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ON NEARLY DISCRETE SPACES

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Nearly discrete and nearly (R) discrete spaces investigated in this note are, roughly speaking, Hausdorff (respectively, regular) spaces the topology of which cannot be modified without either some open set becoming non-open or some non-isolated point becoming isolated. It is immediately seen that discrete (i. e. such that every subset is open) spaces as well as spaces investigated by E. Hewitt [1]*) and by the present author [2] and called maximal (maximal completely regular) in [1], minimal (R-minimal) in [2] are nearly [respectively, nearly (R)] discrete. In this note some properties of nearly discrete and nearly (R) discrete spaces are examined and a close relation is established between nearly (R) discrete spaces and the Čech (bi)compactification of discrete spaces (for instance, every countable nearly (R) discrete space may be imbedded into the Čech compactification of natural numbers).

We begin with some preliminary definitions and lemmas. All spaces considered are Hausdorff topological spaces. The letters P, S always denote spaces. Mapping means a continuous transformation; function means a real-valued function.

Definitions. A space P is called *dense-in-itself* if it contains no isolated point, *dispersed* if it contains no non-void dense-in-itself subspace. A set $Q \subset P$ is called *relatively dense-in-itself* (in P), abbreviated r. d., if it contains no isolated (in Q) points except those isolated already in P, *relatively almost dense-in-itself*, abbreviated almost r. d., if the set of all $x \in Q$ which are isolated in Q without being isolated in P is countable (= finite or countable infinite). A one-to-one mapping φ of a space P_1 into P_2 is called an *ι-mapping* if $\varphi(x)$ is an isolated point in P_2 whenever x is isolated in P_1. If φ is an ι-mapping of P_1 onto P_2, then P_2 is said to be an *ι-image* of P_1.

Lemma 1. *In any space P, the sum of an arbitrary collection of r. d. sets is r. d., every open set is r. d., the intersection of an open set and a r. d.*

*) Numbers in brackets refer to the list at the end of the paper.

set is r. d. A set $Q \subset P$ is r. d. if and only if $\overline{Q}$ is r. d. If $Q \subset P$ is r. d. in P, $M \subset Q$ is r. d. in P, then M is r. d. in P.

Proof is easy and may be omitted.

The following lemma is obvious.

Lemma 2. *Let φ be an ι-mapping of P_1 onto P_2. If $Q \subset P_1$ is r. d., then $\varphi(Q) \subset P_2$ is r. d.*

Lemma 3. *If $A \subset P$ is r. d., $B \subset A$, and no non-void $M \subset B$ is r. d. in P, then $\overline{A - B} \supset A$, $A - B$ is r. d. in P.*

Proof. Denote by C the set of all isolated (in B) points $x \in B$. It is easy to see that $B - \overline{C}$ is r. d., hence void. If $x \in C$, then x is not isolated in A(for otherwise x would be isolated in P, $(x) \subset B$ would be r. d.), and therefore $x \in \overline{A - B}$. Thus $C \subset \overline{A - B}$ whence $B \subset \overline{C} \subset$ $\subset \overline{A - B}$, $A \supset \overline{A - B}$ which proves (cf. Lemma 1) the lemma.

Lemma 4. *Let $A_i \subset P$, $\Sigma_1^n A_i = P$. Then there exist r. d. sets $B_i \subset A_i$ such that $\Sigma_1^n \overline{B_i} = P$.*

Proof. Let B_i be the sum of all r. d. sets $B \subset A_i$; by Lemma 1, B_i is r. d. Suppose $G = P - \Sigma_1^n \overline{B_i} \neq \emptyset$. Then G is open, hence r. d., $G = \Sigma_1^n (GA_i - B_i)$, $GA_i - B_i$ contains no non-void r. d. set. This is easily seen to contradict Lemma 3.

Definitions. A space P is called *nearly discrete* if every ι-mapping of a Hausdorff space onto P is a homeomorphism. A semiregular[1]) [regular, completely regular] space P is called *nearly* (*SR*) [*nearly* (*R*), *nearly* (*SR*)] *discrete* if every ι-mapping of a semiregular [regular, completely regular] space onto P is a homeomorphism.

Theorem 1. *Each of the following three conditions is necessary and sufficient in order that a Hausdorff space P be nearly discrete*: (a) *every relatively dense-in-itself set $Q \subset P$ is open*; (b) *for any bounded function f in P, $\lim\limits_{z \to x} f(z)$ exists, for any non-isolated $x \in P$*; (c) *if $M \subset P$, $x \in \overline{M} - M$, then $M + (x)$ is a neighborhood of x.*

Remark. This theorem is equivalent, for a dense-in-itself P, to Theorem 2 and 3 of [2]. The proof does not differ from the proof given in [2] and is given here for the sake of completeness only.

Proof. I. Let P be nearly discrete and let $Q \subset P$ be r. d. Let P_1 be obtained from P by declaring every set $GQ + H$, G, H open in P, for open in P_1. Clearly, P_1 is a Hausdorff space. It is easy to see (cf. Lemma 1) that the identical mapping of P_1 onto P is an ι-mapping, hence a homeomorphism, and therefore Q is open in P. II. Let (a) hold and let $M \subset P$,

[1]) A space P is called semiregular if, for any open $G \subset P$ and $x \in G$, there exist an open H such that $x \in H \subset \mathrm{Int}\overline{H} \subset G$.

$x \in \overline{M} - M$. By Lemma 4, there exist r. d. sets $A \subset M$, $B \subset P - M$ such that $\overline{A} + \overline{B} = P$. Clearly, $x \in \overline{A}$ (for otherwise we would have $x \in \overline{B}$, $B + (x)$ r. d., hence open, and therefore $x \operatorname{non} \in \overline{M}$), $A + (x)$ is r. d. and therefore open, $M + (x) \supset A + (x)$ is a neighborhood of x. III. Let (c) hold and let φ be an ι-mapping of a space P_1 onto P. Let $M \subset P$, $x \in \overline{M} - M$. Then $M + (x)$ is a neighborhood of x in P and therefore $\varphi^{-1}(M) + \varphi^{-1}(x)$ is a neighborhood of $\varphi^{-1}(x)$ in P. Since $\varphi^{-1}(x)$ is not isolated (φ being an ι-mapping) we obtain $\varphi^{-1}(x) \in \overline{\varphi^{-1}(M)}$. Thus φ is a homeomorphism which proves that P is nearly discrete. IV. Let (c) hold and let f be a bounded function in P. Let $x \in P$ be non-isolated and denote $\liminf\limits_{z\to x} f(z)$ by α. Clearly, for any $\varepsilon > 0$, $x \in \overline{M}$, M being the set of all $z \in P, z \neq x$, such that $|f(z) - \alpha| < \varepsilon$. Therefore $(x) + M$ is a neighborhood of x which implies $\alpha = \lim\limits_{z\to x} f(x)$. V. If (b) holds and $M \subset P, a \in \overline{M} - M$, then putting $f(x) = 1$ if $x \in M$, $f(x) = 0$ if $x \in P - M$ we obtain $\lim\limits_{x\to a} f(x) = 1$ which implies $|f(x) - 1| < \frac{1}{2}$, hence $f(x) = 1$, $x \in M$, for every $x \neq a$ in an appropriate neighborhood G of the point a. Thus (b) implies (c) which completes the proof.

Theorem 2. *The following properties of a semiregular space P are equivalent*: (a) *P is nearly (SR) discrete*; (b) *P is nearly (R) discrete*; (c) *P is nearly (CR) discrete*; (d) *if $Q_1 + Q_2 = P$, $Q_1Q_2 = \emptyset$, Q_1 and Q_2 are relatively dense-in-itself, then Q_1 and Q_2 are open*; (e) *if $Q_1 \subset P$, $Q_2 \subset P$ are relatively dense-in-itself, then Q_1Q_2 is so as well.*

Proof. I. Let (a) [or (b), or (c)] hold. Let $Q_1 \subset P$, $Q_2 = P - Q_1$ be r. d. sets. Let P_1 be defined by declaring every set $Q_1G_1 + Q_2G_2$, G_1, G_2 open in P, for an open subset of P_1. Then P_1 is semiregular [respectively, regular or completely regular] and it is easy to see that the identical mapping of P_1 onto P is an ι-mapping, hence a homeomorphism. Consequently, Q_1 and Q_2 are open in P (being so in P_1). Thus (a), as well as (b) or (c), implies (d). Since, for any open G, Int G and its complement are r. d. (cf. Lemma 1), (d) implies, for a semiregular P, complete regularity of P. Therefore, (a) implies (b)and (c).

II. Suppose that (d) holds and P is semiregular. Let φ be an ι-mapping of a semiregular P_1 onto P and let $G_1 \subset P_1$ be open. Choose $x \in G_1$. Since P_1 is semiregular, there exists a regularly open[2]) set H_1 such that $x \in H_1 \subset G_1$. By Lemma 1, H_1 and $P_1 - H_1$ are r. d. in P_1 and therefore, by Lemma 2, $\varphi(H_1)$ and $\varphi(P_1 - H_1)$ are r. d. in P, hence open. This proves that φ is a homeomorphism. Hence P is nearly (SR) discrete, (d) implies (a). III. From I and II we obtain the equivalence of the conditions (a), (b), (c), (d). We are now going to prove the equivalence

[2]) A set $U \subset P$ is said to be regularly open (in P) if $U = \operatorname{Int}\overline{U}$.

of (d) and (e). Let (d) hold and let Q_i be r. d. Suppose that $a \in Q = Q_1Q_2$ is isolated in Q. There exists an open $G \subset P$ such that $GQ = (a)$. Let $S_1 = Q_1G$, $S_2 = Q_2G - (a)$, $T_1 = S_1 + (\overline{S_1} - S_2)$, $T_2 = S_2 + (\overline{S_2} - T_1)$. It is easy to see that $T_1T_2 = \emptyset$, $T_1 + T_2 = \overline{S_1} + \overline{S_2}$, $S_i \subset T_i \subset \overline{S_i}$. Lemma 1 implies that T_1, T_2, $P - T_1 - T_2$ are r. d., hence, by (d), open. From this we obtain a non $\in \overline{S_2}$; hence a is isolated in Q_2G and therefore in P (for Q_2G is r. d.). Consequently, $Q = Q_1Q_2$ is r. d., (d) implies (e). — Now let (e) hold and let Q_i be r. d., $Q_1 + Q_2 = P$, $Q_1Q_2 = \emptyset$. Choose $x \in \overline{Q_1}$ and put $S_1 = Q_1 + (x)$. Then S_1 is r. d. and therefore Q_2S_1 is so as well. We have either $Q_2S_1 = \emptyset$ or $Q_2S_1 = (x)$. Suppose $x \in Q_2$; then $(x) = Q_2S_1$ is r. d., x is isolated which contradicts $x \in \overline{Q_1} - Q_1$. Therefore $x \in Q_1$, Q_1 is closed which proves the implication (e) $\Rightarrow$ (d).

Remark. Theorem 2 asserts that nearly (SR), nearly (R) and nearly (CR) discrete spaces coincide. Henceforth we shall speak, therefore, of nearly (R) discrete spaces only.

Theorem 3. *Every Hausdorff space is an ι-image of a nearly discrete space.*

Theorem 4. *Every regular space is an ι-image of a nearly (R) discrete space.*

Proof of Theorems 3 and 4 is obtained by a slight modification of corresponding proofs in [1] and [2] (Theorem 1 and 9) and may be omitted here.

Remarks. 1. It is possible to give examples showing that it is not sufficient to suppose, in Theorem 4, semiregularity instead of regularity 2. I do not know whether there exist regular dense-in-itself nearly discrete spaces.

Theorem 5. *Every subspace of a nearly discrete space is nearly discrete.*

Proof. Let P be nearly discrete, $Q \subset P$. Let $M \subset Q$ be r. d. in Q. Denote $M + (P - \overline{Q})$ by S. If $x \in S$ is isolated in S, then either $x \in P - \overline{Q}$ and evidently x is isolated in P, or $x \in M$, x is isolated in Q, hence in $\overline{Q}$, hence in $P = \overline{Q} + (P - \overline{Q})$. Thus $M + (P - \overline{Q})$ is r. d. in P, hence, by Theorem 1, open, and therefore $M = SQ$ is open in Q. This proves, by Theorem 1, that Q is nearly discrete.

Definition. Let $P \subset S$, let S be H-closed[3]) and let P be open in S, $\overline{P} = S$. Let $G + (x)$ be open in S whenever $G \subset P$ is open and $x \in \overline{G} - P$. Then S (which is essentially uniquely determined by the above properties) is said to be a (H-closed) *τ-extension* of P and is denoted by τP (cf. [3] where a formally different definition is given).

[3]) A Hausdorff space P is called H-closed if it is closed in any Hausdorff space $S \supset P$.

It is known (e. g. [3], 3,1) that there exists, for any Hausdorff space P, a τP.

Theorem 6. *Let S be a Hausdorff space, $P \subset S$, $\overline{P} = S$ and let P be nearly discrete. Then S is nearly discrete if and only if $S \subset \tau P$.*

Remark. $S \subset \tau P$ means, of course, that there exists a space $T \supset S$, denoted by τP, which is a H-closed τ-extension of P.

Proof. „If": by Theorem 5, we have only to show that τP is nearly discrete. Let $M \subset \tau P$ be r. d. Then MP is r. d., hence open; clearly, for any $x \in M - P$, we have $x \in \overline{MP}$ (for otherwise x would be isolated in M), $(x) + MP$ is open, and therefore M is open. Hence, by Theorem 1, τP is nearly discrete. — „Only if": let $T = \tau S$. It is easy to show that T is a τ-extension of P.

Corollary. *A dispersed space P is nearly discrete if and only if $P \subset \tau J$ where $J \subset P$ consists of all isolated $x \in P$.*

This follows from the well known fact that, in a dispersed space, the set of isolated points is dense.

Lemma 5. *Let P be nearly or nearly (R) discrete. Then every $M \subset P$ may be represented (uniquely) as a disjoint sum $M = M_1 + M_2$, M_i being open in M, M_1 dense-in-itself, M_2 dispersed.*

Proof. Let the sum of all dense-in-itself subsets of M be denoted by M_1, and put $M_2 = M - M_1$. Clearly, $\overline{M_1}M \subset M_1$, $\overline{M_1}$ and $P - \overline{M_1}$ are r. d., hence, by Theorem 2 or 1, open in P. Therefore, $M_1 = \overline{M_1}M$, $M_2 = (P - \overline{M_1})M$ are open in P. The uniqueness is obvious.

Definition. Let P be completely regular. A compact space $R \supset P$ such that $\overline{P} = R$ and every bounded continuous function in P may be extended to a continuous function in R will be denoted by βP and called a *β-compactification* of P.

It is well known (cf. e. g. [4]) that every completely regular space possesses a β-compactification which is essentially unique.

The following lemma is almost obvious.

Lemma 6. *Let S be completely regular, $P \subset S$, $\overline{P} = S$. Then $S \subset \beta P$ if and only if every bounded continuous function in P may be extended to a continuous function in S.*

Lemma 7. *If P is nearly (R) discrete, $Q \subset P$, $\overline{Q} = P$, then $P \subset \beta Q$.*

Remark. This is, essentially, Theorem 13 of [2].

Proof. Let f be a bounded continuous function in Q. Put $F(x) = f(x)$, for $x \in Q$, $F(x) = \limsup_{z \to x} f(z)$, for $x \in P - Q$. Choose $a \in P$, $\varepsilon > 0$ and denote by A the set of all $x \in Q$ such that $|f(x) - F(a)| < \varepsilon$. Clearly, A is open in Q, hence, by Lemma 1, r. d. in P. By Lemma 1 and Theorem

2, $\overline{A}$ and $P - \overline{A}$ are open. Evidently, $a \in \overline{A}$ and, for any $x \in \overline{A}$, $|F(x) - F(a)| \leq \varepsilon$. Thus F is continuous. Now apply Lemma 6.

Theorem 7. *Let $P \subset S$, $\overline{P} = S$ and let P be nearly (R) discrete, S completely regular. Then S is nearly (R) discrete if and only if $S \subset \beta P$ and $S - P$ is a dispersed space.*

Proof. I. „If": let Q_1, Q_2 be disjoint r. d. subsets of S, $Q_1 + Q_2$; we have to show (see Theorem 2) that Q_1, Q_2 are open. The sets $Q_k - P$ contain no isolated point of S, are dispersed and therefore contain no r. d. (in S) set. Therefore, by Lemma 3, $\overline{PQ_k} \supset Q_k$, the sets PQ_k are r. d. in S, hence in P, and consequently, by Theorem 2, are open in P. Putting $f(x) = k$, for $x \in PQ_k$, we obtain a continuous function f in P. Let f be extended (cf. Lemma 6) to a continuous function F in S. Then $Q_k \subset \subset \overline{PQ_k}$ implies $F(x) = k$, for $x \in Q_k$. Hence Q_k are open. II. „Only if": suppose S nearly (R) discrete; then, by Lemma 7, $S \subset \beta P$ and no non-void $M \subset S - P$ is dense-in-itself, for otherwise M and $S - M$ would be r. d., hence, by Theorem 2, open which is impossible.

Definition. Subsets A, B of a space P are said to be *separated* (in P) if $\overline{A}B + A\overline{B} = 0$.

Lemma 8. *If countable sets A, B are separated in a regular space P, then there exist disjoint open sets $G \supset A$, $H \supset B$.*

This lemma is well known (Urysohn [5], S. 265).

Lemma 9. *If P is nearly (R) discrete, then, for any separated countable A, B, there exist disjoint open $G \supset A$, $H \supset B$ such that $G + H = P$.*

Proof. By Lemma 8, there exist open $G_1 \supset A$, $H_1 \supset B$, $G_1 H_1 = \emptyset$. Put $G = \overline{G_1}$, $H = P - \overline{G_1}$ and apply Lemma 1 and Theorem 2.

Lemma 10. *Let f be a bounded continuous function in a space P such that $f(P)$ contains no interval. Then f may be represented as the sum of a uniformly convergent series $\Sigma_1^\infty g_n$ of continuous functions each assuming two values at most and such that $\Sigma_1^\infty \max|g_n(x)| < \infty$.*

Proof. Let $\alpha = \inf f(P)$, $\beta = \sup f(P)$, choose (if $\alpha \neq \beta$) a real number γ non $\in f(P)$ such that $2\alpha + \beta < 3\gamma < \alpha + 2\beta$, and put $g_1(x) = \frac{1}{2}(\alpha + \gamma)$ if $f(x) < \gamma$, $g_1(x) = \frac{1}{2}(\gamma + \beta)$ if $f(x) > \gamma$, and $f_1(x) = f(x) - g_1(x)$, for any $x \in P$. It is easy to see that starting from f_1 to obtain g_2 and f_2 and proceding indefinitely in this way we obtain a series $\Sigma_1^\infty g_n(x)$ possessing the properties required in the lemma.

Lemma 9 and 10 imply:

Lemma 11. *Let P be nearly (R) discrete and let $M \subset P$ be countable. Then every bounded continuous function in M may be extended to a continuous function in P.*

Theorem 8. *Let P be nearly (R) discrete and let Q be relatively (in P) almost dense-in-itself. Then* (1) *Q is nearly (R) discrete,* (2) *every bounded continuous function in Q may be extended to a continuous function in P.*

Proof. Let H denote the sum of all r. d. (in P) subsets of Q, and let A denote the set of all $x \in P$ which are isolated in Q without being isolated in P; then A is countable, $AH = \emptyset$. $H = \overline{H}Q$ is closed and open in Q, for $\overline{H}$ and $P - \overline{H}$ are r. d., hence open, in P; clearly, $Q - H \subset \overline{A}$. By Lemma 11 and 6, we obtain $A \subset Q - H \subset \beta A$ from which, by Theorem 7, follows that $Q - H$ is nearly (R) discrete (for $Q - H$ is dispersed). Now, if $M_1 \subset H$, $M_2 \subset H$ are r. d. in H, then, by Lemma 1, they are both r. d. in P and therefore, by Theorem 2, M_1M_2 is r. d. in P, hence in M, as well. Thus, by Theorem 2, H is nearly (R) discrete and therefore $Q = H + (Q - H)$ is nearly (R) discrete (for H and $Q - H$ are open and closed in Q). — If f is a bounded continuous function in Q, then f may be continuously extended from A over $P - \overline{H}$, by Lemma 11, and from H over $\overline{H}$, by Lemma 7 (for $\overline{H}$ is clearly nearly (R) discrete). From this the assertion of the theorem follows at once.

Corollary. *Every countably compact*[4]) *subspace of a nearly (R) discrete space is finite.*

Proof. Suppose that $K \subset P$ is countably compact infinite, P being nearly (R) discrete. There exists a countable infinite discrete $A \subset K$. $B = \overline{A}K$ is countably compact and, by Theorem 8, nearly (R) discrete. Therefore, by Theorem 7, $B - A$ is dispersed. Choose a point $x \in B - A$ which is isolated in $B - A$ and choose an open G such that $\overline{G}B - A$ contains a single point x. Clearly, $\overline{G}B$ is countably compact, infinite, consists of isolated points except the point x, and is, by Theorem 8, nearly (R) discrete. This is a contradiction.

We are now going to give some examples.

Example 1. (showing that there exist non-normal nearly (R) discrete spaces and that the property of being nearly (R) discrete is not hereditary). Let I be a countable infinite discrete space. It is known [6] that $\beta I - I$ contains a discrete subspace X of power $\exp\aleph_0$. Denote $I + X$ by P_1. By Theorem 7, P_1 is nearly (R) discrete. Evidently, there exist $\exp\aleph_0$ continuous functions in I, hence in P_1. Since there exist $\exp\exp\aleph_0$ continuous functions in X, it is clear that P_1 is not normal (for otherwise every continuous function in X would admit of a continuous extension over P_1). Choose a point $x \in \overline{X} - X \subset \beta I$ such that $X + (a) \subset \beta X$ does not hold (it is easy to see that such points exist for otherwise we would have $\beta X \subset \beta I$). Denote $P_1 + (x)$ by P_2. Then, by Theorem 7, P_2 is nearly (R) discrete but $X + (x) \subset P_2$ is not so.

[4]) We call a space countably compact if every countable open covering contains a finite subcovering, compact if any open covering contains a finite subcovering.

Example 2. Let P be a countable infinite dense-in-itself nearly (R) discrete space (such a space exists, by Theorem 4). Choose an infinite discrete $A \subset P$ and a point $x \in \beta P$, $x \in \bar{A}$, and put $S = P + (a)$. By Theorem 7, S is nearly (R) discrete; $S - A + (a)$ is dense but is not open. Therefore, by Theorem 1, S is not nearly discrete. By Theorem 3, S is an ι-image of a nearly discrete space S_1. Clearly, S_1 is not regular.

Definition. A space P is called *hypernormal* (cf. [1], Definition 17) if, for any two separated sets $A \subset P$, $B \subset P$, there exist open sets $G \supset A$, $H \supset B$ such that $\bar{G}\bar{H} = \emptyset$.

Hypernormality is clearly hereditary.

Theorem 9. *A space P is hypernormal if and only if every bounded continuous function f in a subspace $M \subset P$ may be extended to a continuous function in P.*

Proof. I. „If": let $A_k \subset P$ $(k = 1, 2)$ be separated. Putting $f(x) = k$, for $x \in A_k$, we obtain a continuous function f in $A_1 + A_2$. Let f be extended to a continuous function F in P and denote by G or, respectively, by H the set of $x \in P$ such that $F(x) < \frac{4}{3}$ or, respectively, $F(x) > \frac{5}{3}$. Clearly, G, H are open, $G \supset A$, $H \supset B$, $\bar{G}\bar{H} = \emptyset$. II. „Only if": let f be a bounded continuous function in $M \subset P$. Denote $\sup_{x \in M}|f(x)|$ by α; denote by A or, respectively, by B the set of $x \in M$ such that $f(x) \leqq -\frac{1}{3}\alpha$ or, respectively, $f(x) \geqq \frac{1}{3}\alpha$, and choose (which is possible, for A, B are separated) open $G \supset A$, $H \supset B$ such that $\bar{G}\bar{H} = \emptyset$. Since G, $P - \bar{G}$ are clearly separated, there exist open $G_1 \supset G$, $H_1 \supset P - \bar{G}$ such that $\bar{G}_1\bar{H}_1 = \emptyset$; evidently, $G_1 \supset A$, $H_1 \supset B$, $\bar{G}_1 + \bar{H}_1 = P$. Putting $F_1(x) = -\frac{1}{3}\alpha$, for $x \in \bar{G}_1$, $F_1(x) = \frac{1}{3}\alpha$, for $x \in \bar{H}_1$, we obtain a continuous function F_1 in P. Clearly, $|F_1(x)| \leqq \frac{1}{3}\alpha$, for any $x \in P$, and $|F_1(x) - f(x)| \leqq \frac{2}{3}\alpha$, for any $x \in M$. It is now easy to find, by induction, continuous functions F_n $(n = 1, 2, \ldots)$ in P such that $|F_n(x)| \leqq 2^{n-1}3^{-n}\alpha$, for any $x \in P$, $|\Sigma_1^n F_k(x) - f(x)| \leqq 2^n 3^{-n}\alpha$, for any $x \in M$. We have now only to put $F(x) = \Sigma_1^\infty F_n(x)$, for any $x \in P$, to prove the theorem.

Remark. A hypernormal compact (= bicompact) space P is finite. For otherwise let $I \subset P$ be infinite discrete. Then, by Theorem 9, $\bar{I} = \beta I$. But βI is not hereditarily normal (cf. Example 1), hence cannot be hypernormal. — It is possible to show more, viz. that an infinite countably compact space cannot be hypernormal.

Theorem 10. *If a space P satisfies one of the following equivalent conditions* (1) *P is nearly (R) discrete and completely normal,* (2) *P is normal and hereditarily nearly (R) discrete, then P is hypernormal.*

Proof. I. If (2) holds, let f be a bounded continuous function in $M \subset P$. Since $\bar{M}$ is nearly (R) discrete, it is possible, by Lemma 7, to extend f to a continuous function in $\bar{M}$, hence (P being normal) in P.

This proves, by Theorem 9, that P is hypernormal and (1) holds. II. If (1) holds, let again f be a bounded continuous function in $M \subset P$. Since P is completely, hence hereditarily normal, f may be extended to a continuous function in $Q = P - (\overline{M} - M)$, hence, by Lemma 7, in $P = \overline{Q}$. Therefore, by Theorem 9, P is hypernormal. We have now only to prove that every $Q \subset P$ is nearly (R) discrete. By Lemma 5, there exist separated Q_i such that $Q_1 + Q_2 = Q$, Q_1 is dense-in-itself, Q_2 is dispersed. Evidently, it suffices to prove that Q_i are nearly (R) discrete. Now, Q_1 is so by Theorem 8, and hypernormality of P implies, by Theorem 9, that $Q_2 \subset \beta I$ (where I denotes the set of all isolated points of Q_2) and therefore, by Theorem 7, Q_2 is nearly (R) discrete.

Remarks. 1. A nearly (R) discrete space need not, of course, be hypernormal (see Example 1). 2. There exist countable hypernormal spaces which are not nearly (R) discrete — see Example 3 below.

Lemma 12. *Let P be regular and let the set of all non-isolated $x \in P$ be countable. Then P is completely normal.*

Proof. Let $A \subset P$, $B \subset P$ be separated. Denote by I the set of all isolated $x \in P$. There exist, by Lemma 8, disjoint open sets $G \supset A - I$, $H \supset B - I$. Let $G_1 = G - \overline{B} + AI$, $H_1 = H - \overline{A} + BI$. Then G_1, H_1 are open, $G_1 \supset A$, $H_1 \supset B$, $G_1 H_1 = \emptyset$.

Lemma 13. *Let I be a discrete space. If $P \subset \beta I$ and $P - I$ is countable, then P is hypernormal.*

Proof. If f is a bounded continuous function in $M \subset P$, then, by Lemma 12, f may be extended to a continuous function in $I + M$ and, therefore, to a continuous function in $P \subset \beta(I + M) = \beta I$. Hence, by Theorem 9, P is hypernormal.

Definition. The least power of a dense subset of P is called the *density character* of P.

Theorem 11. *Any nearly (R) discrete space P may be imbedded into βI, I being a discrete space of power equal to the density character of P.*

Proof. I. Suppose that P is dense-in-itself. Let $Q \subset P$, $\overline{Q} = P$. Let φ be a one-to-one transformation of Q onto a set I, $IP = \emptyset$. Let $S = P + I$ and let the family consisting of all $\overline{H} + \varphi(H) - K$, $H \subset Q$ being r. d. in Q or (which is the same) in P, K finite, and of all (x), $x \in I$, be an open base of the space S. It is easy to see that S is completely regular, $\overline{I} = S$, and P (with its original topology) is imbedded in S. If $x \in P$, $A \subset I$, $x \in \overline{A}$, then $H\varphi^{-1}(A) \neq \emptyset$, for any r. d. $H \subset Q$ such that $x \in \overline{H}$. Denote the sum of all r. d. sets $M \subset \varphi^{-1}(A)$ by B. Then $x \in \overline{B}$, for otherwise, for some open $G \subset P$, $a \in G$, $G\varphi^{-1}(A)$ would contain no non-void r. d. set and therefore, by Lemma 3, $U = G - \varphi^{-1}(A)$ would be r. d., $x \in \overline{U}$, $U\varphi^{-1}(A) = \emptyset$ which is a contradiction. Hence $\overline{B} + A \supset \overline{B} + \varphi(B)$

is a neighborhood of the point x, x non $\in \overline{I - A}$. Thus $\overline{A} \cdot \overline{I - A} = \emptyset$, for any $A \subset I$. From this, it is easy to deduce (cf. [4], p. 833) that $P \subset \subset \beta P = \beta I$. II. For an arbitrary nearly (R) discrete space P, the assertion of the theorem follows from I and Lemmas 5 (with $M = P$) and 7.

Theorem 12. *Let I denote a countable infinite discrete space and let P be a countable space. If P is nearly (R) discrete, then P may be imbedded into βI; if $P \subset \beta I$, then P is hypernormal.*

This follows from Theorem 11 and Lemma 13.

Example 3. Let P be countable dense-in-itself nearly (R) discrete (cf. Theorem 4). By Theorem 12, $P \subset \beta I$, I being countable discrete. Let $A \subset P$ be an infinite discrete subspace. Clearly (cf. Lemma 13) $\overline{A} = \beta A \subset \beta I$. Choose a countable dense-in-itself $B \subset \overline{A} - A$, $BP = \emptyset$ and put $S = P + B$. Then $S \subset \beta I$, S is hypernormal but is not nearly (R) discrete (for P and B are r. d. in S without being open).

Problem. I do not know (not even for countable spaces) whether it is always possible to imbed a hypernormal space into βI, for an appropriate discrete I.

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O skoro isolovaných prostorech

(Obsah předešlého článku).

Nazýváme AHF-prostor *skoro isolovaným*, když jeho topologii nelze zeslabiti, aniž by se některý neisolovaný bod stal isolovaným (anebo prostor přestal vyhovovat axiomům A, H, F); regulární prostor nazýváme *skoro* (R) *isolovaným*, když jeho topologii nelze zeslabit, aniž by se některý neisolovaný bod stal isolovaným anebo prostor přestal být regulárním. V článku jsou studovány vlastnosti těchto prostorů. Mimo jiné, v článku je dokázána věta: *každý skoro* (R) *isolovaný prostor P lze vnořit do prostoru βI, kde βI značí Čechův bikompaktní obal isolovaného prostoru I, jehož mohutnost se rovná nejmenší mohutnosti husté části prostoru P.*