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# A note on the simplicity of skew polynomial rings of derivation type

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**Abstract.** In earlier papers we have obtained necessary and sufficient conditions for the simplicity of an iterated skew polynomial ring defined over a ring $R$ with respect to a finite set of derivations of $R$ commuting to each other.
In the present paper a sufficient condition is obtained for the simplicity of an iterated skew polynomial ring defined over a ring $R$ of characteristic zero with respect to a finite set of derivations of $R$ not necessarily commuting to each other.
As an application we construct such a ring over the coordinate ring of the real sphere.

## 1. Iterated skew polynomial rings of derivation type

All the rings considered in this paper are with identity.
Let $R$ be a ring, and let $D = \{d_1, d_2, \dots, d_n\}$ be a finite set of derivations of $R$. Consider the set $S_n$ of all polynomials in $n$ variables, say $x_1, x_2, \dots, x_n$ over $R$. Define addition in $S_n$ in the usual way, and define multiplication by the distributive law and by the rules

$$x_i r = r x_i + d_i(r) \tag{1}$$

and

$$x_i x_j = x_j x_i, \tag{2}$$

for all $r$ in $R$ and each $i, j = 1, 2, \dots, n$. Then $S_i$ is a skew polynomial ring over $S_{i-1}$ for each $i = 1, 2, \dots, n$, if, and only if, $d_i$ commutes with $d_j$ for all $i, j = 1, 2, \dots, n$ ([4] ; Theorem 2.2).
The ring constructed above is called an iterated skew polynomial ring of derivation type over $R$ and we shall denote it by $S_n = R[x, D]$.

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Notice that, in order to have that $x_ix_j = x_jx_i$, it is necessary that $d_i \circ d_j = d_j \circ d_i$ for all $i, j = 1, 2, \ldots, n$. In fact, given $r$ in $R$, we have that

$$x_ix_jr = x_i[rx_j + d_j(r)] = (x_ir)x_j + x_id_j(r) = [rx_i + d_i(r)]x_j + d_j(r)x_i + (d_i \circ d_j)(r) = rx_ix_j + d_i(r)x_j + d_j(r)x_i + (d_i \circ d_j)(r) \quad (3)$$

In the same way one finds

$$x_jx_ir = rx_jx_i + d_j(r)x_i + d_i(r)x_j + (d_j \circ d_i)(r), \quad (4)$$

and the result follows by equating the right hand sides of (3) and (4). Further it is easy to check that the above set $S_n$ with addition defined in the usual way, and multiplication defined by the distributive law and by the rule (1) is a ring, even if the rule (2) does not hold. For example such kind of rings have been considered by Kishimoto [2].

In order to distinguish between the two cases, we shall denote the ring of the later case by $S_n^*$ . Obviously in this case we need not to have that $d_i \circ d_j = d_j \circ d_i$ for $i, j = 1, 2, \ldots, n$.

## 2. Simplicity of $S_n^*$.

Let $R$ be a ring and let D be a set of derivations of $R$, then an ideal I of $R$ is called a D–ideal if $d(I) \subseteq I$ for each d in D, and $R$ is called a D–simple ring if it has not non trivial D–ideals. If $D = \{d\}$, then $R$ is called for simplicity a d–simple ring.

In [4] and [5] we have obtained necessary and sufficient conditions for the simplicity of $S_n = R[x, D]$, where D is a finite set of derivations of $R$ commuting to each other.

In this section, assuming that R is of characteristic zero, we shall obtain a sufficient condition for the simplicity of $S_n^*$. Namely we shall prove the following theorem:

**2.1. Theorem.** *Let $R$ be a D-simple ring of characteristic zero, where $D = \{d_1, d_2, \ldots, d_n\}$ is a finite set of derivations of $R$. Assume further that $d_i(C(S_{i-1}^*) \cap R) \neq 0$ , where $C(S_{i-1}^*)$ denotes the center of $S_{i-1}^*$ and $S_0^* = R$ for all $i = 1, 2, \ldots, n$. Then $S_n^*$ is a simple ring.*

*Proof.* Assume that $S_n^*$ is not a simple ring and let I be a nonzero proper ideal of $S_n^*$ . Denote by k be the least non zero degree with respect to $x_n$ of polynomials in I and let $f = \sum_{i=0}^{k} f_ix_n^i$ be such a polynomial, with $f_i$ in $S_{n-1}^*$ for each $i = 0, 1, \ldots, k$. Choose a non zero r in $C(S_{n-1})^* \cap R$ such that $d_n(r) \neq 0$, then $fr = f_kx_n^kr + \cdots + f_1x_nr + f_0r = f_k\left[\sum_{i=0}^{k} \binom{k}{i} d_n^k(r)x^{k-i}\right] r + \cdots + f_1[rx + d_n(r)]r + f_0r = rf + g$, where g is a non zero polynomial having degree less than k with respect to $x_n$. Thus $fr - rf = g$ is in I, a contradiction to the minimality of k. Therefore there exists a polynomial of zero degree with respect to $x_n$ in I and so $I_{n-1} = I \cap S_{n-1}^*$ is a non zero ideal of $S_{n-1}^*$.

Repeating the same argument one finds that $I_{n-2} = I_{n-1} \cap S_{n-2}^*$ is a non zero ideal of $S_{n-2}^*$ and so on. Finally, and after n steps, one finds that $I_0 = I_1 \cap R = I \cap R$ is a non zero ideal of $R$.

But, given a non zero t in $I \cap R$, $d_i(t) = x_it - tx_i$ is also in $I \cap R$ for each

$i = 1, 2, \ldots, n$ and therefore, since $R$ is a D–simple ring, $I \cap R = R$. Thus $I = S_n^*$, a contradiction. $\square$

The above theorem gives for $n = 1$ the following corollary:

**2.2. Corollary.** *Let $R$ be a d–simple ring of characteristic zero, where d is a derivation of $R$, such that $d[C(R)] \neq 0$. Then the skew polynomial ring $S = R[x, d]$ is a simple ring.*

REMARK: Obviously, if $d_i(C(S_{i-1}^*) \cap R) \neq 0$, then $d_i$ is an outer derivation of $S_{i-1}^*$. Thus, if the elements of D commute to each other, then Theorem 2.1 is a weaker form of Theorem 3.4 in [4] . For the same reason Corollary 2.2 is a weaker form of Corollary 3.6 (i) in [4].

## 3. An application.

In this section, and in order to illustrate Theorem 2.1, we shall construct a simple skew polynomial ring over the coordinate ring $R = \frac{\mathbb{R}[x,y,z]}{x^2+y^2+z^2-1}$ of the real sphere (with $\mathbb{R}$ we denote the field of the real numbers).
For this, observe first that, since there is no derivation d of R such that R is a d–simple ring ([1]; section 3, example (iii)), one cannot construct simple skew polynomial rings in one variable over R ([4]; Corollary 3.6 (i)).
Next, consider the $\mathbb{R}$–derivations $d_1$ and $d_2$ of the polynomial ring $\mathbb{R}[x, y, z]$ defined by $d_1 = (y + z, z - x, -x - y)$ and $d_2 = (y + 2z, xyz - x, -xy^2 - 2x)$ respectively. It is straightforward to check that $d_1 \circ d_2 \neq d_2 \circ d_1$. Further, since $d_i(x^2 + y^2 + z^2 - 1) = 0$ for $i = 1, 2$ , $d_i$ induces an $\mathbb{R}$–derivation of R, denoted also by $d_i$.
Set $D = \{d_1, d_2\}$.Using the above notation we shall prove first the following lemma:

**3.1. Lemma.** *The ring $R$ is a D–simple ring.*

*Proof.* Given $F$ in $\mathbb{R}[x, y, z]$ we shall denote by $\bar{F}$ the image of $F$ in $R$.
To prove the lemma it suffices to show that $R$ has no no zero prime D–ideals ([3]; Corollary 1.5).
For this, let $\bar{P}$ be a nonzero prime D–ideal of R, then $\bar{P}$ lifts to a prime D–ideal $P$ of $\mathbb{R}[x, y, z]$ containing $x^2 + y^2 + z^2 - 1$.
We shall show that $z$ is not in $P$.
In fact, if $z$ is in $P$, then $d_1(z) = -x - y$ and $d_2(z) = -x(y^2 + 2)$ are also in $P$. Then, if $x$ is in $P$, $y$ is also in $P$ and therefore $x^2 + y^2 + z^2$ is in $P$, i.e. 1 is in $P$, a contradiction. Thus $y^2 + 2$ is in $P$ and therefore $d_1(y + 2) = 2yd_1(y)$ is in $P$. But $y$ is not in $P$, therefore $d_1(y) = z - x$ is in $P$, i.e. $x$ is in $P$, a contradiction.
Given $\bar{g}$ in $\bar{P}$ , we can write $\bar{g} = f_1(\bar{x}, \bar{y}) + \bar{z}f_2(\bar{x}, \bar{y})$. Then, if $\bar{f}_1 = \bar{0}$ , $\bar{z}\bar{f}_2$ is in $\bar{P}$ . But $z$ is not in $P$, therefore $f_2$ is in $P$. Also, if $\bar{f}_1 \neq \bar{0}$ , then $(\bar{f}_1 + \bar{z}\bar{f}_2)(\bar{f}_1 - \bar{z}\bar{f}_2) = \bar{f}_1^2 - \bar{z}^2\bar{f}_2^2 = \bar{f}_1^2 + (\bar{x}^2 + \bar{y}^2 - \bar{1})\bar{f}_2^2$ is in $\bar{P}$ , therefore $f_1^2 + (x^2 + y^2 - 1)f_2^2$ is in $P$. Thus $Q = P \cap \mathbb{R}[x, y] \neq 0$.
Further, if $f$ is a non zero polynomial in $Q$, then $d_2(f) - d_1(f) = z[\frac{\partial f}{\partial x} + (xy - 1)\frac{\partial f}{\partial y}]$ is in $P$, therefore $\frac{\partial f}{\partial x} + (xy - 1)\frac{\partial f}{\partial y}$ is in $P$.
Consider the $\mathbb{R}$–derivation $d$ of $\mathbb{R}[x, y]$ defined by $d = (1, xy - 1)$ . Then the polynomial ring $\mathbb{R}[x]$ is d–simple because it is a PID and $d$ lowers the degree. Further the equation $-1 + d(g) = gx$ (1) has no solution g in $\mathbb{R}[x]$ because, if there

exists a polynomial $g$ of degree $n$ in $\mathbb{R}[x]$ satisfying (1), then the degree of $d(g)$ is at most $n$, a fact which contradicts (1). Thus, by Lemma 3.1 in [6], $\mathbb{R}[x,y]$ is a d-simple ring.
Also, for any $f$ in $Q$, $d(f) = \frac{\partial f}{\partial x} + (xy-1)\frac{\partial f}{\partial y}$ is also in $Q$. Therefore $Q = 0$, a contradiction. □

Keeping the same notation we are ready now to prove:

**3.2. Theorem.** *The skew-polynomial ring $S_2^*$ defined over $R$ with respect to $D$ is simple.*

*Proof.* We have that $d_1(R) \neq 0$ (e.g. $d_1(\bar{x}) = \bar{y} + \bar{z} \neq 0$) .
Also, by relation (1) of section 1, given $r$ in R, $r$ is in $C(S_1^*)$, if, and only if, $d_1(r) = 0$.
Set $r = \bar{y} - \bar{x} - \bar{z}$, then it is straightforward to check that $d_1(r) = 0$, while $d_2(r) \neq 0$ . Thus the result follows by Theorem 2.1. □

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