Zeqing Liu; Lokeneth Debnath; Shin Min Kang; Jeong Sheok Ume
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On the generalized nonlinear quasivariational inclusions

Z. Liu, L. Debnath, S. M. Kang and J. S. Ume

ABSTRACT. In this paper, we introduce and study a new class of generalized nonlinear quasivariational inclusions for multivalued mappings and construct some new iterative algorithms for finding the approximate solutions of this class of quasivariational inclusions. We establish the existence of solutions for this generalized nonlinear quasivariational inclusions involving both relaxed Lipschitz and strongly monotone and generalized pseudo-contractive mappings and obtain the convergence of iterative sequences generated by the algorithms.

1. Introduction

In 1996, Noor [7] and Huang [4] introduced and studied the generalized multivalued strongly nonlinear quasi-variational inequalities for compact valued mappings and the set-valued nonlinear generalized variational inclusions for closed and bounded valued mappings, respectively, they constructed a few algorithms for finding the approximate solutions of their quasi-variational inequalities and variational inclusions and established the convergence of iterative sequences generated by these algorithms. Afterwards, Bai-Tang-Liu [2], Ding [3], Huang [5], Noor [8], Verma [11]-[14] and others have extended and generalized the resules due to Noor [7] and Huang [4] in various different aspects. In 2000, Ahmad-Ansari [1] considered the generalized nonlinear variational inclusions with nonclosed and nonbounded valued mappings, constructed an algorithm without using Hausdorff metric, proved the existence of solutions for the generalized nonlinear variational inclusions involving relaxed Lipschitz mappings and the convergence of iterative sequences generated by the algorithm. We point out that the multivalued mapping in [1, Theorem 4.1] is, in fact, a single-valued mapping.

Inspired and motivated by recent research works [1]-[5], [7]-[16], in this paper, we introduced a new class of generalized nonlinear quasivariational inclusions for multivalued mappings and construct some new iterative algorithms for finding the approximate solutions of the quasivariational inclusions. We establish the existence

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of solutions for this generalized nonlinear quasivariational inclusions involviong both relaxed Lipschitz and strongly monotone and generalized pseudo-contractive mappings and show the convergence of iterative sequences generated by the algorithms. Our results clarify, extend and unify the corresponding results in [1]-[5], [7]-[16] and others.

2. Preliminaries

Let H be a Hilbert space endowed with a norm $\|\cdot\|$ and a inner product $\langle \cdot, \cdot \rangle$, respectively, 2^H, $CB(H)$ and $CC(H)$ denote the families of all the nonempty subsets, all the nonempty closed bounded subsets and all the nonempty closed convex subsets of H, respectively, and $\varphi : H \times H \to \mathbb{R} \cup \{+\infty\}$ be such that for each fixed $y \in H$, $\varphi(\cdot, y) : H \to \mathbb{R} \cup \{+\infty\}$ is a proper convex lower semicontinuous function on H and $g(H) \cap dom\, \partial\varphi(\cdot, y) \neq \emptyset$ for each $y \in H$. Let I denote the identity mapping on H.

Given mappings $A, B, C, D : H \to 2^H$, $g, a, b, c, d : H \to H$, $N : H \times H \times H \to H$ and $f \in H$, we consider the following problem:

Find $u \in H$, $x \in Au$, $y \in Bu$, $z \in Cu$, $w \in Du$ such that $gu \in dom\, \partial\varphi(\cdot, dw)$ and

$$\langle gu - N(ax, by, cz) - f, v - gu \rangle \geq \varphi(gu, dw) - \varphi(v, dw), \quad \forall\, v \in H, \tag{2.1}$$

which is called the *generalized nonlinear quasivariational inclusion.*

Special cases. (a) If $a = b = c = d = C = D = I$, $f = 0$ and $N(x, y, z) = -x + y + gz$ for all $x, y, z \in H$, then problem (2.1) is equivalent to finding $u \in H$, $x \in Au$, $y \in Bu$ such that $gu \in dom\, \partial\varphi(\cdot, u)$ and

$$\langle x - y, v - gu \rangle \geq \varphi(gu, u) - \varphi(v, u), \quad \forall\, v \in H, \tag{2.2}$$

which is known as the *generalized quasivariational inclusion*, introduced and studied by Ding [3].

(b) If $b = c = d = B = C = D = I$, $f = 0$, $N(x, y, z) = x$ and $\varphi(x, y) = \varphi(x)$ for all $x, y, z \in H$, then problem (2.1) is equivalent to finding $u \in H$, $x \in Au$ such that $gu \in dom\, \partial\varphi$ and

$$\langle gu - ax, v - gu \rangle \geq \varphi(gu) - \varphi(v), \quad \forall\, v \in H, \tag{2.3}$$

which is called the *generalized nonlinear variational inclusion.* introduced and studied by Ahmad-Ansari [1].

(c) If $K : H \to CC(H)$ is a mapping such that for each fixed $y \in H$, $\varphi(\cdot, y) = I_{K(y)}(\cdot)$ is the indicator function of $K(y)$, that is,

$$I_{K(y)}(x) = \begin{cases} 0, & \text{if } x \in K(y), \\ +\infty, & \text{otherwise}, \end{cases}$$

$c = d = C = I$, $f = 0$ and $N(x, y, z) = -x - y + gz$ for all $x, y, z \in H$, then problem (2.1) is equivalent to finding $u \in H$, $x \in Au$, $y \in Bu$, $w \in Du$ such that $gu \in K(w)$ and

$$\langle ax + by, v - gu \rangle \geq 0, \quad \forall\, v \in K(w), \tag{2.4}$$

which is called the *generalized strongly nonlinear implicit quasivariational inequality,* introduced and studied by Huang [5].

(d) If $a = b = c = d = C = D = I$, $f = 0$, $N(x, y, z) = -N(x, y) + gz$ and $\varphi(x, y) = I_{K(y)}(x)$ for all $x, y, z \in H$, then problem (2.1) is equivalent to finding $u \in H$, $x \in Au$, $y \in Bu$ such that $gu \in K(u)$ and

$$\langle N(x, y), v - gu\rangle \geq 0, \quad \forall v \in K(u), \tag{2.5}$$

which is called the *generalized multivalued quasi-variational inequality*, introduced and studied by Noor [8].

Example 1. *Reamrk 2.1 For appropriate and suitable choices of the mappings g, a, b, c, d, A, B, C, D, N, φ and the element f, a number of known classes of variational inequalities and quasivariational inequalities, studied previously by a few authors including Huang [4], Bai er al. [5], Noor [7], Siddiqi-Ansari [9], [10], Verma [11]-[14], Yao [15] and Zhang [16], can be obtained as special cases of problem (2.1).*

Definition 2.1. Let H be a Hibert space and $G : H \to 2^H$ be a maximal monotone mapping. For any fixed $\rho > 0$, the mapping $J^G_\rho : H \to H$ defined by

$$J^G_\rho(x) = (I + \rho G)^{-1}(x), \quad \forall x \in H,$$

is said to be the *resolvent operator* of G.

It is known that the resolvent operator J^G_α is singlevalued and nonexpansive. Notice that the subdifferential $\partial\varphi$ of a proper, convex and lower semicontinuous function $\varphi : H \to \mathbb{R} \cup \{+\infty\}$ is a maximal monotone multivalued mapping. It follows that the resolvent operator $J^{\partial\varphi}_\rho$ of $\partial\varphi$ is given by

$$J^{\partial\varphi}_\rho(x) = (I + \rho\partial\varphi)^{-1}(x), \quad \forall x \in H.$$

Definition 2.2. A mapping $g : H \to H$ is said to be *strongly monotone* and *Lipschitz continuous* if there exist constants $\alpha > 0$, $\beta > 0$ such that

$$\langle gx - gy, x - y\rangle \geq \alpha\|x - y\|^2 \quad \text{and} \quad \|gx - gy\| \leq \beta\|x - y\|, \quad \forall x, y \in H,$$

respectively.

Definition 2.3. A mapping $N : H \times H \times H \to H$ is said to be *Lipschitz continuous with respect to the first argument* if there exists a constant $s > 0$ such that

$$\|N(x, u, v) - N(y, u, v)\| \leq s\|x - y\|, \quad \forall x, y, u, v \in H.$$

In a similar way, we can define Lipschitz continuity of the mapping $N(\cdot, \cdot, \cdot)$ with respect to the second or third argument.

Definition 2.4. A multivalued mapping $B : H \to CB(H)$ is said to be *strongly monotone with respect to the mapping $b : H \to H$* and the *second argument of $N : H \times H \times H \to H$* if there exists a constant $t > 0$ such that

$$\langle N(p, bx, q) - N(p, by, q), u - v\rangle \geq t\|u - v\|^2, \quad \forall u, v, p, q \in H,\ x \in Bu,\ y \in Bv.$$

Definitions 2.5. A multivalued mapping $A : H \to CB(H)$ is said to *relaxed Lipschitz with respect to the mapping* $a : H \to H$ and the *first argument* of $N : H \times H \times H \to H$ if there exists a constant $t > 0$ such that

$$\langle N(ax, p, q) - N(ay, p, q), u - v\rangle \le -t\|u - v\|^2, \quad \forall\, u, v, p, q \in H,\ x \in Au,\ y \in Av.$$

Definitions 2.6. A multivalued mapping $C : H \to CB(H)$ is said to *generalized pseudo-contractive with respect to the mapping* $c : H \to H$ and the *third argument* of $N : H \times H \times H \to H$ if there exists a constant $t > 0$ such that

$$\langle N(p, q, cx) - N(p, q, cy), u - v\rangle \le t\|u - v\|^2, \quad \forall\, u, v, p, q \in H,\ x \in Cu,\ y \in Cv.$$

Definition 2.7. A multivalued mapping $A : H \to CB(H)$ is said to be *H-Lipschitz continuous* if there exists a constant $t > 0$ such that

$$H(Ax, Ay) \le t\|x - y\|, \quad \forall\, x, y \in H,$$

where $H(\cdot, \cdot)$ is the Hausdorff metric on $CB(H)$.

3. Main Results

Lemma 3.1. *Let ρ and t be positive parameters. Then the following conditions are equivalent:*

(i) *the generalized nonlinear quasivariational inclusion* (2.1) *has a solution $u \in H$, $x \in Au$, $y \in Bu$, $z \in Cu$ and $w \in Du$ with $gu \in dom\, \partial\varphi(\cdot, dw)$;*

(ii) *there exist $u \in H$, $x \in Au$, $y \in Bu$, $z \in Cu$ and $w \in Du$ satisfying*

$$gu = J_\rho^{\partial\varphi(\cdot,dw)}((1 - \rho)gu + \rho N(ax, by, cz) + \rho f), \tag{3.1}$$

where $J_\rho^{\partial\varphi(\cdot,dw)}$ denotes the resolvent operator of $\partial\varphi(\cdot, dw)$;

(iii) *the multivalued mapping $F : H \to 2^H$ defined by*

$$\begin{aligned} Fq = \cup_{x\in Aq, y\in Bq, z\in Cq, w\in Dq}[(1 - t)q + t(q - gq \\ + J_\rho^{\partial\varphi(\cdot,dw)}((1 - \rho)gq + \rho N(ax, by, cz) + \rho f))], \quad \forall\, q \in H, \end{aligned} \tag{3.2}$$

has a fixed point $u \in H$.

Proof. Note that (3.1) holds if and only if

$$(1 - \rho)gu + \rho N(ax, by, cz) + \rho f \in gu + \rho\partial\varphi(gu, dw),$$

which is equivalent to

$$N(ax, by, cz) + f - gu \in \partial\varphi(gu, dw).$$

The relation holds if and only if

$$\langle N(ax, by, cz) + f - gu, v - gu\rangle \le \varphi(v, dw) - \varphi(gu, dw), \quad \forall\, v \in H.$$

On the other hand, F has a fixed point $u \in H$ if and only if there exist $x \in Au$, $y \in B(u)$, $z \in Cu$ and $w \in Du$ such that

$$u = (1 - t)u + t(u - gu + J_\rho^{\partial\varphi(\cdot,dw)}((1 - \rho)gu + \rho N(ax, by, cz) + \rho f)),$$

which is equivalent to (3.1). This completes the proof .□

Example 2. *Remark 3.1 Lemma 3.1 extends Lemma 3.3 in [2], Theorem 3.1 in [3], Lemma 2.1 in [4], Lemma 3.4 in [5], Lemma 3.1 in [7]-[11] and [16], Lemma 3.2 in [12] and [13], and Lemmas 2.1 and 2.2 in [14].*

Based on Lemma 3.1 and Nadler's result, we suggest the following algorithms for the generalized nonlinear quasivariational inclusion (2.1).

Algorithm 3.1. *Let $g, a, b, c, d, : H \to H$, $A, B, C, D : H \to CB(H)$, $N : H \times H \times H \to H$ and $f \in H$. Given $u_0 \in H$, $x_0 \in Au_0$, $y_0 \in Bu_0$, $z_0 \in Cu_0$ and $w_0 \in Du_0$, compute u_{n+1} by the iterative scheme*

$$\begin{aligned} u_{n+1} = (1-t)u_n + t(u_n - gu_n \\ + J_\rho^{\partial\varphi(\cdot,dw_n)}((1-\rho)gu_n + \rho N(ax_n, by_n cz_n) + \rho f))), \end{aligned} \tag{3.3}$$

$$\begin{aligned} x_n \in Au_n, \quad \|x_n - x_{n+1}\| \le (1+(n+1)^{-1})H(Au_n, Au_{n+1}), \\ y_n \in Bu_n, \quad \|y_n - y_{n+1}\| \le (1+(n+1)^{-1})H(Bu_n, Bu_{n+1}), \\ z_n \in Cu_n, \quad \|z_n - z_{n+1}\| \le (1+(n+1)^{-1})H(Cu_n, Cu_{n+1}), \\ w_n \in Du_n, \quad \|w_n - w_{n+1}\| \le (1+(n+1)^{-1})H(Du_n, Du_{n+1}) \end{aligned} \tag{3.4}$$

for all $n \ge 0$, where t and ρ are positive parameters with $t \le 1$.

Algorithm 3.2. *Let $g, a, b, c, d : H \to H$, $A, B, C, D : H \to CB(H)$, $N : H \times H \times H \to H$ and $f \in H$. Given $u_0 \in H$, $x_0 \in Au_0$, $y_0 \in Bu_0$, $z_0 \in Cu_0$ and $w_0 \in Du_0$, compute u_{n+1} by the iterative scheme*

$$gu_{n+1} = J_\rho^{\partial\varphi(\cdot,dw_n)}((1-\rho)gu_n + \rho N(ax_n, by_n, cz_n) + \rho f) \tag{3.5}$$

for all $n \ge 0$, where $\{x_n\}_{n\ge0}$, $\{y_n\}_{n\ge0}$, $\{z_n\}_{n\ge0}$ and $\{w_n\}_{n\ge0}$ satisfy (3.4) and $\rho > 0$ is a parameter.

Example 3. *Remark 3.2 Algorithms 3.1 and 3.2 include the algorithms in [1]-[5], [7]-[16] as special cases.*

Theorem 3.1. *Let $\varphi : H \times H \to \mathbb{R} \cup \{+\infty\}$ be such that for each fixed $y \in H$, $\varphi(\cdot, y)$ is a proper convex lower semicontinuous function on H, $g(H) \cap dom\, \partial(\cdot, y) \neq \emptyset$ and there exists a constant $\mu > 0$ satisfying*

$$\|J_\rho^{\partial\varphi(\cdot,x)}(z) - J_\rho^{\partial\varphi(\cdot,y)}(z)\| \le \mu\|x - y\|, \quad \forall x, y, z \in H,\ \rho > 0. \tag{3.6}$$

Let $f \in H$ and $g, a, b, c, d, : H \to H$ be Lipschitz continuous with constants $l, \alpha, \beta, \gamma, \delta$, respectively, and g be strongly monotone with constant h. Let $N : H \times H \times H \to H$ be Lipschitz continuous with constants ξ, η, ζ with respect to the first, second and third arguments, respectively. Assume that $A, B, C, D : H \to CB(H)$ are H-Lipschitz continuous with constants p, q, r, s, respectively, A is relaxed Lipschitz with constant σ with respect to a and the first argument of N, B is strongly monotone *with constant ν with respect to b and the second argument of N, and C is generalized pseudo-contractive with constant m with respect to c and the third argument of N. Let*

$$\begin{aligned} &k = 2\sqrt{1-2h+l^2} + \mu\delta s, \quad i = 1 + 2\sigma + \xi^2\alpha^2 p^2, \\ &j = \sqrt{1-2\nu+\eta^2\beta^2 q^2} + \sqrt{1+2m+\zeta^2\gamma^2\tau^2} - \sqrt{1-2h+l^2} > 0. \end{aligned} \tag{3.7}$$

Suppose that there exists $\rho \in (0,1]$ such that

$$k + \rho j < 1 \tag{3.8}$$

holds and at least one of the following conditions

$$\begin{aligned} &i > j^2, \quad |1+\sigma-(1-k)j| > \sqrt{(2k-k^2)(i-j^2)}, \\ &\left|\rho - \frac{1+\sigma-(1-k)j}{i-j^2}\right| \\ &\quad < \frac{\sqrt{(1+\sigma-(1-k)j)^2-(2k-k^2)(i-j^2)}}{i-j^2}; \end{aligned} \tag{3.9}$$

$$i = j^2, \quad 1+\sigma > (1-k)j, \quad \rho > \frac{2k-k^2}{2(1+\sigma-(1-k)j)}; \tag{3.10}$$

$$\begin{aligned} &i < j^2, \quad \left|\rho - \frac{(1-k)j-\sigma-1}{j^2-i}\right| \\ &\quad > \frac{\sqrt{((1-k)j-\sigma-1)^2+(2k-k^2)(j^2-i)}}{j^2-i} \end{aligned} \tag{3.11}$$

is fulfilled. Then the generalized nonlinear quasivariational inclusion (2.1) has a solution $u \in H$, $x \in Au$, $y \in Bu$, $z \in Cu$, $w \in Du$ and the sequences $\{u_n\}_{n\geq 0}$, $\{x_n\}_{n\geq 0}$, $\{y_n\}_{n\geq 0}$, $\{z_n\}_{n\geq 0}$, $\{w_n\}_{n\geq 0}$ defined in Algorithm 3.1 converge strongly to u, x, y, z, w, respectively.

Proof. Put $E_n = (1-\rho)gu_n + \rho N(ax_n, by_n, cz_n) + \rho f$ and $E = (1-\rho) + \rho N(ax, by, cz) + \rho f$. Since $J_\rho^{\partial\varphi(\cdot,y)}$ is nonexpansive, by (3.3), (3.4) and the assumptions of Theorem 3.1 we deduce that

$$\begin{aligned}
&\|u_{n+1} - u_n\| \\
&\le (1-t)\|u_n - u_{n-1}\| + t\|u_n - u_{n-1} - (gu_n - gu_{n-1})\| \\
&\quad + t\|J_\rho^{\partial\varphi(\cdot,dw_n)}(E_n) - J_\rho^{\partial\varphi(\cdot,dw_n)}(E_{n-1})\| \\
&\quad + t\|J_\rho^{\partial\varphi(\cdot,dw_n)}(E_{n-1}) - J_\rho^{\partial\varphi(\cdot,dw_{n-1})}(E_{n-1})\| \\
&\le (1 - t + t\sqrt{1-2h+l^2})\|u_n - u_{n-1}\| + t\mu\|dw_n - dw_{n-1}\| \\
&\quad + t\|E_n - E_{n-1}\| \\
&\le (1 - t + t\sqrt{1-2h+l^2} + t\mu\delta s(1+n^{-1}))\|u_n - u_{n-1}\| \\
&\quad + t(1-\rho)\|gu_n - gu_{n-1} - (u_n - u_{n-1})\| \\
&\quad + t\|(1-\rho)(u_n - u_{n-1}) + \rho(N(ax_n, by_n, cz_n) \\
&\quad - N(ax_{n-1}, by_n, cz_n))\| + t\rho\|N(ax_{n-1}, by_n, cz_n) \\
&\quad - N(ax_{n-1}, by_{n-1}, cz_n) - (u_n - u_{n-1})\| \\
&\quad + t\rho\|N(ax_{n-1}, by_{n-1}, cz_n) - N(ax_{n-1}, by_{n-1}, cz_{n-1}) \\
&\quad + u_n - u_{n-1}\| \\
&\le (1 - (1-\theta_n)t)\|u_n - u_{n-1}\|,
\end{aligned} \tag{3.12}$$

where

$$\begin{aligned}
\theta_n &= (2-\rho)\sqrt{1-2h+l^2} + \mu\delta s(1+n^{-1}) \\
&\quad + \sqrt{(1-\rho)^2 - 2\rho(1-\rho)\sigma + \rho^2\xi^2\alpha^2p^2(1+n^{-1})^2} \\
&\quad + \rho\sqrt{1-2\nu+\eta^2\beta^2q^2(1+n^{-1})^2} + t\rho\sqrt{1+2m+\zeta^2\gamma^2r^2(1+n^{-1})^2} \\
&\to \theta = k + \sqrt{(1-\rho)^2 - 2\rho(1-\rho)\sigma + \rho^2\xi^2\alpha^2p^2} + \rho j,
\end{aligned}$$

as $n \to \infty$. From (3.8), we get that

$$\begin{aligned}
\theta < 1 &\Leftrightarrow \sqrt{(1-\rho)^2 - 2\rho(1-\rho)\sigma + \rho^2\xi^2\alpha^2p^2} < 1 - k - \rho j \\
&\Leftrightarrow (i - j^2)\rho^2 - 2\rho(1 + \sigma - (1-k)j) < -(2k - k^2).
\end{aligned} \tag{3.13}$$

Since at least one of (3.9), (3.10) and (3.11) is satisfied, by (3.13) we conclude easily that $\theta < 1$. Put $L = \frac{1}{2}(1+\theta)$. Then there exists a positive integer M such that $\theta_n < L < 1$ for all $n \ge M$. It follows from (3.12) that

$$\|u_{n+1} - u_n\| \le (1 - (1-L)t)\|u_n - u_{n-1}\|, \quad \forall n \ge M, \tag{3.14}$$

which implies that $\{u_n\}_{n\ge0}$ is a Cauchy sequence. (3.4) and (3.14) yield that $\{x_n\}_{n\ge0}$, $\{y_n\}_{n\ge0}$, $\{z_n\}_{n\ge0}$ and $\{w_n\}_{n\ge0}$ are Cauchy sequences. Consequently there exist $u, x, y, z, w \in H$ satisfying $u_n \to u$, $x_n \to x$, $y_n \to y$, $z_n \to z$, $w_n \to w$ as $n \to \infty$. Clearly,

$$d(x, Au) \le \|x - x_n\| + H(Au_n, Au) \le \|x - x_n\| + p\|u_n - u\| \to 0,$$

as $n \to \infty$. That is, $x \in Au$. Similarly, we have $y \in Bu$, $z \in Cu$ and $w \in Du$. By virtue of (3.6) and the nonexpansivity of $J_\rho^{\partial\varphi(\cdot,y)}$, we know that

$$\begin{aligned}
&\|J_\rho^{\partial\varphi(\cdot,dw_n)}(E_n) - J_\rho^{\partial\varphi(\cdot,dw)}(E)\| \\
&\leq \|J_\rho^{\partial\varphi(\cdot,dw_n)}(E_n) - J_\rho^{\partial\varphi(\cdot,dw)}(E_n)\| + \|J_\rho^{\partial\varphi(\cdot,dw)}(E_n) - J_\rho^{\partial\varphi(\cdot,dw)}(E)\| \\
&\leq \mu\|dw_n - dw\| + (1-\rho)\|gu_n - gu\| \\
&\quad + \rho\|N(ax_n, by_n, cz_n) - N(ax_{n-1}, by_n, cz_n)\| \\
&\quad + \rho\|N(ax_{n-1}, by_n, cz_n) - N(ax_{n-1}, by_{n-1}, cz_n)\| \\
&\quad + \rho\|N(ax_{n-1}, by_{n-1}, cz_n) - N(ax_{n-1}, by_{n-1}, cz_{n-1})\| \\
&\leq \mu\delta\|w_n - w\| + (1-\rho)l\|u_n - u\| + \rho(\xi\alpha\|x_n - x\| \\
&\quad + \eta\beta\|y_n - y\| + \zeta\gamma\|z_n - z\|),
\end{aligned}$$

which implies that

$$\lim_{n\to\infty} J_\rho^{\partial\varphi(\cdot,dw_n)}(E_n) = J_\rho^{\partial\varphi(\cdot,dw)}(E). \tag{3.15}$$

It follows from (3.3) and (3.15) that

$$u = (1-t)u + t(u - gu + J_\rho^{\partial\varphi(\cdot,dw)}((1-\rho)gu + \rho N(ax, by, cz) + \rho f)).$$

¿From the above equation and Lemma 3.1 we obtain that $u \in H$, $x \in Au$, $y \in Bu$, $z \in Cu$ and $w \in Du$ are a solution of the generalized nonlinear quasivariational inclusion (2.1). This completes the proof.□

Example 4. *Remark 3.3 Theorem 3.1 is an improvement and generalization of Theorem 4.1 in [1], [7] and [8], Theorem 3.1 in [4], [10] and [16], Theorems 4.1 and 4.2 in [5], and Theorem 2.1 in [12].*

Theorem 3.2. *Let $\varphi, f, g, a, b, c, d$ and N be as in Theorem 3.1. Let A, B, C, $D : H \to CB(H)$ be H-Lipschitz continuous with constants p, q, r, s, respectively, A be relaxed Lipschitz with constant σ with respect to a and the first argument of N, and C be generalized pseudo-contractive with constant m with respect to c and the third argument of N. Let*

$$\begin{aligned}
&k = \mu\delta s + \sqrt{1 - 2h + l^2}, \quad i = 1 + 2(\sigma - m) + (\alpha\xi p + \gamma\zeta r)^2, \\
&j = \eta\beta q - \sqrt{1 - 2h + l^2} > 0.
\end{aligned} \tag{3.16}$$

Assume that there exists p e (0,1] *such that k* 4 pj < *l holds and at least one of the following conditions*

i > j^2, (i + <r - m - (i - *k)j)*^{2} >*(i~ j*^{2}*)(l - (l ~ k)*^{2}*),*

l *i* _j_ ,j _ *m* _ ^ _ &)j j

r ^ J 2 l (^{3}-^{17})

***i* _p**

<«,', l + , - m > (l - * w , p > ^ ^ ^ (3·18)

< < J'^{2}, ((« - *)j - 1 - tr + m)^2 > (1 - (1 - A;)^2)(j^2 - i),

I (I — fe)j — 1 - *a* 4- m t

r j ^{2} - r ~ 1 (3.19)

^ v^r^jTTi o-+m)^2 + (g - g - *i)(j*^{2} - o

is *satisfied. Then the generalized nonlinear quasivariational inclusion* (2.1) *has a solution u* € *H, x* € Au, t/ € J3u, 2 € Cu, u> G Du and í/ie *sequences* {u$_n$}$_n$>o, *{x$_n$}n>o, {yn}n>o,* {^n}n>0t *{w$_n$}n>o defined in Algorithm 3.2 converges strongly to u, x, y, z, w, respectively.*

Proof. Since *g* is strongly monotone and Lipschitz continuous, it follows that

IK+i - unii < *rHgun+i - gu$_n$*\\. (3.20)

As in the proof of Theorem 3.1, by (3.5), (3.20) and the assurnptions of Theorem 3.2 we infer that

||«n+l — Un||

< *r*^{l}*ll\\dw$_n$ - dWn~.l*\\ + $^{i-1}$(1 ~ *P)\\9U$_n$* ™ *9U$_n$-i - (u$_n$* ~ U$_n$-1)||

+ /"^{1}||(1 - *p)(u$_n$* - w$_n$-i) + *p(N(ax$_n$,hy$_n$,cz$_n$)* ~ *N(ax$_n$-i,by$_{nt}$cz$_n$)*

4- *N(ax$_n$-i,by$_n$-i,cz$_n$)* - *N(ax$_n$-i,byn-uczn-i)*\\

+ *r*^{l}*p**N(ax$_{rl}^{i}_j$by$_n$,cz$_n$)* . . . .- *N(ax$_n$-i,by$_n$-i,cz$_n$)*\\

< *9$_n$u$_n$* -Un-.l||,**

where

9$_n$ - l^lfiósil + n'^{1}) 4 (1 - *p)y/'l-2h + l*^{2}

+ V(l - P)^2 - 2(f^ p)p(a - *m)* 4- p^2(c*CP + C7^)^2(l 4- *rr^*

+ pC/?g(l+n-^{1})] '

_* *9* = r*(fc 4- x/(l - *p)*^{2} ~ 2(1 - *p)p(a- m)* 4- *p*^{2}*(aip + j(r)*^{2} 4- *pj)*

as n —+ 00. The remaining portion of the proof can be derived as in Theorem 3.1 and is therefore omitted. This completes the proof.D

Example 5. *Remark 3.4 Theorem 3.2 extends, improves and unifies Theorems 4.1-4.3 in [2], Theorem 3.1 in [9], [11] and [13], and Theorem 3.6 in [15].*

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References

[1] R. Ahmad and Q. H. Ahsari*An iterative algorithm for generalized nonlinear varitational inclusions* Appl. Math. Lett. **13**, 2000, 23–26.

[2] M. R. Bai, Y. Y. Tang and Y. P. Liu*Generalized implicit quasivariational inequalities with relaxed Lipschitz and relaxed monotone mappings* Appl. Math. Lett.**12**, 1999, 69–76.

[3] X. P. Ding*Perturbed proximal point algorithms for generalized quasivariational inclusions* J. Math. Anal. Appl. **210**, 1997, 88–101.

[4] N. J. Huang*Generalized nonlivear variational inclusions with noncompact valued mappings* Appl. Math. Lett. **9**, 1996, 25–29.

[5] N. J. Huang*On the generalized implicit quasivariational inequalities* J. Math. Anal. Appl. **216**, 1997, 197–210.

[6] S. B. Nadler, Jr.*Multi-valued contraction mappings* Pacific J. Math. **30**, 1969, 475–488.

[7] M. A. Noor*Generalized multivalued quasi-variational inequalities* Computers Math. Applic. **31**, 1996, 1–13.

[8] M. A. Noor*Generalized multivalued quasi-variational inequalities* (II) Computers Math. Applic.**35**, 1998, 63–78.

[9] A. H. Siddiqi and Q. H. Ansari*Strongly nonlinear quasivariational inequalities* J. Math. Anal. Appl. **149**, 1990, 444–450.

[10] A. H. Siddiqi and Q. H. Ansari*General strongly nonlinear variational inequalities* J. Math. Anal. Appl. **166**, 1992, 386–392.

[11] R. U. Verma*On generalized variational inequalities involving relaxed Lipschitz and relaxed monotone operators* J. Math. Anal. Appl.**213**, 1997, 387–392.

[12] R. U. Verma*Generalized variational inequalities involving multivalued relaxed monotone operators* Appl. Math. Lett. **10**, 1997, 107–109.

[13] R. U. Verma*Generalized variational inequalities and associated nonlinear equations* Czechoslovak Math. J. **48(123)**, 1998, 413–418.

[14] R. U. Verma*The solvability of a class of generalized nonlinear variational inequalities based on an iterative algorithm* Appl. Math. Lett. **12**, 1999, 51–53.

[15] J. C. Yao*Applications of variational inequalities to nonlinear analysis* Appl. Math. Lett. **4**, 1991, 89–92.

[16] J. H. Zhang*General strongly nonlinear variational inequalities for multifunctions* Appl. Math. Lett.**8**, 1995, 75–80.

Zeqing Liu: Department of Mathematics, Liaoning Normal University, P. O. Box 200, Dalian, Liaoning 116029, People's Republic of China
E-mail: zeqingliui@sina.com.cn

Lokenath Debnath: Department of Mathematics, University of Texas - Pan American, Edinburg, Texas 78539, U. S. A.
E-mail: debnathl@panam.edu

Shin Min kang: Department of Mathematics and Research Institute of Natural Science, Gyeongsang National University, Chinju 660-701, Korea
E-mail: smkang@nongae.gsnu.ac.kr

Jeong Sheok Ume: Department of Applied Mathematics, Changwon National University, Changwon 641-773, Korea
E-mail: jsume@sarim.changwon.ac.kr