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DYNAMICAL MAGNIFICATION OF A SEISMOGRAPH EXCITED BY A SHOCK OF THE FORM $\lambda^n e^{-\lambda\tau} \tau^n$

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The purpose of the present paper is to transform the curves of dynamical magnification and those of phase lag given in a previous paper [1] to a form directly utilisable in the analysis of seismographic records.

The following symbols will be used:

y = displacement of the indicator free from solid friction;
α = reduced damping constant of the seismograph, $\beta = \sqrt{1 - \alpha^2}$;
τ = $2\pi t/T_0$ = dimensionless time variable, t = time;
T_0 = free period of the undamped instrument;
V_0 = static magnification;
η = $\eta(\tau)$ = ground displacement; $Y(\tau) = V_0\,\eta(\tau) = Ae^{-\lambda\tau}\tau^n$;
A, λ = constants, $\lambda > 0$; in this paper $A = \lambda^n$;
n = integer number greater than 1;
M, N = constants of integration;

$$K_k = \binom{k+1}{0}\gamma^{k+1}\beta^0 - \binom{k+1}{2}\gamma^{k-1}\beta^2 + \binom{k+1}{4}\gamma^{k-3}\beta^4 - \dots$$

$$L_k = \binom{k+1}{1}\gamma^{k}\beta^0 - \binom{k+1}{3}\gamma^{k-2}\beta^2 + \binom{k+1}{5}\gamma^{k-4}\beta^4 - \dots$$

$$k = 0, 1, 2, \dots, n$$

γ = $\alpha - \lambda$; exponents of γ in K_k and L_k are positive integers or zero;
R = $\gamma^2 + \beta^2 = \lambda^2 - 2\alpha\lambda + 1$;
τ^* = $n : \lambda$ = abscissa of the maximum of $Y(\tau)$; the corresponding time is t^*;
y_1 = the first extreme value of y, different from zero („first maximum“);
$\mathfrak{V}_r$ = $y_1/y_{\max}$ = reduced dynamical magnification $\mathfrak{V} : V_0$; $\mathfrak{V}$ dynamical magnification;
τ_1^* = abscissa of the first maximum; the corresponding time is t_1^*;
τ_1 = smallest root $\neq 0$ of the equation $y = 0$; the corresponding time is t_1;
u^* = $2n : (\pi\lambda)$;

$u_1^* \quad = 2\tau_1^* : \pi = 4t_1^* : T_0;$
$U \quad = \tau_1 : \pi = 2t_1 : T_0.$

A brief summary of the results obtained in the first part of [1] may be given: The writer has given a general solution[1]) of the equation

$$\frac{d^2y}{d\tau^2} + 2\alpha\frac{dy}{d\tau} + y = -\frac{d^2}{d\tau^2}(\lambda^n e^{-\lambda\tau}\tau^n), \tag{1}$$

which holds for the displacement of the indicator of a seismograph with stationary mass moving without solid friction under the influence of the ground shock

$$\eta(\tau) = \frac{A}{V_0} e^{-\lambda\tau}\tau^n \tag{2}$$

where we put, by choosing suitably the unit of length, $A = \lambda^n$. For $\tau = 0$ the initial conditions required were as follows:

$$y = 0, \left(\frac{dY}{d\tau}\right)_0 = 0. \tag{3}$$

Thus the general solution of (1) may be written, if $0 \leqq \alpha < 1$,

$$\begin{aligned} y = e^{-\alpha\tau}(M \sin\beta\tau + N \cos\beta\tau) + \lambda^n e^{-\lambda\tau}\{-\tau^n + \\ + \sum_{k=0}^{n}(-1)^k \binom{n}{k} \frac{k!\tau^{n-k}}{R^{k+1}}[2\alpha K_k + (1-2\alpha^2)L_k]\}, \end{aligned} \tag{4}$$

with

$$\begin{aligned} M = \frac{(-1)^{n+1}\lambda^n n!}{\beta R^{n+1}}[(1-2\alpha^2)K_n - 2\alpha\beta^2 L_n], \ N = \\ = \frac{(-1)^{n+1}\lambda^n n!}{R^{n+1}}[2\alpha K_n + (1-2\alpha^2)L_n]. \end{aligned}$$

For the critical damping $\alpha = 1$ there is

$$\begin{aligned} y_{er} = \frac{(-1)^n\lambda^n n!}{(1-\lambda)^{n+1}} e^{-k}\left(\tau + \frac{2\lambda + n - 1}{1-\lambda}\right) + \lambda^n e^{-\lambda\tau}[-\tau^n + \\ + \sum_{k=0}^{n}(-1)^n\binom{n}{k}\frac{k!\tau^{n-k}}{(1-\lambda)^{k+2}}(1-k-2\lambda)], \end{aligned} \tag{4'}$$

and for $\lambda = 1$

$$\bar{y}_{er} = -\frac{\lambda^n e^{-\tau}\tau^n}{(n+2)(n+1)}[\tau^2 + 2(n+2)\tau + (n+2)(n+1)]. \tag{4''}$$

The resulting form of forced vibration was then discussed in the case $n = 3$ for different values of parameters α and λ respectively. The corresponding curves for λ variable and $\alpha = 0{,}5$, i. e. the damping ratio equal

[1]) See also *Fysika v technice*, **3**, pp. 113 ff., Praha 1948.

to 5,7, are plotted in Fig. 1. The curves of reduced dynamical magnification $\mathfrak{V}_r$ were constructed for the first maximum y_1 of the forced vibration. They are very similar to those obtained for other forms of shock motions,

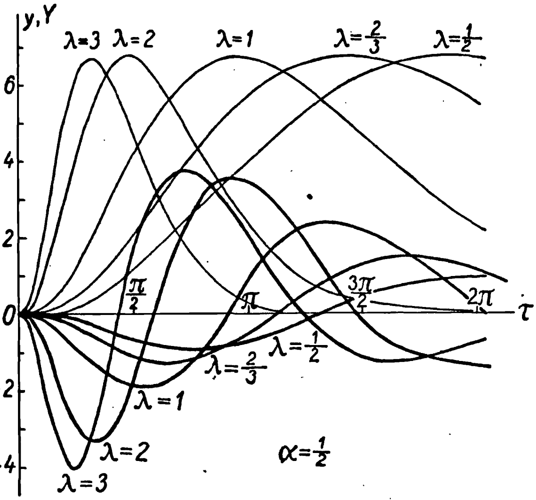

Fig. 1. Ground displacement of the type $A\bar{e}^{\lambda\tau}\tau^3$ (above) with the corresponding forced vibration of the seismograph (below) for different values of λ; damping constant $\alpha = 0{,}5$.

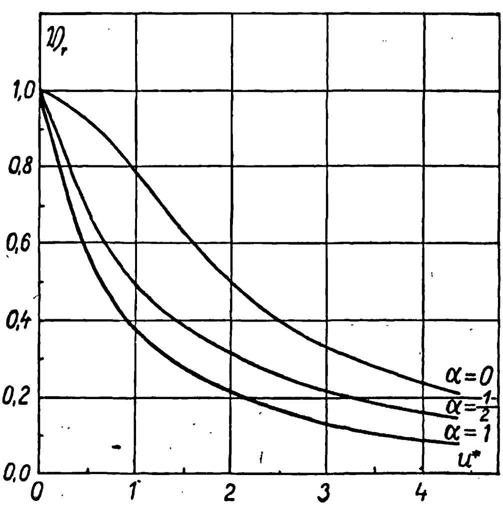

Fig. 2. Reduced dynamical magnification for the first maximum as function of $u^* = 6/(\pi\lambda)$.

as may be seen in [2, 3, 4, 5, 6, 7]. In our case we have for them the general formula

$$\mathfrak{B}_r = (e/n)^n \, y_1^*, \tag{5}$$

i. e. if $n = 3$,

$$\mathfrak{B}_r = (e/3)^3 \cdot y_1^* = 0{,}744 y_1^*.$$

These curves were plotted in [1] as functions of u^*, defined generally by the equation $u^* = 2n/(\pi\lambda)$, and for $n = 3$, $u = 6/(\pi\lambda) = 1{,}910/\lambda$ (see

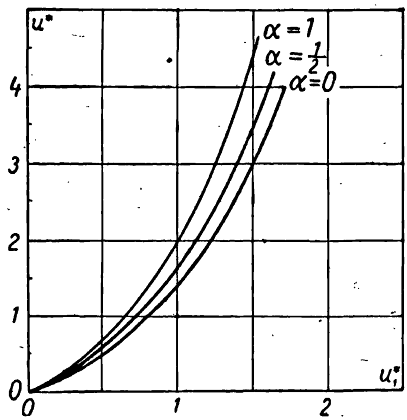

Fig. 3. Phase lag for the first maximum; $u_1^* = 2\tau_1^*/\pi$, $u^* = 6/(\pi\lambda)$.

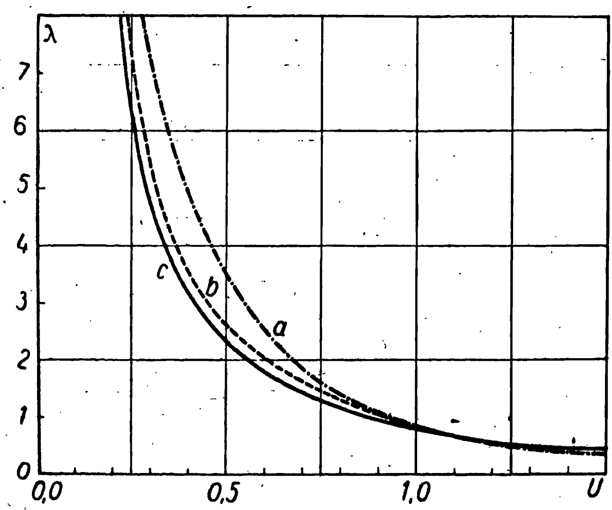

Fig. 4. $U = 2\tau_1/\pi$ as function of λ.

Fig. 2). Similarly the phase lag defined by the difference $\tau^* - \tau_1^*$ was represented graphically in the usual manner, where the abscissa was $u_1{}^* = 2\tau_1{}^*/\pi = 0{,}637\tau_1{}^*$ and the ordinate u^*; see Fig. 3.

The curves drawn in Figs. 2 and 3 are only of a theoretical value, but they are not utilisable in the analysis of seismograms, because we are unable to determine u^* directly from the diagram recorded by the pendulum.

Let us now consider the smallest and from zero different root τ_1 of the equation $y = 0$. Its value is directly determinable from the seismogram by means of the corresponding t_1. If we hold n for a given integer — e. g. $n = 3$ — then τ_1 becomes for a certain instrument (i. e. for a given α) a function of λ only. The curves representing such a function may be constructed by calculating a sufficient number of their points; α is the parameter of the resulting set of curves. In this paper I have introduced a new variable $U = \tau_1/\pi = 2t_1/T_0$ instead of τ_1 in order to obtain a certain uniformity of results, as H. P. BERLAGE [5] and C. TSUBOI [3] have done, making use of the first „half-period" observed on the seismogram. The curves representing the mutual relation between U and λ are plotted in Fig. 4 successively for $\alpha = 0$, 0,5 and 1. We see here how if U increases the values of λ are decreasing from $\lambda = \infty$ corresponding to $U = 0$ to a value of about $\frac{2}{3}$, corresponding roughly to $U \approx 1{,}1$. There the curves are crossing, which means that in the neighbourhood of $U = 1{,}1$ the position of the first root of the equation $y = 0$ is practically independent of the damping of the system. For great values of U this dependence reappears, but in the inverse sense. It is clear that for a shock of the type considered the ground displacement may be reconstructed if we possess a set of curves represented by Fig. 4.

Therefore it is important to have some information about the applicability of our results in a concrete case. In fact there exists a number of relations by which the applicability of the theory may be examined. One of them is the relation between $u_1{}^*$ and U represented by Fig. 5. This relation is particularly useful, as it contains both quantities determinable immediately from the records. Another check would be given for example by a comparison of the measured ratio of amplitudes of the first and second maximum with its calculated value as function of λ. In the author's opinion this last characteristic is not very convenient in practice, because the second maximum may be (and usually is) already considerably affected by onsets of the following disturbances.

The curves of the reduced dynamical magnification taken as functions of U are represented by Fig. 6, again for the three characteristic vaues of α. It is interesting to compare them with a figure drawn for $Y = 3\sin\mu\tau - \sin 3\mu\tau$ by KAWASUMI ([4], p. 327). He has plotted $\mathfrak{V}_r$ as function of t_1/T_0, i. e. $\frac{1}{2}U$. The values of KAWASUMI are somewhat, but not too much, higher than those of the present paper; the general aspect of both sets is of course the same. This similarity may be followed in many other directions for all the kinds of shocks which so far have been mathematically investigated. In the just mentioned paper we find e. g. that the crossing of curves, giving the position of τ_1, is quite similar to that in our paper.

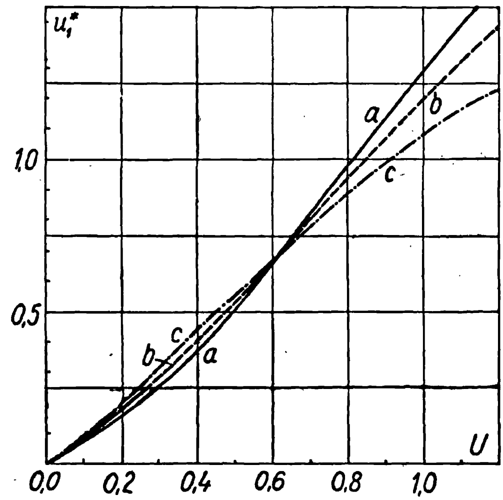

Fig. 5. Time of the first maximum expressed by u_1^* as function of U.

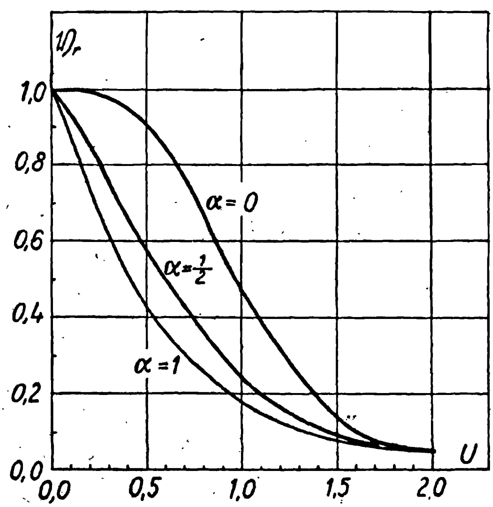

Fig. 6. Reduced dynamical magnification for the first maximum expressed as function of U.

The phase lag $\tau^* - \tau_1^*$ between the first maximum y_1 and the maximum of the ground displacement may be qualitatively followed in Fig. 1. As we now know λ as a function of U by means of Fig. 4, we can also calculate τ^*. From Fig. 3 we find the corresponding value of τ_1^*, which can also be directly determined from the seismogram. So we have a further control of the applicability of the theory. The relation between u^* as function of U is graphically represented in Fig. 7; the difference

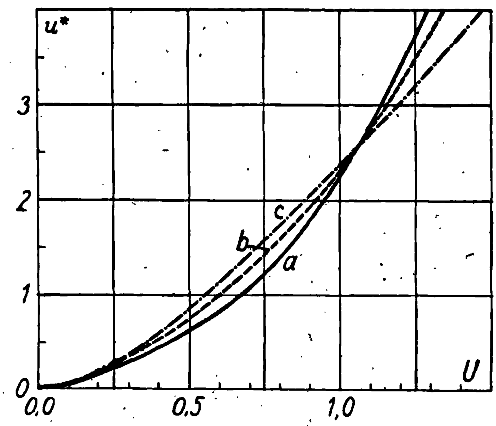

Fig. 7. Time of the maximum of ground displacement expressed by u^* as function of U.

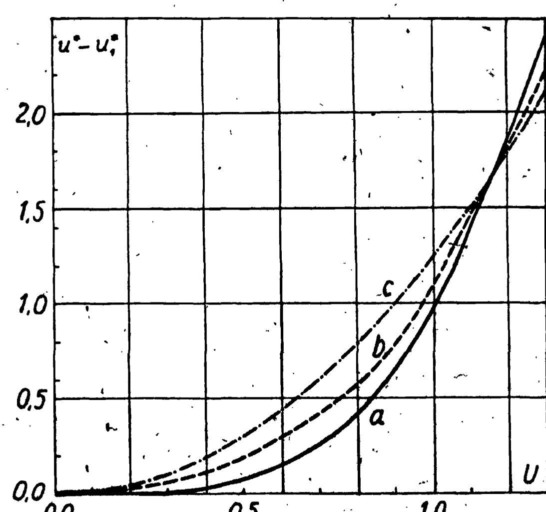

Fig. 8. Phase lag of the first maximum given by $u^* - u_1^*$ as function of U.

$u^* - u_1^* = 2(\tau^* - \tau_1^*)/\pi = (75{,}4\lambda^{-1} - 4t_1^*)/T_0$ is shown in Fig. 8 as a function of U.

As example was examined the onset of the P wave in the NS component of the Turkestan earthquake on March 4, 1949, 10^h19^m M. G. T., recorded by the Wiechert horizontal seismograph at the station of Praha. Here t_1 was found to be $2{,}8_5$ sec and t_1^* $1{,}8_0$ sec; the first maximum was 630 μ. The constants of the instrument were $T_0 = 9{,}9$ sec, $V_0 = 230$ and $\alpha = 0{,}45$. From these data we have $U = 0{,}58$, $u_1^* = 0{,}64$; from Fig. 5 it follows that $u_1^* = 0{,}63$, which may be considered a certain justification for the use of the theory. From Fig. 6 we read $\mathfrak{V}_r \doteq 0{,}53$. This gives $\mathfrak{V} = 122$ and for the maximum of the true displacement of the soil we get about 5,2 μ. This result was compared with the amplitudes derived by aid of the corresponding curves of dynamical magnification published in the papers of KAWASUMI [4] for $Y = 3 \sin\mu\tau - \sin 3\mu\tau$, $Y = 2 \sin\mu\tau - \sin 2\mu\tau$, $Y = e^{-\lambda^2\tau^2}$ (worked out theoretically by T. SUZUKI [8]), BERLAGE jr. [5] for $Y = \tau e^{-\lambda\tau} \sin\mu\tau$, TOPERCZER [6] for $Y = \tau^2 e^{-\lambda\tau} \sin\mu\tau$ and KIRNOS [7] for $Y = \sin\mu\tau$, $Y = e^{-\lambda\tau} \sin\mu\tau$ and $Y = \tau e^{-\lambda\tau}$.

Though the analytic expressions of the shock just quoted are so different, the same value of about 5 microns was always obtained for the true amplitude of the ground displacement, whereas the use of the common magnification curves holding for the stationary harmonic vibration led to a value of 2,2 microns only. In this way one sees that the true amplitude of the first motion in the seismogram, especially the important amplitude of the onset of P, may be determined with a certain accuracy by curves of the dynamical magnification for any kind of shock mentioned above, if the first half-period of the record is short enough with regard to the period of the undamped pendulum. It is by no means justified to use the curves of dynamical magnification holding for stationary harmonic vibrations for calculating the amplitudes of the first moation recorded and those of rapid onsets at all.

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Dynamické zvětšení seismografu buzeného nárazem tvaru $\lambda^n e^{-\lambda\tau} \tau^n$.

(Obsah předchozího článku.)

Řešení rovnice (1) pro pohyb indikátoru seismografu se setrvačnou hmotou a bez vlečného tření za vtištěného pohybu půdy tvaru $\lambda^n e^{-\lambda\tau} \tau^n$ s počátečními podmínkami (3) dává integrál (4) resp. (4′) nebo (4″). Křivky redukovaného dynamického zvětšení na obr. 2 a křivky fázového posunutí maxima skutečného pohybu půdy vzhledem k prvnímu maximu záznamu na obr. 3 nejsou přímo prakticky použitelné. Proto byl zaveden parametr U, který se dá určiti z přímého měření délky první půlperiody záznamu. Transformované křivky redukovaného dynamického zvětšení $\mathfrak{V}_r$ pro první maximum (obr. 6) a křivky fázového posunutí $u^* - u_1^*$ v závislosti na parametru U (obr. 8) dovolují nalézti pro uvažovaný druh nárazu skutečný pohyb půdy z jeho seismografického záznamu poměrně velmi jednoduše. Počítati amplitudy při nasazení zemětřesení a nárazech z obvyklých křivek platných pro ustálené kmitání harmonické je nepřípustné.