag{3,29}$$

uniformly for $x \in [\alpha, \alpha + \pi]$ *and* $t \in [\alpha, \alpha + \pi]$.

Proof. Put $|x - t| = q$ and $\varrho = \max\limits_{|z|=n+\frac{1}{2}} |\varrho(z)|$. Using the notations of the previous lemma, it is immediately seen from (2,6) that

$$|\varrho_n(x, t)| < \varrho[I_n(q) + I_n(-q)].$$

Observing that in virtue of (1,2) $n + \frac{1}{2} \in M$ for almost all values of n, the last expression is less than $\frac{k_1}{\eta}$ in the case (i) by (3,20), (3,26) and (3,13), and less than $k_2n^{-1}n = k_2$ in the case (ii) by (3,22), (3,25) and (3,13).

Lemma 7. *Let* t_1 *and* t_2 *be any two numbers in* $[\alpha, \alpha + \pi]$. *Then*

$$\text{(a)}\quad \int_{t_1}^{t_2} \varrho_n(x, t)\,dt = o(1) \text{ for } n \to \infty \tag{3,30}$$

uniformly for $x \in [\alpha + \eta, \alpha + \pi - \eta]$.

(b) *When* $\varrho(z) \in P$ *and* $\{l_\nu\} \in A_3$, *(3,30) holds uniformly in* $[\alpha, \alpha + \pi]$. (3,31)

Proof. The primitive function of $\varrho_n(x, t)$ being

$$J_n(x, t) = \frac{1}{4\pi i} \int_{(n+\frac{1}{2})} z^{-1} \varrho(z) \, l^{-1}(z) \sin [(t-x) z] \, dz,$$

we deduce easily, retaining the notations of the preceding proof and supposing that $t \in [\alpha, \alpha + \pi]$

$$|J_n(x, t)| < n^{-1}\varrho[I_n(q) + I_n(-q)].$$

Now, the results are obtained by employing (3,19), (3,26) and (3,5) for the case (a) and by employing (3,21), (3,25) and (3,13) for the case (b).

Proof of the theorem.

Suppose that $f(t)$ is a non-decreasing finite function in $[\alpha, \alpha + \pi]$. By applying the second mean-value theorem of the integral calculus to (2,7), we see that (2,7) tends to zero uniformly in $(\alpha + \eta, \alpha + \pi - \eta)$ in virtue of (3,30). A function of bounded variation being a difference of two such functions, (A) is established.

In order to prove (B) and (D), we observe that both statements are true by (A) and (3,31) respectively when $f(x)$ is a polynomial $P(x)$ in $[\alpha, \alpha + \pi]$, for a polynomial is of bounded variation. Now, $f(t)$ being any function defined by (2,1), we have by (3,28) and (3,29) respectively

$$\left| \int_{\alpha}^{\alpha+\pi} \varrho_n(x, t) \, [f(t) - P(t)] \, dt \right| < k_1 \int_{\alpha}^{\alpha+\pi} |f(t) - P(t)| \, dt,$$

where k_1 is independent of $x \in (\alpha + \eta, \alpha + \pi - \eta)$ and of $x \in [\alpha, \alpha + \pi]$ respectively. The last integral can be made arbitrarily small by a suitable choice of $P(t)$. Combining these results we see that (B) and (D) are established.

C is proved by combining the methods of the proofs of A and of B.

4. Theorem on the coefficients.

In this chapter the following theorem will be proved:

The coefficients a_n and b_n given by (2,3) tend to zero for $n = \pm \infty$, provided that $\{l_n\} \in A_2$ and that $f(t)$ is a function defined by (2,1).

Since by the Riemann-Lebesgue theorem $\int_{\alpha}^{\alpha+\pi} f(t) \, e^{il_n t} \, dt \to 0$ for $n \to \pm \infty$, it is sufficient to prove

$$k(l_n) \, \{l'(l_n)\}^{-1} = O(1).$$

Suppose $|n|$ so large that $|\lambda_n| < \frac{1}{2}$.

When $l_n \neq n$, we have

$$k(l_n) = \varrho(l_n) = O(1) - \frac{l(n)}{\pi \lambda_n} + \frac{1}{\pi} \sum_{\substack{\nu=-\infty \\ \nu \neq n}}^{\infty} \frac{l(\nu)}{\nu - l_n}.$$

Arguing as in the proof of Lemma 4 we see that the last sum is $O(1)$ for $n = \pm\infty$. Further, it follows by (3,17)

$$\frac{l(n)}{\lambda_n} = O(1).$$

Accordingly, $k(l_n) = O(1)$.

When $l_n = n$, we have

$$k(l_n) = \frac{1}{\pi} l'(n) + \frac{1}{\pi} \sum_{\substack{\nu=-\infty \\ \nu \neq n}}^{\infty} \frac{l(\nu)}{\nu - n} + b.$$

whence we see that

$$|k(l_n)| < |l'(n)| + k_1.$$

Accordingly, our problem reduces to the proof of the inequalities

$$k_2 < |l'(l_n)| < k_3. \tag{4,1}$$

Observing that by (1,2) $l_n \epsilon M(-\frac{1}{2})$ for almost all values of n, (4,1) follows immediately from (4,10) and (3,13).

In order to prove (4,10), i. e. Lemma 10, two more lemmas are necessary.

Lemma 8. *If $l(z) \epsilon L_2$ and $I = (\vartheta_1 r, \vartheta_2 r)$ where $\vartheta_1 < 1$ and $\vartheta_2 > 1$ are constants independent of r and φ, then for $z = re^{i\varphi}$ and $r \epsilon (2, \infty) M$*

$$|\lambda'(z)| < c_1 \sum_{|\nu| \epsilon I} \frac{|\lambda_\nu|}{(r - |\nu|)^2} + c_2 r^{-1}. \tag{4,2}$$

Hence

$$|\lambda'(z)| < c_3 \log^{-1} r. \tag{4,3}$$

Proof. Taking logarithms of (3,14) for $a = 0$ and differentiating, we obtain

$$\frac{\lambda'(z)}{\lambda(z)} = \sum_{\nu=-\infty}^{\infty} \frac{\lambda_\nu}{(z - \nu)(z - l_\nu)}.$$

Using (3,13), (1,2) and (1,3), it is easily seen that for all sufficiently large values of $r \epsilon M$

$$|\lambda'(re^{i\varphi})| < c_4 \left[\sum_{|\nu| \leqq \vartheta_1 r} + \sum_{|\nu| \epsilon I} + \sum_{|\nu| \geqq \vartheta_2 r} \frac{|\lambda_\nu|}{(r - |\nu|)^2} \right] <$$

$$< c_5 r r^{-2} + c_6 \sum_{|\nu| \epsilon I} \frac{|\lambda_\nu|}{(r - |\nu|)^2} + c_7 r^{-1} < c_8 r^{-1} + c_9 \log^{-1} r$$

$$< c_{10} \log^{-1} r.$$

Lemma 9. *The zeros l'_ν of the function $l'(z)$ where $l(z) \epsilon L_2$ are all real and either*

$$\{l'_\nu\} \epsilon A(\tfrac{1}{2}) \text{ or } \{l'_\nu\} \epsilon A(-\tfrac{1}{2}).$$

Proof. The order of $l(z)$ is 1, for $l(z)$ is a canonical product and the exponent of convergence of its zeros is equal to 1. Therefore, by a well-known theorem of Laguerre l'_ν are all real and are separated from each other by the zeros of $l(z)$.

In order that $l'_0 \geq 0$ and $l'_{-1} < 0$ we have to fix the numbering so that either

$$l_{\nu-1} < l'_\nu < l_\nu$$

or

$$l_\nu < l'_\nu < l_{\nu+1}.$$

Put $l_\nu^* = l'_\nu$ in the first case and $l_\nu^* = l'_{\nu-1}$ in the second case.

Denote by I_n the open interval, the end-points of which are $n - \frac{1}{2}$ and l_n^*. It is obvious that for $u \in I_n$

$$|l(n - \tfrac{1}{2})| < |l(u)|; \tag{4,6}$$

furthermore

$$|l(n - \tfrac{1}{2})| \leq |l(l_n^*)|. \tag{4,7}$$

From (1,2) we see that, given any fixed $\eta_0 \in (0, 1)$, we can choose δ and a positive $N(\eta_0)$ so that

$$I'_n(\eta_0) = [n - 1 + \eta_0,\ n - \eta_0] \subset M$$

for all values of $|n| > N(\eta_0)$.

Put $\lambda_n^* = l_n^* - n + \frac{1}{2}$. We can easily show that there is a fixed $\eta_0 \in (0, 1)$ such that

$$|\lambda_n^*| \leq \tfrac{1}{2} - \eta_0 \tag{4,8}$$

for all values of $|n| > N(\eta_0)$. For, if (4,8) were false we could choose $u_0 \in I_n$ so that

$$|u_0 - n + \tfrac{1}{2}| = \tfrac{1}{2} - \eta_0,$$

and since also $u_0 \in I'_n(\eta_0) \subset M$, (3,13) and (4,6) would yield

$$k_1 < \left|\frac{\lambda(n - \frac{1}{2})}{\lambda(u_0)}\right| = \left|\frac{l(n - \frac{1}{2})}{l(u_0)}\right| \frac{\sin \pi u_0}{\sin [\pi (n - \frac{1}{2})]} < \sin \pi\eta_0,$$

which is impossible if η_0 is small enough.

Since by (4,8) $\cos \pi\lambda_n^* > 0$ and $l'_n \in M$ for almost all values of n, we deduce from (4,7) in a similar manner

$$\left|\frac{\lambda(n - \frac{1}{2})}{\lambda(l_n^*)}\right| \leq \cos \pi\lambda_n^*$$

for all sufficiently large values of $|n|$, whence

$$\cos \pi\lambda_n^* \geq 1 - \left|\frac{\lambda(n - \frac{1}{2}) - \lambda(l_n^*)}{\lambda(l_n^*)}\right|,$$

or by (3,13)

$$\sin^2 \frac{\pi \lambda_n^*}{2} < k_2 \, |\lambda_n^* \, \lambda'(l''_n)|,$$

where $l''_n \in I_n$.

Hence by (4,3)

$$|\lambda_n^*| < k_3 \, |\lambda'(l''_n)| = O(\log^{-1} |n|). \tag{4,9}$$

Hereby the first required property of $\{l'_\nu\}$ is proved.

In order to prove also

$$\sum_{\nu=-\infty}^{\infty}{}' \left|\frac{\lambda_\nu^*}{\nu}\right| < \infty,$$

we employ (4,2), (4,9) and (1,3). Observing that $l''_s \in M$ for almost all values of s, we have for $N \to \infty$

$$\sum_{|s|>N} \left|\frac{\lambda_s^*}{s}\right| = O\left(\sum_{|s|>N} \left|\frac{\lambda'(l''_s)}{s}\right|\right) =$$

$$= O(\sum_{|s|>N} |s|^{-2}) + O[\sum_{|s|>N} |s|^{-1} \sum_{\frac{1}{2}|s|<|\nu|<2|s|} |\lambda_\nu| (|l''_s| - |\nu|)^{-2}] =$$

$$= o(1) + O[\sum_{|\nu|>\frac{1}{2}N} |\lambda_\nu| \sum_{|s|>\frac{1}{2}|\nu|} |s|^{-1} (|l''_s| - |\nu|)^{-2}] =$$

$$= o(1) + O\left[\sum_{|\nu|>\frac{1}{2}N} |\lambda_\nu| \, |\nu|^{-1} \sum_{s=-\infty}^{\infty} \left(\frac{1}{s+\frac{1}{2}}\right)^2\right] = o(1),$$

whence the result.

Lemma 10. *Putting*

$h = -\dfrac{l'(0)}{l_0}$ *when* $l_0 \neq 0$, *and* $h = l''(0)$ *when* $l_0 = 0$, *we have*

$$h^{-1} \, l'(z) \in L_2(\pm \tfrac{1}{2}), \tag{4,10}$$

provided that $l(z) \in L_2$.

Proof. $l'(z)$ being an integral function of order 1, it can be written in the form

$$l'(z) = he^{\gamma z}(z - l'_0) \prod_{\nu=-\infty}^{\infty}{}' \left(1 - \frac{z}{l'_\nu}\right) e^{\frac{z}{\nu}} = he^{\gamma z} t(z) = he^{\gamma z} \, \tau(z) \cos \pi z.$$

γ is a real number, for $l'(z)$ is real for real values of z.

The zeros l'_ν of the function $t(z)$ belonging to $A_2(\pm \frac{1}{2})$, $t(z) \in L_2(\pm \frac{1}{2})$ and Lemma 2 yields for all real values of $u \in M(-\frac{1}{2})$

$$|\tau(u)| > k_1^{*})$$

and consequently also

$$|t(u)| > k_2.$$

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*) k_i are here positive constants independent of u under consideration.

On the other hand, we obtain from (3,13) and (4,3)

$$|l'(u)| = |\pi\lambda(u) \cos \pi u + \lambda'(u) \sin \pi u| < k_3$$

for $u \in M$.

Were now $\gamma > 0$, it would follow for $u \in (1, \infty)\, MM(-\frac{1}{2})$

$$k_3 > |l'(u)| = |h|\, e^{\gamma u}\, |t(u)| > k_2 h e^{\gamma u} \to \infty$$

for $u \to \infty$, which is impossible. The impossibility of $\gamma < 0$ is shown similarly. Accordingly, $\gamma = 0$ and (4,10) is proved.

*

O jistém zobecnění Fourierových řad.

(Obsah předešlého článku.)

Funkci reálné proměnné $f(x)$ s variací konečnou lze pro $x \in (\alpha, \alpha + \pi)$,*) kde α je libovolné číslo reálné, rozvinouti v řadu

$$\sum_{\nu=-\infty}^{\infty}(a_\nu \cos l_\nu x + b_\nu \sin l_\nu x) = \tfrac{1}{2}\,[f(x+0) + f(x-0)],$$

při čemž l_ν jsou čísla hovící podmínkám

$$l_\nu < l_{\nu+1},\ l_{-1} < 0 \leqq l_0,\ l_\nu = \nu + \lambda_\nu,$$
$$\limsup_{\nu=\pm\infty} |\lambda_\nu| < \tfrac{1}{12}$$

a koeficienty a_ν a b_ν jsou dány vzorci (2,3) (viz (1,7)—(1,10)).

Předpokládáme-li ještě (1,2) a (1,3), je naše řada pro jakoukoli funkci v intervalu $(\alpha, \alpha + \pi)$ integrace schopnou**) v tomto intervalu ekvikonvergentní s řadou Fourierovou s koeficienty (2,5). Za tohoto předpokladu platí také

$$a_\nu \to 0,\ b_\nu \to 0 \text{ pro } \nu = \pm\infty.$$

Za dalších předpokladů pro l_ν platí výše zmíněná ekvikonvergence též pro $x = \alpha$ a $x = \alpha + \pi$.

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*) Interval otevřený.

**) Ve smyslu Lebesgueově; i co do absolutní hodnoty.